REFLECTIVE CLASSES OF SEQUENTIALLY BASED CONVERGENCES, SEQUENTIAL CONTINUITY AND SEQUENCE-RICH FILTERS

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ABSTRACT. We investigate under what conditions sequentially continuous maps between convergence spaces are continuous. Along the way, we provide a new characterization of the Urysohn property for convergence of sequences in terms of a functorial inequality, and introduce a new class of filters, called sequence-rich, intermediate between first-countable and Fréchet α_2.

1. INTRODUCTION

Definitions and notations concerning convergence spaces follow [8]. We gather this information in an appendix for the sake of completeness.

Beattie and Butzmann make in [2] a compelling argument that *convergence vector spaces* offer a more convenient framework for functional analysis than topological vector spaces. A convergence vector space is a vector space equipped with a convergence structure making the addition and scalar multiplication continuous. Convergent vector spaces are particularly convenient for analysis in part because countability conditions are more often present than in the topological setting (e.g., spaces of test functions and of distributions are second-countable when viewed as convergence vector spaces), allowing frequent sequential arguments. However, one needs to be cautious because even among first countable convergence spaces, continuity and sequential continuity are different notions. Recall that a convergence is first-countable if whenever $x \in \lim\mathcal{F}$, there exists a countably based filter $\mathcal{D} \leq \mathcal{F}$ such that $x \in \lim\mathcal{D}$. In [3] *sequentially determined* convergences were introduced and shown to be –among first countable convergences– exactly the class of convergences for which sequential continuity and continuity coincide.

In the present paper, we investigate the question of when sequential continuity and continuity coincide, and more generally when $\mathbb{D}$-continuity and continuity coincide, where $\mathbb{D}$ is a class of filters and a map $f : (X, \xi) \to (Y, \tau)$ is $\mathbb{D}$-*continuous* if $f(\lim_\xi \mathcal{D}) \subset \lim_\tau f(\mathcal{D})$ for every $\mathcal{D} \in \mathbb{D}$. In this context, we do not need to restrict ourselves to first-countable spaces. However various instances of $\mathbb{D}$-*based convergences*, that is, convergences for which whenever $x \in \lim\mathcal{F}$, there exists a filter $\mathcal{D}$ of $\mathbb{D}$ coarser than $\mathcal{F}$ such that $x \in \lim\mathcal{D}$, play a fundamental role. In particular, in results on sequential continuity, countably based filters can be replaced by the larger class of *sequence-rich filters*, which is in some sense the largest class that could be used. Doing so, we improve some results of [3] and [2]. The class of sequence-rich filters, studied in Section 5, is a proper subclass of Fréchet α_2-filters ([1]).

[1]A filter is *Fréchet* if $A\#\mathcal{F}$ implies that there is a sequence on A finer than $\mathcal{F}$. Notice that a topological space has the Fréchet-Urysohn property if and only if each of its neighborhood filters is Fréchet. See Section 5 for the definition of α_2.

In many arguments, *sequentially based convergences*, that is, convergences based in the class $\mathbb{E}$ of filters generated by sequences, as well as *sequentially based modifications* of convergences, play an important role. An essential ingredient is to know when a convergence of sequences (a sequentially based convergence) is the convergence of sequences for a topology, a pretopology, a paratopology, a pseudotopology. We give characterizations of such sequentially based convergences in terms of functorial inequalities in Section 2. Similar characterizations were known, in particular from S. Dolecki and G. Greco [9]. However, the characterization of convergence of sequences for paratopologies as *Urysohn convergences*, that is, convergences for which a sequence whose every subsequence has a subsequence converging to x converges to x, is of particular interest and seems entirely new.

2. Reflective classes of sequentially based convergence spaces

In this section, we investigate the conditions under which a convergence of sequences can be considered as the convergence of sequences for a convergence in a specific reflective class of convergences. In particular we consider this property for the reflective subcategories $\mathbf{S}$ of pseudotopological, $\mathbf{P}_\omega$ of paratopological, $\mathbf{P}$ of pretopological, and $\mathbf{T}$ of topological spaces. It can be interpreted by a functorial inequality. Recall that Seq denotes the sequentially based coreflector of **Conv**.

Proposition 1. *Let J be a reflector of* **Conv**. *The convergence* $\mathrm{Seq}\xi$ *is the convergence of sequences for a* $\mathbf{J}$*-convergence if and only if*

$$\xi \le \mathrm{Seq}J\mathrm{Seq}\xi. \tag{1}$$

In particular, a sequentially based convergence τ is the convergence of sequences for a $\mathbf{J}$-convergence if and only if $\tau \le \mathrm{Seq}J\tau$.

Proof. If ξ satisfies (1) then $\mathrm{Seq}\xi = \mathrm{Seq}J\mathrm{Seq}\xi$ so that $\mathrm{Seq}\xi$ is the convergence of sequences for the $\mathbf{J}$-convergence $J\mathrm{Seq}\xi$. Conversely, if there is $\sigma = J\sigma$ such that $\mathrm{Seq}\xi = \mathrm{Seq}\sigma$, then $\xi \le \mathrm{Seq}\xi = \mathrm{Seq}\sigma = \mathrm{Seq}J\sigma$ and $J\sigma \le J\mathrm{Seq}\sigma = J\mathrm{Seq}\xi$. Hence, $\xi \le \mathrm{Seq}J\mathrm{Seq}\xi$. □

This general fact –a particular case of a general scheme based on Galois connections presented in [9]– takes more specific forms in the case where the reflector J is the pseudotopologizer S and when J is the paratopologizer P_ω.

The following observation is a consequence of [9, Theorem 5.2].

Lemma 2.

$$\lim_{\mathrm{Seq}(S\xi)} (\mathcal{F} \wedge \mathcal{G}) = \lim_{\mathrm{Seq}(S\xi)}\mathcal{F} \cap \lim_{\mathrm{Seq}(S\xi)}\mathcal{G}.$$

Proposition 3. *For every convergence ξ,*

$$S(\mathrm{Seq}\xi) \ge \mathrm{Seq}(S\xi).$$

Proof. Let $x \in \lim_{S(\mathrm{Seq}\xi)}\mathcal{F}$. Then for every $\mathcal{U} \in \mathbb{U}(\mathcal{F})$, there exists a sequence $(y_n^{\mathcal{U}})_n \le \mathcal{U}$ such that $x \in \lim_\xi (y_n^{\mathcal{U}})_n$. If $\mathcal{U}$ is free, we can assume $(y_n^{\mathcal{U}})_n$ to be a free sequence. Otherwise, $\mathcal{U} = \{u\}^\uparrow$ is a constant sequence converging to x. Either way, $\mathcal{U}$ contains a countable (possibly finite) set $E_{\mathcal{U}}$. Thus, by compactness of $\mathbb{U}(\mathcal{F})$ in $\beta|\xi|$ ([2]), there exists a finite collection $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of ultrafilters finer than $\mathcal{F}$ such

[2] If X is a set, βX ($\beta_* X$) denotes the set of its (free) ultrafilters endowed with its Stone topology (which is compact Hausdorff). A base for this topology is formed by $\{\beta U = \{\mathcal{U} \in \beta X : U \in \mathcal{U}\} : \mathcal{U} \subset X\}$ (with $\beta\varnothing = \varnothing$).

that $\bigcup_{i=1}^{n} E_{\mathcal{U}_i} \in \mathcal{F}$. Hence $\mathcal{F}$ contains a countable set. If this set is finite, let $(w_n)_{n\omega}$ be a sequence whose sequential filter is $\bigcup_{i=1}^{n} E_{\mathcal{U}_i}$. If $\bigcup_{i=1}^{n} E_{\mathcal{U}_i}$ is infinite, let $(w_n)_n$ denote the cofinite filter of this countable set. We show that $x \in \lim_{S\xi}(w_n)_n$. Indeed, if $\mathcal{W} \in \mathbb{U}((w_n)_n)$, then $\mathcal{W} \in \mathbb{U}(\bigcup_{i=1}^{n} E_{\mathcal{U}_i})$ and therefore, there is $i \in \{1 \ldots n\}$ such that $E_{\mathcal{U}_i} \in \mathcal{W}$. If $E_{\mathcal{U}_i} = \{u_i\}$, then $\mathcal{W} = \{u_i\}^{\uparrow}$ converges to x. Otherwise, $\mathcal{W} \geq (y_n^{\mathcal{U}_i})_n$ so that $x \in \lim_{\xi}\mathcal{W}$.

If $\mathcal{F}$ is free then $\mathcal{F} \geq (w_n)_n$, so that $x \in \lim_{\mathrm{Seq}(S\xi)}\mathcal{F}$. If $\mathcal{F}$ is principal then it is the principal filter of a countable set, and therefore is a sequential filter ([3]). Hence $x \in \lim_{\mathrm{Seq}(S\xi)}\mathcal{F}$ because $x \in \lim_{S\xi}\mathcal{F}$. Now,

$$\begin{aligned}
\lim{}_{S(\mathrm{Seq}\xi)}\mathcal{F} &= \lim{}_{S(\mathrm{Seq}\xi)}\mathcal{F}^{\circ} \wedge \mathcal{F}^{\bullet} \\
&= \lim{}_{S(\mathrm{Seq}\xi)}\mathcal{F}^{\circ} \cap \lim{}_{S(\mathrm{Seq}\xi)}\mathcal{F}^{\bullet} \\
&\subset \lim{}_{\mathrm{Seq}(S\xi)}\mathcal{F}^{\circ} \cap \lim{}_{\mathrm{Seq}(S\xi)}\mathcal{F}^{\bullet} \\
&\subset \lim{}_{\mathrm{Seq}(S\xi)}\mathcal{F},
\end{aligned}$$

the last inclusion following from Lemma 2. □

Corollary 4. *The convergence* Seqξ *is a pseudotopology if and only if*

$$\xi \leq \mathrm{Seq}S\mathrm{Seq}\xi.$$

In particular, $\mathbf{Seq} \cap \mathbf{S}$ *is the class of convergence of sequences for a pseudotopology.*

Proof. If $\xi \leq \mathrm{Seq}S\mathrm{Seq}\xi$, then $\mathrm{Seq}\xi \leq \mathrm{Seq}S\mathrm{Seq}\xi$, and, in view of Proposition 3, $\mathrm{Seq}\xi \leq S\mathrm{Seq}\mathrm{Seq}\xi = S\mathrm{Seq}\xi$. Hence $\mathrm{Seq}\xi$ is a pseudotopology. Conversely, if $\mathrm{Seq}\xi$ is pseudotopological, then $\mathrm{Seq}\xi \leq S\mathrm{Seq}\xi \leq \mathrm{Seq}S\mathrm{Seq}\xi$. Hence, $\xi \leq \mathrm{Seq}S\mathrm{Seq}\xi$.

In particular $\tau = \mathrm{Seq}\tau$ is pseudotopological if and only if $\tau \leq \mathrm{Seq}S\tau$. In this case, τ is the convergence of sequences for a pseudotopology: itself. Conversely, if $\tau = \mathrm{Seq}\sigma$ for some $\sigma = S\sigma$, then $\tau = \mathrm{Seq}S\sigma \leq S\mathrm{Seq}\sigma = S\tau$, so that τ is pseudotopological. □

Recall that a convergence space is *Urysohn* ([4]) if a sequence converges to x whenever every subsequence has a subsequence which converges to x. This property is also called *sequentially Choquet* in [2], and *sequentially maximal* in [3]. A characterization of this property by an inequality of the type (1) was unknown, despite the extensive study of both the Urysohn property and functorial inequalities in [9]. Hence, the following new result completes the general scheme.

Proposition 5. *A convergence* ξ *is Urysohn if and only if*

$$\xi \leq \mathrm{Seq}P_{\omega}\mathrm{Seq}\xi.$$

In particular, a sequentially based convergence τ *is Urysohn if and only if*

$$\tau \leq \mathrm{Seq}P_{\omega}\tau \tag{2}$$

if and only if it is the convergence of sequences for a paratopology.

[3]If $A = \{a_i : i \in \omega\}$, then $A^{\uparrow}$ coincides with the sequential filter of the sequence $a_1, a_2, a_1, a_3, a_1, a_2, a_4, a_1, a_2, a_3, a_5, \ldots$

[4]This is not to be confused with the notion of a $T_{2\frac{1}{2}}$ topological, sometimes also called Urysohn topological space, which means that two disjoint points always have open neighborhood with disjoint closures (e.g., [11]).

Proof. Assume that ξ is Urysohn and let $x \in \lim_{P_\omega \mathrm{Seq}\xi}(x_n)_n$. Every subsequence $(x_{n_k})_k$ of $(x_n)_n$ is a countably based filter meshing with $(x_n)_n$ so that $x \in \mathrm{adh}_{\mathrm{Seq}\xi}(x_{n_k})_k$. Hence, there exists a subsequence of $(x_{n_{k_p}})_p$ that converges to x. By the Urysohn property, $x \in \lim_\xi (x_n)_n$.

Conversely, assume that $\xi \leq \mathrm{Seq}P_\omega\mathrm{Seq}\xi$ and that $(x_n)_n$ has the property that each of its subsequences has a subsequence converging to x. Then $x \in \lim_{P_\omega \mathrm{Seq}\xi}(x_n)_n \subset \lim_\xi (x_n)_n$. Indeed, if $\mathcal{H}$ is a countably based filter such that $\mathcal{H}\#(x_n)_n$ then there is a subsequence $(y_n)_n$ of $(x_n)_n$ which is finer than $\mathcal{H}$. Since $(y_n)_n$ admits a subsequence that converges to x, we have that $x \in \mathrm{adh}_{\mathrm{Seq}\xi}\mathcal{H}$. □

While the characterization (16) of countably Choquet spaces immediately implies the observation [2] that pseudotopological spaces (called Choquet spaces in [2]) are countably Choquet, the characterization of the Urysohn property obtained above shows that ([5]):

Corollary 6. *Every paratopology is Urysohn.*

The converse of Corollary 6 is false, as shows the following example.

Example 7 (Non paratopological Urysohn convergence)**.** *Let τ be a non first-countable strongly Fréchet topology. In view of Appendix* (13)*, we have $\tau = P_\omega \mathrm{Seq}\tau$. As τ is not first-countable, $\tau < \mathrm{Seq}\tau$, so that* $\mathrm{Seq}\tau$ *is not paratopological. However,* $\mathrm{Seq}\tau$ *is the convergence of sequences for a topology and is therefore Urysohn.*

The following example improves [9, Example 5.6] which gives a sequentially based pseudotopological space that is not Urysohn. Indeed, the convergence constructed below is also Hausdorff, which was not the case of the cited example.

Example 8 (Hausdorff non-Urysohn sequentially based pseudotopology)**.** *Let X be a countably infinite set and let ∞ be an element of X. Consider a free ultrafilter $\mathcal{W}$ and define a convergence ξ on X in which ∞ is the only non-isolated point by $\lim_\xi \mathcal{F} = \{\infty\}$ if $\mathcal{F}$ is either the principal ultrafilter of ∞ or is free and does not mesh $\mathcal{W}$. The convergence ξ is not Urysohn, because each sequence on X contains a subsequence, the range of which does not belong to $\mathcal{W}$, hence converging to ∞, but the sequence that generates the cofinite filter of X does not converge, because it meshes $\mathcal{W}$. The convergence ξ is a pseudotopology, so that* $\mathrm{Seq}\xi$ *is also a pseudotopology in view of Corollary 4, because if $\infty \in \lim_\xi \mathcal{U}$ for each ultrafilter $\mathcal{U} \geq \mathcal{F}$, then for each such $\mathcal{U}$ there is $U_\mathcal{U} \in \mathcal{U} \backslash \mathcal{W}^\#$, hence by the compactness of $\mathbb{U}(\mathcal{F})$ in $\beta\omega$ there exist $n < \omega$ and $\mathcal{U}_1, \ldots, \mathcal{U}_n$ finer than $\mathcal{F}$ such that $U_{\mathcal{U}_1} \cup \ldots \cup U_{\mathcal{U}_1} \in \mathcal{F} \backslash \mathcal{W}^\#$ proving that $\infty \in \lim_\xi \mathcal{F}$.*

The following is another example of a Hausdorff non-Urysohn sequentially based pseudotopology.

Example 9. [3, Example 2.17 (i)] *Let $\mathcal{A} = \{A \subset \mathbb{R} : \sum_{a \in A} |a| < \infty\}$ and let ξ be the convergence on $\mathbb{R}$ in which all points but 0 are isolated and $0 \in \lim_\xi \mathcal{F}$ if $0 \in \lim_\mathbb{R} \mathcal{F}$ and $\mathcal{F} \cap \mathcal{A} \neq \varnothing$. The convergence ξ is Hausdorff because it is finer*

[5]The claim that Choquet spaces are Urysohn [2] turns out to be false, as shows Example 8 or [3, Example 2.17(i)]. Also the terminology "sequentially Choquet" seems inconsistent. In view of Proposition 3, it seems more consistent to call *sequentially Choquet* convergences that are pseudotopological and sequentially based. As shown in Example 8, it does not coincide with sequential convergences with the Urysohn property. In view of Proposition 5, a more coherent alternative name for the Urysohn property can be *sequentially paratopological.*

than the usual topology of $\mathbb{R}$. *It is pseudotopological because if each element of* $\mathbb{U}(\mathcal{F})$ *contains a summable set, so does* $\mathcal{F}$ *by compactness of* $\mathbb{U}(\mathcal{F})$ *in* $\beta\mathbb{R}$. *It is not Urysohn because each subsequence of* $(\frac{1}{n})_{n\in\mathbb{N}}$ *has a convergent subsequence, but* $(\frac{1}{n})_{n\in\mathbb{N}}$ *does not converge. In view of Proposition 3,* $\mathrm{Seq}\xi$ *is a Hausdorff sequentially based pseudotopology that is not Urysohn.*

In view of Proposition 1, a sequentially based convergence τ is the convergence of sequences for a topology if and only if $\tau \leq \mathrm{Seq}T\tau$. Because $T \leq P_\omega$, it is obvious that such convergences must be Urysohn. Among Hausdorff convergences (but not in general), the converse is true (e.g., [15], [6], [9, Corollary 7.4]). Hence, in view of Proposition 5,

Corollary 10. *If* ξ *is a Hausdorff paratopology then the convergent sequences for* ξ, $P\xi$ *and* $T\xi$ *are the same.*

The examples above show that this is not true for a Hausdorff pseudotopology.

Further conditions characterizing convergence of sequences for pseudotopologies, pretopologies and topologies can be found in [9].

3. Modified continuity

It is well known (e.g., [11]) that a sequentially continuous map between two topological spaces is continuous provided that the domain is a sequential space, that is, sequentially closed sets are closed. We shall see that this classical fact not only extends to convergence spaces but is an instance of a general but simple scheme.

Proposition 11. *Let* $F : \mathbf{Conv} \to \mathbf{Conv}$ *be a (concrete) functor. If* $f : F\xi \to F\tau$ *is continuous,* $\xi \geq F\xi$ *and* $\tau \leq F\tau$, *then* $f : \xi \to \tau$ *is continuous.*

Proof. As $f : F\xi \to F\tau$ is continuous, $F\tau \leq f(F\xi)$. Under our assumptions

$$\tau \leq F\tau \leq f(F\xi) \leq f\xi,$$

so that $f : \xi \to \tau$ is continuous. □

If a map $f : |\xi| \to |\tau|$ is sequentially continuous then $f : \mathrm{Seq}\xi \to \mathrm{Seq}\tau$ is continuous. If J is a functor then $f : J\mathrm{Seq}\xi \to J\mathrm{Seq}\tau$ is also continuous, so that Proposition 11 applies with $F = J\mathrm{Seq}$. In particular when J runs over T, P, P_ω, S, we obtain, in view of Appendix (13):

Corollary 12.
(1) *A sequentially continuous map* $f : |\xi| \to |\tau|$ *from a sequential convergence (*$\xi \geq T\mathrm{Seq}\xi$*) to a convergence*

$$\tau \leq T\mathrm{Seq}\tau$$

(in particular to a topology) is continuous;

(2) *A sequentially continuous map* $f : |\xi| \to |\tau|$ *from a Fréchet convergence (*$\xi \geq P\mathrm{Seq}\xi$*) to a convergence*

$$\tau \leq P\mathrm{Seq}\tau$$

(in particular to a pretopology) is continuous;

(3) *A sequentially continuous map* $f : |\xi| \to |\tau|$ *from a strongly Fréchet convergence (*$\xi \geq P_\omega\mathrm{Seq}\xi$*) to a convergence*

$$\tau \leq P_\omega\mathrm{Seq}\tau \tag{3}$$

(in particular to a paratopology) is continuous;

(4) *A sequentially continuous map $f : |\xi| \to |\tau|$ from a sequentially based pseudotopology to a convergence*

$$\tau \leq S\mathrm{Seq}\tau$$

(in particular a pseudotopology) is continuous.

Notice that given a projector J and a coprojector E, properties of the type $\tau \leq EJ\tau$ and of the type $\tau \leq JE\tau$ are of different nature. In particular, (2) and (3) should be carefully distinguished. For instance the convergence $\mathrm{Seq}\tau$ in Example 7 satisfies (2) but not (3). However, one easily sees that a convergence satisfying (3) must be Urysohn by applying the expansive modifier Seq to (3).

A convergence space is *sequentially determined* [3] if a countably based filter converges to x whenever each finer sequence does.

One of the main motivations for the introduction of *sequentially determined convergence spaces* by R. Beattie and H.P. Butzmann is that in general sequential continuity of a map between two convergence spaces does not imply continuity, even if these convergence spaces are first-countable. However

Theorem 13. [3, Theorem 2.10] *If (X, ξ) is first-countable and (Y, τ) is sequentially determined, then $f : (X, \xi) \to (Y, \tau)$ is continuous if and only if it is sequentially continuous.*

Among the large classes of convergence spaces shown to be sequentially determined are all first-countable pretopological spaces, second-countable convergence spaces and web-spaces (see [3] and [2]). It is interesting to note that every first-countable convergence is Fréchet and every first-countable pretopological space (even every pretopological space!) is a $\mathbf{PSeq}^{\geq I}$-convergence. Hence Corollary 12 (2) gives a useful alternative to Theorem 13. However,

Proposition 14. *Every $\mathbf{PSeq}^{\geq I}$-convergence is sequentially determined.*

Proof. If $\mathcal{F}$ is countably based, then $\mathcal{F} = \bigwedge_{(x_n)_n \geq \mathcal{F}} (x_n)_n$. If each $(x_n)_n$ finer than $\mathcal{F}$ converges to x for ξ, then $\mathcal{F} = \bigwedge_{(x_n)_n \geq \mathcal{F}} (x_n)_n$ converges for $P\mathrm{Seq}\xi$, hence for ξ because $P\mathrm{Seq}\xi \geq \xi$. □

Recall that a map $f : |\xi| \to |\tau|$ is called $\mathbb{J}$-*continuous* if $f : \mathrm{Base}_{\mathbb{J}}\xi \to \tau$ is continuous. The class $\mathbb{J}$ is *transferable* if $f(\mathcal{J}) \in \mathbb{J}(Y)$ whenever $f : X \to Y$ and $\mathcal{J} \in \mathbb{J}(X)$. If $\mathbb{J}$ is transferable then $f : |\xi| \to |\tau|$ is $\mathbb{J}$-continuous if and only if $f : \mathrm{Base}_{\mathbb{J}}\xi \to \mathrm{Base}_{\mathbb{J}}\tau$ is continuous. In the next section, we study conditions that ensure that $\mathbb{J}$-continuous maps are continuous.

4. Convergences determined by finer filters

Let $\mathbb{J}$ be a class of filters. A convergence ξ is called *determined by finer $\mathbb{J}$-filters* if

$$\lim_{\xi}\mathcal{F} = \bigcap_{\mathcal{J} \in \mathbb{J}(\mathcal{F})} \lim_{\xi}\mathcal{J}.$$

Proposition 15. *The class of convergences determined by finer $\mathbb{J}$-filters is projective and the associated projector is given by*

$$\lim_{U_{\mathbb{J}}\xi}\mathcal{F} = \bigcap_{\mathcal{J} \in \mathbb{J}(\mathcal{F})} \lim_{\xi}\mathcal{J}.$$

Proof. It is easy to verify that $U_{\mathbb{J}}\xi \leq U_{\mathbb{J}}\tau$ whenever $\xi \leq \tau$, that $U_{\mathbb{J}}\xi \leq \xi$, and that $U_{\mathbb{J}}$ idempotent. $\square$

Note that if $\mathbb{J}(\mathcal{F}) = \varnothing$ then $\lim_{U_{\mathbb{J}}\xi}\mathcal{F} = |\xi|$. Hence, $U_{\mathbb{J}}\xi$ typically fails to be Hausdorff.

Recall [3] that a convergence is called *sequentially determined* if a countably based filter converges to a point whenever every finer sequence does, and that $\mathbb{E}$ denotes the class of filters generated by sequences. It is now immediate that

Proposition 16. *A convergence ξ is sequentially determined if and only if*

$$\mathrm{Base}_{\mathbb{F}_1}U_{\mathbb{E}}\xi \geq \xi.$$

Example 17. *If ν denotes the usual topology of the real line, then* $\mathrm{Seq}\nu$ *is not sequentially determined. However, ν is sequentially determined by Proposition 14.*

Proposition 18. *Let $\mathbb{J}$ be a class of filters. Then:*

1. $\mathcal{J} \in \mathbb{J} \implies \lim_{U_{\mathbb{J}}\xi}\mathcal{J} = \lim_{\xi}\mathcal{J}$;
2. $\mathrm{Base}_{\mathbb{J}}U_{\mathbb{J}} = \mathrm{Base}_{\mathbb{J}}$;
3. $U_{\mathbb{J}}\mathrm{Base}_{\mathbb{J}} = U_{\mathbb{J}}$.

A filter $\mathcal{D}$ on X is $\mathbb{J}$-*rich* if for every $f : X \to Y$ and every $\mathcal{J} \in \mathbb{J}(f(\mathcal{D}))$, there exists $\mathcal{G} \in \mathbb{J}(\mathcal{D})$ such that $\mathcal{J} \geq f(\mathcal{G})$. Recall [14] that a class $\mathbb{J}$ of filters is called *steady* if for every meshing $\mathbb{J}$-filters $\mathcal{J}$ and $\mathcal{G}$, the filter $\mathcal{J} \vee \mathcal{G}$ is also in $\mathbb{J}$. A class $\mathbb{D}$ of filters is called $\mathbb{J}$-*composable* if $\mathcal{J}(\mathcal{D}) = \{J(D) : J \in \mathcal{J}, D \in \mathcal{D}\}$ is a (possibly degenerate) $\mathbb{D}$-filter (on Y) whenever $\mathcal{D}$ is a $\mathbb{D}$-filter (on X) and $\mathcal{J}$ is a $\mathbb{J}$-filter (on $X \times Y$). Here we assume that every class of filters contains the degenerate filter on each set.

Lemma 19. *If $\mathbb{J}$ is $\mathbb{F}_0$-composable and steady, then every $\mathbb{J}$-filter is $\mathbb{J}$-rich.*

Proof. Let $f : X \to Y$, $\mathcal{F} \in \mathbb{J}(X)$ and $\mathcal{J} \in \mathbb{J}(f(\mathcal{F}))$. Then for each $F \in \mathcal{F}$, there exists $A_F \subset F$ such that $f(A_F) \in \mathcal{J}$. By $\mathbb{F}_0$-composability of $\mathbb{J}$, $f^{-}\mathcal{J} \in \mathbb{J}$. Moreover $f^{-}\mathcal{J}\#\mathcal{F}$. Since $\mathbb{J}$ is steady, $f^{-}\mathcal{J} \vee \mathcal{F} = \mathcal{G} \in \mathbb{J}(\mathcal{F})$. For each $J \in \mathcal{J}$ and $F \in \mathcal{F}$, there exists $A_F \subset f^{-}J \cap F$. Hence $f(f^{-}J \cap F) \in \mathcal{J}$. Thus $\mathcal{J} \geq f(\mathcal{G})$. $\square$

Proposition 20. *$U_{\mathbb{J}}$ is a functor (hence, a reflector) if and only if every filter is $\mathbb{J}$-rich.*

Proof. Consider $f : X \to (Y, \tau)$, and let $x \in \lim_{U_{\mathbb{J}}f^{-}\tau}\mathcal{F}$, that is, $x \in \lim_{f^{-}\tau}\mathcal{G}$ for every $\mathcal{G} \in \mathbb{J}(\mathcal{F})$. Hence $f(x) \in \lim_{\tau} f(\mathcal{G})$ for every $\mathcal{G} \in \mathbb{J}(\mathcal{F})$. Let $\mathcal{J} \in \mathbb{J}(f(\mathcal{F}))$. Because $\mathcal{F}$ is $\mathbb{J}$-rich, there exists $\mathcal{G} \in \mathbb{J}(\mathcal{F})$ such that $\mathcal{J} \geq f(\mathcal{G}) \geq f(\mathcal{F})$. But $f(x) \in \lim_{\tau} f(\mathcal{G})$, thus $f(x) \in \lim_{\tau}\mathcal{J}$. Therefore $f(x) \in \lim_{U_{\mathbb{J}}\tau} f(\mathcal{F})$, that is, $x \in \lim_{f^{-}(U_{\mathbb{J}}\tau)}\mathcal{F}$.

Conversely, assume that there exists a non $\mathbb{J}$-rich filter $\mathcal{F}$ on X, that is, there exists $f : X \to Y$ and $\mathcal{J} \in \mathbb{J}(f(\mathcal{F}))$ such that for every $\mathcal{G} \in \mathbb{J}(\mathcal{F})$, $\mathcal{J} \notin \mathbb{J}(f(\mathcal{G}))$. Let $x_0 \in X$ and let ξ denote the atomic convergence on X defined by $x_0 \in \lim_{\xi}\mathcal{H}$ if there exists $\mathcal{G} \in \mathbb{J}(\mathcal{F})$ such that $\mathcal{H} \geq \mathcal{G}$. Then $f : \xi \to f\xi$ is continuous but $f : U_{\mathbb{J}}\xi \to U_{\mathbb{J}}(f\xi)$ is not. Indeed, $x_0 \in \lim_{U_{\mathbb{J}}\xi}\mathcal{F}$, that is, $f(x_0) \in \lim_{f(U_{\mathbb{J}}\xi)} f(\mathcal{F})$. But $f(x_0) \notin \lim_{U_{\mathbb{J}}(f\xi)} f(\mathcal{F})$ because $f(x_0) \notin \lim_{f\xi}\mathcal{J}$. $\square$

Recall that $\mathbb{F}$ denotes the class of all filters and $\mathbb{U}$ denotes the class of ultrafilters. It is obvious that

Lemma 21. *Every filter is $\mathbb{F}$-rich and $\mathbb{U}$-rich. The reflector $U_{\mathbb{F}}$ is the identity functor; and the reflector $U_{\mathbb{U}}$ is the pseudotopologizer.*

Moreover,

Lemma 22. *If $\mathbb{J}$ contains principal ultrafilters and there exists a filter $\mathcal{F}$ such that $\mathbb{J}(\mathcal{F})=\varnothing$, then $\mathcal{F}$ is not $\mathbb{J}$-rich. In particular, there exist non $\mathbb{F}_0$-rich, non $\mathbb{F}_1$-rich, non $\mathbb{E}$-rich filters.*

Proof. Under these assumptions on $\mathbb{J}$, let $f : X \to \{*\}$. Then $\{*\}^{\uparrow} \in \mathbb{J}(f(\mathcal{F}))$ but $\mathbb{J}(\mathcal{F})=\varnothing$. □

Theorem 23. *Let $\mathbb{D}$ be a class of $\mathbb{J}$-rich filters. Then $f : \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\xi \to U_{\mathbb{J}}\tau$ is continuous whenever f is $\mathbb{J}$-continuous.*

Proof. Assume that $f : \mathrm{Base}_{\mathbb{J}}\xi \to \tau$ is continuous and that $x \in \lim_{U_{\mathbb{J}}\xi}\mathcal{D}$, where $\mathcal{D} \in \mathbb{D}$. Let $\mathcal{J} \in \mathbb{J}(f(\mathcal{D}))$. Because $\mathcal{D}$ is $\mathbb{J}$-rich, there exists $\mathcal{G} \in \mathbb{J}(\mathcal{D})$ such that $\mathcal{J} \geq f(\mathcal{G}) \geq f(\mathcal{D})$. But $x \in \lim_{\mathrm{Base}_{\mathbb{J}}\xi}\mathcal{G}$ so that $f(x) \in \lim_{\tau}\mathcal{J}$ by $\mathbb{J}$-continuity of f. Hence, $f(x) \in \lim_{U_{\mathbb{J}}\tau}\mathcal{D}$. □

Corollary 24. *Let $\mathbb{D}$ be an $\mathbb{F}_0$-composable class of $\mathbb{J}$-rich filters. Let $\xi \geq \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\xi$ and $\tau \leq \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\tau$. If $f : |\xi| \to |\tau|$ is $\mathbb{J}$-continuous, then $f : \xi \to \tau$ is continuous.*

Proof. If f is $\mathbb{J}$-continuous, then $f : \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\xi \to U_{\mathbb{J}}\tau$ is continuous by Theorem 23, hence $f : \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\xi \to \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\tau$ is continuous because $\mathbb{D}$ is $\mathbb{F}_0$-composable. In view of the assumptions, we have

$$f\xi \geq f(\mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\xi) \geq \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\tau \geq \tau$$

so that $f : \xi \to \tau$ is continuous. □

If R is a (symmetric) relation on X and $A \subset X$, we denote by A^R the *polar* of A, that is, the set $\{x \in X : a \in A \Longrightarrow xRa\}$. Consider the relation $\triangle$ on $\mathbb{F}(X)$ introduced in [14] by $\mathcal{F}\triangle\mathcal{H}$ if

$$\mathcal{F}\#\mathcal{H} \Longrightarrow \exists \mathcal{L} \in \mathbb{F}_1 : \mathcal{L} \geq \mathcal{F} \vee \mathcal{H}.$$

Recall that $\mathbb{F}_0$ denotes the class of principal filters and that $\mathbb{F}_1$ denotes the class of countably based filters. Therefore, in this notation $\mathbb{F}_0^{\triangle}$ is the class of Fréchet filters and $\mathbb{F}_1^{\triangle}$ is the class of *strongly Fréchet filters*. Let also $\mathcal{R}(\mathbb{E})$ denote the class of $\mathbb{E}$-rich filters, where $\mathbb{E}$ is the class of filters generated by sequences. In view of the results of Section 5, we have $\mathbb{F}_1 \subset \mathcal{R}(\mathbb{E}) \subset \mathbb{F}_1^{\triangle} \subset \mathbb{F}_0^{\triangle}$ so that

$$U_{\mathbb{F}_0^{\triangle}} \geq U_{\mathbb{F}_1^{\triangle}} \geq U_{\mathcal{R}(\mathbb{E})} \geq U_{\mathbb{F}_1} \geq U_{\mathbb{E}}$$

and

$$\mathrm{Seq} = \mathrm{Base}_{\mathbb{E}} \geq \mathrm{Base}_{\mathbb{F}_1} \geq \mathrm{Base}_{\mathcal{R}(\mathbb{E})} \geq \mathrm{Base}_{\mathbb{F}_1^{\triangle}} \geq \mathrm{Base}_{\mathbb{F}_0^{\triangle}}.$$

Note also that $U_{\mathbb{F}_0^{\triangle}}\xi = U_{\mathbb{E}}\xi$ if $\xi = P\xi$. Moreover, Proposition 14 can be generalized as follows

Proposition 25. *Let $\mathbb{J}$ be a class of filters such that $\mathbb{J}^{\triangle} \neq \varnothing$. Then*

$$\mathrm{Base}_{\mathbb{J}^{\triangle}}U_{\mathbb{E}} \geq \mathrm{Adh}_{\mathbb{J}}\mathrm{Seq}. \tag{4}$$

Proof. Let $\mathcal{F} \in \mathbb{J}^{\triangle}$ such that $x \in \lim_{U_{\mathbb{E}}}\mathcal{F}$ and let $\mathcal{J} \in \mathbb{J}$ such that $\mathcal{J}\#\mathcal{F}$. Then, there exists a countably based and hence a sequence $(x_n)_{n\in\mathbb{N}}$ finer than $\mathcal{J} \vee \mathcal{F}$. In view of $x \in \lim_{U_{\mathbb{E}}}\mathcal{F}$, we conclude that $x \in \lim_{\xi}(x_n)_n$. Therefore, $x \in \mathrm{adh}_{\mathrm{Seq}\xi}\mathcal{J}$. □

In particular, when $\mathbb{J} = \mathbb{F}_0$, we have $\mathrm{Base}_{\mathbb{F}_0^{\triangle}}U_{\mathbb{E}} \geq P\mathrm{Seq}$, so that:

Corollary 26. (1) *If $\xi \leq P\mathrm{Seq}\xi$ (in particular a pretopology) then $\xi \leq \mathrm{Base}_{\mathbb{F}_0^{\triangle}}U_{\mathbb{E}}\xi$;*

(2) *If* $\xi \geq \mathrm{Base}_{\mathbb{F}_0^{\triangle}} U_{\mathbb{E}} \xi$ *then* $\xi \geq P\mathrm{Seq}\xi$.

When $\mathbb{J} = \mathbb{F}_1$, we have $\mathrm{Base}_{\mathbb{F}_1^{\triangle}} U_{\mathbb{E}} \geq P_\omega \mathrm{Seq}$, so that:

Corollary 27. (1) *If* $\xi \leq P_\omega \mathrm{Seq}\xi$ *(in particular a paratopology) then* $\xi \leq \mathrm{Base}_{\mathbb{F}_1^{\triangle}} U_{\mathbb{E}} \xi$ *(in particular,* $\xi \leq .\mathrm{Base}_{\mathcal{R}(\mathbb{E})} U_{\mathbb{E}} \xi$ *and* ξ *is sequentially determined);*

(2) *If* $\xi \geq \mathrm{Base}_{\mathbb{F}_1^{\triangle}} U_{\mathbb{E}} \xi$ *(in particular if* $\xi \geq \mathrm{Base}_{\mathcal{R}(\mathbb{E})} U_{\mathbb{E}} \xi$*) then* $\xi \geq P_\omega \mathrm{Seq}\xi$.

Notice that Corollary 27(1) generalizes [3, Proposition 2.3] stating that a pretopology is sequentially determined in both directions: it weakens significantly the assumption and yields a stronger conclusion.

In connection with Proposition 25, note that

$$\mathbb{J} \subset \mathbb{F}_1^{\triangle} \Longrightarrow \mathrm{Adh}_{\mathbb{J}} \mathrm{Seq} = \mathrm{Adh}_{\mathbb{J}} \mathrm{Base}_{\mathbb{F}_1}.$$

Moreover, the inequality (4) cannot be reversed, even in the simplest case ($\mathbb{J} = \mathbb{F}_0$) as shows the following example.

Example 28 (A sequentially determined convergence such that $\xi \nleq P\mathrm{Seq}\xi$ and $\xi \nleq \mathrm{Base}_{\mathbb{F}_0^{\triangle}} U_{\mathbb{E}} \xi$)**.** *The convergence defined in Example 9 has these properties. Indeed, if a countably based filter converging to* 0 *in the usual topology of the real line does not converge in* ξ*, then its countable base* $(A_n)_{n\in\omega}$ *is made of unsummable sets. In* A_1*, pick finitely many terms* $x_1, x_2 ..., x_{n_1}$ *such that* $\sum_{i=1}^{i=n_1} |x_i| \geq 1$*. Similarly, pick finitely many terms that add up to at least one in each* A_n*, and form a sequence* $(x_p)_{p\in\omega}$ *by concatenation. Note that* $(x_p)_{p\in\omega} \geq (A_n)_{n\in\omega}$ *and* $(x_p)_{p\in\omega}$ *does not converge for* ξ*, so that* ξ *is sequentially determined. Also, if we had* $\xi \leq P\mathrm{Seq}\xi$ *then we would have* $\xi \leq \mathrm{Seq}P_\omega\mathrm{Seq}\xi$ *and* ξ *would be Urysohn. Take uncountably many sequences* $(x_n^\alpha)_{n\in\omega}$ ξ*-convergent to* 0 *with disjoint supports. Then any sequence* $(y_n)_{n\in\omega} \geq \bigwedge_{\alpha\in I} (x_n^\alpha)_{n\in\omega}$ *must be finer than the infimum of finitely many of the sequences* $(x_n^\alpha)_{n\in\omega}$ *and is therefore convergent to* 0*. But* $\bigwedge_{\alpha\in I} (x_n^\alpha)_{n\in\omega}$ *has a basis of unsummable sets and therefore does not converge to* 0*. Hence* $\xi \nleq \mathrm{Base}_{\mathbb{F}_0^{\triangle}} U_{\mathbb{E}} \xi$.

By comparison, we have:

Proposition 29. *Example 8 is sequentially determined if and only if the ultrafilter* $\mathcal{W}$ *is a* P*-point in* $\beta_*\omega$ *(*[6]*).*

Proof. This convergence is sequentially determined if every non-convergent countably based filter admits a non-convergent finer sequence. In other words, for each $\mathcal{H} \in \mathbb{F}_1$ such that $\mathcal{H} \leq \mathcal{W}$ there is a sequence $(x_n)_{n\in\omega}$ such that $\mathcal{H} \leq (x_n)_{n\in\omega} \leq \mathcal{W}$, that is, each countable subfamily of $\mathcal{W}$ has a pseudo-intersection. □

Therefore, we can always chose $\mathcal{W}$ to make this convergence non sequentially determined, and we can consistently chose $\mathcal{W}$ to make the convergence sequentially determined.

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[6]A *P-point* in a topological space is a point at which every countable intersection of neighborhoods is a neighborhood. It is consistent that P-points do not exist in $\beta_*\omega$, but there are P-points in $\beta_*\omega$ under (CH). See e.g., [20].

Example 30 (A convergence τ such that $\tau \leq \mathrm{Base}_{\mathbb{F}_0^\triangle} U_{\mathbb{E}}\tau$ but $\tau \not\leq P\mathrm{Seq}\tau$)**.** *Consider the convergence ξ defined in Example 9, and let $\tau = \mathrm{Base}_{\mathbb{F}_0^\triangle} U_{\mathbb{E}}\xi$. Then $\tau = \mathrm{Base}_{\mathbb{F}_0^\triangle} U_{\mathbb{E}}\tau$ by definition. Notice that a sequence converges for τ if and only if it converges for ξ, that is $\mathrm{Seq}\tau = \mathrm{Seq}\xi$. The infimum of all sequences convergent to 0 for ξ (or τ) is a Fréchet filter with some non convergent finer sequences (for instance $(\frac{1}{n})_{n\in\omega}$). Hence it does not converge in τ. Therefore $\tau \not\leq P\mathrm{Seq}\tau$.*

When $\mathbb{J} = \mathbb{E}$ and $\mathbb{D} = \mathbb{F}_1$, Corollary 24 particularizes to the following:

Corollary 31. *Let $\xi \geq \mathrm{Base}_{\mathbb{F}_1} U_{\mathbb{E}}\xi$ (in particular, a first-countable convergence) and let τ be a sequentially determined convergence, that is, $\tau \leq \mathrm{Base}_{\mathbb{F}_1} U_{\mathbb{E}}\tau$ (in particular, a paratopology). Then every sequentially continuous map $f : |\xi| \to |\tau|$ is continuous*

When $\mathbb{J} = \mathbb{E}$ and $\mathbb{D} = \mathcal{R}(\mathbb{E})$, Corollary 24 particularizes to the following:

Corollary 32. *Let $\xi \geq \mathrm{Base}_{\mathcal{R}(\mathbb{E})} U_{\mathbb{E}}\xi$ (in particular, an $\mathcal{R}(\mathbb{E})$-based convergence) and let $\tau \leq \mathrm{Base}_{\mathcal{R}(\mathbb{E})} U_{\mathbb{E}}\tau$ (in particular a paratopology). Then every sequentially continuous map $f : |\xi| \to |\tau|$ is continuous.*

Notice that, in view of Example 33, the first Corollary 31 refines [2, Theorem 1.5.12] that states the same result for a first-countable domain. Corollary 32 provides a new variant with a weaker assumption on the domain and a stronger assumption on the range, even though this assumption is still satisfied by each $\mathbf{PSeq}^{\geq I}$-convergence and each paratopology. Moreover, in view of Example 30, it applies to cases that cannot be handled by Corollary 12 (2).

Example 33 (A non first-countable convergence $\xi \geq \mathrm{Base}_{\mathbb{F}_1} U_{\mathbb{E}}\xi$)**.** *Let $\mathcal{H}$ be a uniform ([7]) countably based filter on an uncountable set X. Define on $X \cup \{\infty\}$ the convergence ξ in which every point but ∞ is isolated and $\infty \in \lim_\xi \mathcal{F}$ if either $\mathcal{F}$ is finer than a sequence finer than $\mathcal{H}$, or $\mathcal{F}$ is a uniform ultrafilter of $\mathcal{H}$. Then, ξ is not first-countable, because uniform ultrafilters of $\mathcal{H}$ are not finer than any sequence, and therefore are not finer than any countably based convergent filter. On the other hand, by definition of ξ, $\infty \in \lim_{U_{\mathbb{E}}\xi} \mathcal{H}$, since every (non principal) ξ-convergent filter is finer than $\mathcal{H}$. It shows that $\xi \geq \mathrm{Base}_{\mathbb{F}_1} U_{\mathbb{E}}\xi$.*

Also, Corollary 24 and therefore also its instances Corollaries 31 and 32, are best possible in the following sense:

Proposition 34.
(1) *If there exists a non $\mathbb{J}$-rich $\mathbb{D}$-filter on X then there exists $\xi \geq \mathrm{Base}_{\mathbb{D}}\xi$ on X, $\tau \leq U_{\mathbb{J}}\tau$, and $f : |\xi| \to |\tau|$ that is $\mathbb{J}$-continuous but not continuous.*
(2) *Assume $\mathbb{J} \subset \mathbb{D}$. If $\xi \not\geq \mathrm{Base}_{\mathbb{D}} U_{\mathbb{J}}\xi$ then there exists $\tau = \mathrm{Base}_{\mathbb{D}} U_{\mathbb{J}}\tau$ and $f : |\xi| \to |\tau|$ that is $\mathbb{J}$-continuous but not continuous.*
(3) *Assume $\mathbb{J} \subset \mathbb{D}$. If $\tau \not\leq \mathrm{Base}_{\mathbb{D}} U_{\mathbb{J}}\tau$ then there exists $\xi = \mathrm{Base}_{\mathbb{D}} U_{\mathbb{J}}\xi$ and $f : |\xi| \to |\tau|$ that is $\mathbb{J}$-continuous but not continuous.*

Note that the assumption $\mathbb{J} \subset \mathbb{D}$ is natural in view of Lemma 19.

Proof. (1). Assume that there exists a non $\mathbb{J}$-rich filter $\mathcal{D} \in \mathbb{D}$, that is, there exists $f : X \to Y$, $\mathcal{D} \in \mathbb{D}(X)$, and $\mathcal{J} \in \mathbb{J}(f(\mathcal{D}))$ such that for every $\mathcal{G} \in \mathbb{J}(\mathcal{D})$, $\mathcal{J} \notin \mathbb{J}(f(\mathcal{G}))$. Let $x_0 \in X$ and let ξ denote the atomic topology on X defined by $\mathcal{N}_\xi(x_0) =$

[7]A filter on X is called *uniform* if all of its elements have the cardinality of X.

$\mathcal{D}\wedge\{x_0\}$. By definition, $\xi \geq \mathrm{Base}_{\mathbb{D}}\xi$ (and therefore $\xi \geq \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\xi$). Let τ be the atomic convergence on Y for which $f(x_0) \in \lim_\tau \mathcal{F}$ if for every $\mathcal{J} \in\mathbb{J}(\mathcal{F})$, there exists $\mathcal{G} \in\mathbb{J}(\mathcal{D})$ such that $\mathcal{J} \geq f(\mathcal{G})$. Then $\tau \leq U_{\mathbb{J}}\tau$ (and therefore $\tau \leq \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\tau$). Moreover, $f : |\xi| \to |\tau|$ is $\mathbb{J}$-continuous. But f is not continuous, because $f(x_0) \notin \lim_\tau f(\mathcal{D})$.

(2). If $\xi \ngeq \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\xi$, let $\tau = \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\xi$. Then

$$\mathrm{Base}_{\mathbb{J}}\tau = \mathrm{Base}_{\mathbb{J}}\mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\xi = \mathrm{Base}_{\mathbb{J}}U_{\mathbb{J}}\xi = \mathrm{Base}_{\mathbb{J}}\xi,$$

because $\mathrm{Base}_{\mathbb{J}} \geq \mathrm{Base}_{\mathbb{D}}$ and because of Proposition 18. Hence the identity map $i : |\xi| \to |\tau|$ is $\mathbb{J}$-continuous, but not continuous.

(3). If $\tau \nleq \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\tau$, let $\xi = \mathrm{Base}_{\mathbb{D}}U_{\mathbb{J}}\tau$. Then $\mathrm{Base}_{\mathbb{J}}\xi = \mathrm{Base}_{\mathbb{J}}\tau$ so that the identity map $i : |\xi| \to |\tau|$ is $\mathbb{J}$-continuous, but not continuous. □

5. $\mathbb{E}$-rich filters

In view of the results of Section 4, the class $\mathcal{R}(\mathbb{E})$ of $\mathbb{E}$-rich filters is the biggest class of filters $\mathbb{J}$ such that $\mathrm{Base}_{\mathbb{J}}U_{\mathbb{E}}$ is a functor and therefore plays an essential role in investigating the range of results akin to Theorem 13 (see Corollary 31). In this section, we characterize $\mathbb{E}$-rich filters and compare them with other types of (Fréchet) filters.

Let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of X. We say that $\mathcal{A}$ *almost meshes with* $\mathcal{B}$, in symbol $\mathcal{A}\#_*\mathcal{B}$, if every $B \in \mathcal{B}$ meshes with all but finitely many elements of $\mathcal{A}$. Recall that a filter $\mathcal{F}$ is *substantial* if $\mathbb{U}(\mathcal{F})$ is infinite.

Theorem 35. *Let $\mathcal{F}$ be a filter on X. The following are equivalent:*

1. *$\mathcal{F}$ is $\mathbb{E}$-rich;*
2. *$\mathcal{F}$ is Fréchet and for every sequence $(A_n)_{n\in\omega}$ of* disjoint *non empty subsets of X*
$$(A_n)_{n\in\omega}\#_*\mathcal{F} \Longrightarrow \exists x_n \in A_n : (x_n)_{n\in\omega} \geq \mathcal{F}; \qquad (5)$$
3. *For each $A\#\mathcal{F}$, the filter $\mathcal{F}\vee A$ is substantial or principal and* (5) *is satisfied for every sequence $(A_n)_{n\in\omega}$ of* disjoint *non empty subsets of X.*

Notice a filter that satisfies (5) for every sequence of disjoint sets may not be $\mathbb{E}$-rich. Therefore the additional assumptions are essential. Indeed, in view of Lemma 22, a free ultrafilter is never $\mathbb{E}$-rich. However, on a set of measurable cardinality, there is a free countably deep ultrafilter. Such an ultrafilter $\mathcal{U}$ satisfies (5) for every sequence of disjoint sets because no such sequence can almost mesh with $\mathcal{U}$.

Before we prove Theorem 35, let us point out a property of filters satisfying (5) for every sequence $(A_n)_{n\in\omega}$ of *disjoint* subsets. Recall ([8]) that a filter$\mathcal{F}$ is α_i ($i = 1, 2, 3, 4$) if for each countable collection $(\mathcal{E}_i)_{i\in\omega}$ of sequences finer than $\mathcal{F}$, there is a sequence $\mathcal{E}' \geq \mathcal{F}$ whose range intersects

- α_1 : the range of each sequence $\mathcal{E}_i$ in a cofinite set;
- α_2 : the range of each sequence $\mathcal{E}_i$ in an infinite set;
- α_3 : the range of infinitely many sequences $\mathcal{E}_i$ in an infinite set;
- α_4 : the range of infinitely many sequences $\mathcal{E}_i$.

Lemma 36. *Every filter on X satisfying* (5) *for every sequence $(A_n)_{n\in\omega}$ of disjoint subsets of X is α_2.*

[8]Properties α_i (i=1,2,3,4) were introduced in [1] for spaces. They correspond to neighborhood filters being α_i-filters.

Proof. Let $(B_n)_{n\in\omega}$ be the ranges of countably many sequences finer than $\mathcal{F}$. By [19, Lemma 1.2], we can assume the B_n's to be pairwise disjoint. Partition each B_n into countably many infinite subsets B_n^k. Each B_n^k is the range of a sequence finer than $\mathcal{F}$. Hence $\{B_n^k : n,k\in\omega\}$ is the collection of ranges of countably many disjoint sequences finer than $\mathcal{F}$; so that they all mesh with $\mathcal{F}$. By (5), we can pick one element in each B_n^k to form a sequence $(x_n)_{n\in\omega}$ finer than $\mathcal{F}$. By construction, the range of that sequence intersects each B_n in an infinite set. □

Proof of Theorem 35. ($1 \Longrightarrow 2$) Suppose $\mathcal{F}$ is an $\mathbb{E}$-rich filter on X. Let $(A_n)_{n\in\omega}$ be a sequence of non empty pairwise disjoint sets such that $(A_n)_{n\in\omega}\#_*\mathcal{F}$. Pick $x_n\in A_n$ for each $n\in\omega$. Define $Y=\{x_n\colon n\in\omega\}\cup\left(X\setminus\left(\bigcup_{n\in\omega}A_n\right)\right)$. Define $f\colon X\to Y$ so that $f(A_n)=\{x_n\}$ for every n and $f(x)=x$ for all $x\in X\setminus\left(\bigcup_{n\in\omega}A_n\right)$. Since $(A_n)_{n\in\omega}\#_*\mathcal{F}$, $(x_n)_{n\in\omega}\geq f(\mathcal{F})$. Since $\mathcal{F}$ is $\mathbb{E}$-rich, there is a sequence $(w_n)_{n\in\omega}\geq\mathcal{F}$ such that $(f(w_n))_{n\in\omega}\leq(x_n)_{n\in\omega}$. Let $N\in\omega$ be large enough that $\{x_n\colon n\geq N\}\subseteq\{f(w_n)\colon n\geq\in\omega\}$. So, for each $n\geq N$ there is $k_n\in\omega$ such that $f(w_{k_n})=x_n$. By the definition of f, $w_{k_n}\in A_n$ for all $n\geq N$. Let $(z_n)_{n\in\omega}$ be the sequence defined by $z_n=w_{n_k}$ if $n\geq N$ and $z_n=x_n$ if $n<N$. It is easily verified that $(z_n)_{n\in\omega}$ satisfies (5).

Moreover, $\mathcal{F}$ is Fréchet. Indeed, if $A\#\mathcal{F}$ define $f:X\to(X\setminus A)\cup\{\infty\}$ by $f(x)=x$ if $x\notin A$ and $f(x)=\infty$ if $x\in A$. Then $\{\infty\}^\uparrow$ is a sequence finer than $f(\mathcal{F})$ so that there exists $(x_n)_{n\in\omega}\geq\mathcal{F}$ such that $(f(x_n))_{n\omega}=\{\infty\}^\uparrow$, that is, $x_n\in A$.

($2\Longrightarrow 3$) because every Fréchet filter satisfies $A\#\mathcal{F}\Longrightarrow\mathcal{F}\vee A$ is substantial or principal.

($3\Longrightarrow 1$) Suppose $\mathcal{F}$ is a filter on a set X satisfying (5) such that $\mathcal{F}\vee A$ is substantial or principal whenever $A\#\mathcal{F}$. We first show that $\mathcal{F}$ is Fréchet. Indeed, if $\mathcal{F}\vee A$ is principal, there is a (constant) sequence finer than $\mathcal{F}\vee A$. If $\mathcal{F}\vee A$ is substantial then the free filters of $\mathbb{U}(\mathcal{F}\vee A)$ form an infinite subset without isolated points of the Hausdorff topological space βX. Hence, we can find a sequence $(\beta A_n)_n$ of pairwise disjoint open subsets of $\mathbb{U}(\mathcal{F}\vee A)$. The sequence $(A_n)_{n\in\omega}$ is a sequence of pairwise disjoint subsets of A meshing with $\mathcal{F}$. By (5), there is a sequence finer than $\mathcal{F}\vee A$. Thus $\mathcal{F}$ is Fréchet.

Let Y be a set and $f\colon X\to Y$ be a function. Suppose $(y_n)_{n\in\omega}$ is a sequence on Y finer than $f(\mathcal{F})$. Since $(y_n)_{n\in\omega}\geq f(X)$, we may assume that $y_n\in f(X)$ for all n. In particular, $f^{-1}(y_n)\neq\emptyset$ for all n. Let T be the elements of $\{y_n\colon n\in\omega\}$ that appear infinitely many times in the sequence $(y_n)_{n\in\omega}$ and $S=\{y_n\colon n\in\omega\}\setminus T$.

Notice that $f^{-1}(y)\#\mathcal{F}$ for each $y\in T$. Since $\mathcal{F}$ is Fréchet, there is for each $y\in T$ a sequence $(x_n^y)_{n\in\omega}\geq f^{-1}(y)\bigvee\mathcal{F}$. Let T_1 be the set of all $y\in T$ such that $(x_n^y)_{n\in\omega}$ has a term x^y that is repeated infinitely often. Let $\mathcal{G}=\{x^y\colon y\in T_1\}^\uparrow$. Notice that $\mathcal{G}\geq\mathcal{F}$.

Let $T_2=T\setminus T_1$. If T_2 is finite, let $\mathcal{E}_1=\bigwedge_{y\in T_2}\{x_n^y\colon n\in\omega\}$. Notice that $\mathcal{E}_1\geq\mathcal{F}$ and $f^{-1}(y)$ has infinite intersection with the range of $\mathcal{E}_1$ for all $y\in T_2$. Suppose T_1 is infinite. In this case, since $\mathcal{F}$ is α_2, there is a sequence $\mathcal{E}_2\geq\mathcal{F}$ such that the range of $\mathcal{E}_2$ has infinite intersection with $f^{-1}(y)$ for every $y\in T_2$. In either case, we can find a sequence $\mathcal{E}_3$ such that $\mathcal{E}_3\geq\mathcal{F}$ such that the range of $\mathcal{E}_3$ has infinite intersection with $f^{-1}(y)$ for every $y\in T_2$. Clearly, $\mathcal{E}_3\geq\mathcal{F}$. Let $(y_{n_l})_{l\in\omega}$ be the subsequence of $(y_n)_{n\in\omega}$ consisting of terms from T_2. Let J be a tail of $\mathcal{E}_3$. Since every element of $f^{-1}(T_2)$ appears infinitely often in $\mathcal{E}_3$, $f(J)=T_2=\{y_{n_l}\colon l\in\omega\}$. Thus, $f(\mathcal{E}_3)\leq(y_{n_l})_{l\in\omega}$. Let $\mathcal{E}_4=\mathcal{E}_3\bigwedge\mathcal{G}$. Clearly, $\mathcal{E}_4\geq\mathcal{F}$. Notice that for every tail J of $\mathcal{E}_4$ we have $f(J) = T_2 \cup T_1$. It follows that $f(\mathcal{E}_4)$ is coarser than the subsequence $\mathcal{E}_5$ of $(y_n)_{n\in\omega}$ consisting of elements of T.

If S is finite, then $y_n \notin S$ for almost all $n \in \omega$. In this case, $f(\mathcal{E}_4) \leq \mathcal{E}_5 = (y_n)_{n\in\omega}$. So, we may assume that S is infinite. Let $\mathcal{E}_6 = (y_n)_{n\in\omega} \bigvee S$. Let $(y_{n_k})_{k\in\omega}$ be a subsequence of $\mathcal{E}_6$ in which each element of S appears exactly once. Since each element of S appears at most finitely many times in $(y_{n_k})_{k\in\omega} = \mathcal{E}_6$. Notice that $(f^{-1}(y_{n_k}))_{k\in\omega}\#\mathcal{F}$. By (5), there is a sequence $(w_{n_k})_{k\in\omega} \geq \mathcal{F}$ such that $w_{n_k} \in f^{-1}(w_{n_k})$ for all $k \in \omega$. Let I be a tail of $(w_{n_k})_{k\in\omega}$. Clearly, $f(I)$ is a tail of $(y_{n_k})_{k\in\omega} = \mathcal{E}_6$. So, $(f(w_{n_k}))_{k\in\omega} \leq \mathcal{E}_6$. Now, $\mathcal{E}_4 \bigwedge (w_{n_k})_{k\in\omega} \geq \mathcal{F}$ and $(f(w_{n_k}))_{k\in\omega} \bigwedge \mathcal{E}_4 \leq \mathcal{E}_6 \bigwedge \mathcal{E}_5 = (y_n)_{n\in\omega}$. Thus, $\mathcal{F}$ is $\mathbb{E}$-rich. □

Note that in particular neighborhood filters in a subsequential ([9]) topological space satisfy $A\#\mathcal{F} \Longrightarrow \mathcal{F}\vee A$ is substantial or principal.

Example 37. *Every first-countable and every cofinite filter is $\mathbb{E}$-rich.*

Let X be a metric space with metric d. Given subsets A and B of X we define $\mathrm{d}(A,B) = \inf\{\mathrm{d}(x,y)\colon x \in A \text{ and } y \in B\}$. We define the diameter of a set A by $\mathrm{diam}(A) = \sup\{\mathrm{d}(x,y)\colon x,y \in A\}$. Finally, by $B(x,r)$ we denote the open ball about x with radius r. Following [13], we denote by $\Gamma(X)$ the filter generated on the set $\mathcal{O}(X)$ of open subsets of X by sets of the form $\{U \in \mathcal{O}(X) : U \supset F\}$ where F ranges over finite subsets of X.

Proposition 38. *If X is a metric space, then $\Gamma(X)$ is $\mathbb{E}$-rich if and only if X is countable.*

Proof. Suppose X is metric and $\Gamma(X)$ is $\mathbb{E}$-rich. We will prove that X is countable. For every $k \in \omega$ define $\mathcal{U}_k$ to be the collection of all open sets U such that U is the union of finite collection $\mathcal{I}_U$ of open balls I such that $\mathrm{diam}(I) \leq 1/2^k$ and $\mathrm{d}(J,I) > 1/2^{k-1}$ for every pair of distinct element $I, J \in \mathcal{I}_U$. When we refer to a ball of U we mean an element of the finite collection $\mathcal{I}_U$.

We claim that $(\mathcal{U}_k)_{k\in\omega}$ almost meshes with $\Gamma(X)$. Let $F \subseteq X$ be finite. Suppose $F = \{x_1, \ldots, x_n\}$. Let l be large enough that $\mathrm{d}(x_i, x_j) > 1/2^l$ for all $1 \leq i < j \leq n$. Suppose $k \geq l$. For each $1 \leq i \leq n$ let $U_i = B(x_i, 1/2^{k+3})$. and $U = \bigcup_{i=1}^{n} U_i$. Now $\mathrm{d}(U_i, U_j) > 1/2^l - 1/2^{k+2} = (2^{k+2-l} - 1)/2^{k+2} > 1/2^{k+1}$ for each $1 \leq i < j \leq n$. Notice $\mathrm{diam}(U_i) \leq 1/2^{k+2}$ for each $1 \leq i \leq n$. So, $U \in \mathcal{U}_{k+2}$ and $F \subseteq U$. Thus, $(\mathcal{U}_k)_{k=1}^{\infty}$ almost meshes $\Gamma(X)$.

Let $k \geq 1$. Suppose $U \in \mathcal{U}_k$ and I is a ball of U. Let $V \in \mathcal{U}_{k+1}$ and J be a ball of V. Suppose $I \cap J \neq \emptyset$. Assume that K is a ball of V distinct from J. Since $\mathrm{d}(J,K) > 1/2^k \geq \mathrm{diam}(I)$, $K \cap I = \emptyset$. Thus, any ball of any $U \in \mathcal{U}_k$ has non empty intersection with at most one ball of any $V \in \mathcal{U}_{k+1}$.

Since $-(X)$ is $\mathbb{E}$-rich, there is a selection $U_k \in \mathcal{U}_k$ such that $(U_k)_{k=1}^{\infty} \geq \Gamma(X)$. So, $X = \underline{\lim} U_k$. Let $X_n = \bigcap_{n\leq k} U_k$. Let I_n be a ball of U_n. Assume that $x, w \in X_n \cap I_n$. Since $x, w \in U_{n+1} \cap I_n$, by the previous paragraph there is a single ball I_{n+1} of X_{n+1} that contains x and w. Continuing inductively, we may construct a sequence of balls $(I_k)_{k\geq n}$ such that $x, w \in I_k$ and I_k is a ball of U_k. Since $\lim \mathrm{diam}(I_k) = 0$, $x = w$. Since U_n has finitely many balls, X_n is finite. Thus, X is countable. □

The converse of Theorem 36 is (consistently) false, as shows the following example.

[9] i.e., a subspace of a sequential topological space.

Example 39 (An α_1 and Fréchet filter that is not $\mathbb{E}$-rich under $\omega_1 < \mathbf{p}$)**.** *Here* $\mathbf{p}$ *denotes the pseudo-intersection number and* $\mathbf{b}$ *denotes the bounding number* [16, p. 115]*. Under this assumption, there exists* [12] *an uncountable (of cardinality* ω_1*)* γ*-set* X *in* $\mathbb{R}$*. By* [13, Lemma 21]*,* $\Gamma(X)$ *is Fréchet. Because* $|X| < \mathbf{p} \leq \mathbf{b}$*,* $\Gamma(X)$ *is* α_1 *by* [19, Theorem 1.8]*. In view of Proposition 38,* $\Gamma(X)$ *is not* $\mathbb{E}$*-rich.*

6. Appendix

Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X *mesh*, in symbols $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \varnothing$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A filter on X is a family of subsets of X that is closed under finite intersection and supersets. The only filter containing the empty set is said to be *degenerate.* Filters on a set X are partially ordered by inclusion of families of sets. We denote the infimum of two filters $\mathcal{F}$ and $\mathcal{G}$ by $\mathcal{F} \wedge \mathcal{G}$ and the supremum (which exists only if $\mathcal{F}\#\mathcal{G}$) by $\mathcal{F} \vee \mathcal{G}$. We frequently identify subsets of X with their principal filters. We also identify a sequence $(x_n)_{n\in\omega}$ with the corresponding *sequential filter* $\{\{x_n : n \geq k\} : k \in \omega\}^{\uparrow}$. Elements of a class $\mathbb{D}$ of filters are called $\mathbb{D}$*-filters.* The set of filters on X of the class $\mathbb{D}$ is denoted $\mathbb{D}(X)$. In particular $\mathbb{F}$, $\mathbb{U}$, $\mathbb{F}_1$, $\mathbb{E}$, $\mathbb{F}_0$ denote the classes of all filters, of ultrafilters, of countably based, of sequential, and of principal filters respectively. If $\mathcal{F}$ is a filter on X, we denote by $\mathbb{D}(\mathcal{F})$ the set of filters of $\mathbb{D}(X)$ that are finer than $\mathcal{F}$.

Each filter can be decomposed as $\mathcal{F} = \mathcal{F}^{\circ} \wedge \mathcal{F}^{\bullet}$ where $\mathcal{F}^{\circ}$ is free, that is, if $\cap\mathcal{F} = \varnothing$, and $\mathcal{F}^{\bullet}$ is principal. Namely, the *principal part of a filter* $\mathcal{F}$ is $\mathcal{F}^{\bullet} = \cap\mathcal{F}$. This filter is the degenerate filter only if $\mathcal{F}$ is free. The *free part* $\mathcal{F}^{\circ}$ of a filter $\mathcal{F}$ is $\mathcal{F} \vee (\cap\mathcal{F})^c$. This filter is the degenerate filter only if $\mathcal{F}$ is principal.

A *convergence structure* on a set X is a relation lim between X and the set $\mathbb{F}(X)$ of filters on X that satisfies $x \in \lim\{x\}^{\uparrow}$ ([10]) for every $x \in X$ and $\lim\mathcal{F} \subset \lim\mathcal{G}$ whenever $\mathcal{F} \leq \mathcal{G}$. A map $f : (X, \xi) \to (Y, \tau)$ between two convergence spaces is *continuous* if $f(\lim_{\xi}\mathcal{F}) \subset \lim_{\tau} f(\mathcal{F})$ ([11]) for every $\mathcal{F} \in \mathbb{F}(X)$.

Let **Conv** denote the category of convergence spaces and continuous maps. If ξ is a convergence space (an object of **Conv**), we denote by $|\xi|$ its underlying set (that is, $|.|$ denotes the forgetful functor to **Set**). If ξ and τ are such that $|\xi| = |\tau|$, we say that ξ is *finer than* τ or that τ is *coarser than* ξ, in symbols $\xi \geq \tau$, if the identity map $i_{|\xi|} : \xi \to \tau$ is continuous. This partial order makes the set of convergences on a given set a complete lattice for which $\lim_{\vee_{i\in I}\xi_i}\mathcal{F} = \bigcap_{i\in I} \lim_{\xi_i}\mathcal{F}$ and $\lim_{\wedge_{i\in I}\xi_i}\mathcal{F} = \bigcup_{i\in I} \lim_{\xi_i}\mathcal{F}$.

We call *modifier of* **Conv** a map $M : Ob(\mathbf{Conv}) \to Ob(\mathbf{Conv})$ such that $|M\xi| = |\xi|$ for every ξ and $\xi \leq \tau \Longrightarrow M\xi \leq M\tau$. A class of convergence spaces that is closed under supremum is called *projective.* Dually, a class of convergences closed under infima is called *coprojective.* If $\mathbf{S}$ is a projective class of convergences containing indiscrete convergences, then for every convergence ξ, there exists the finest convergence $M\xi$ (on $|\xi|$) coarser than ξ. The map M is a contractive ($M\xi \leq \xi$) and idempotent ($MM\xi = M\xi$) modifier. Such modifiers are called *projectors*, and the class of fixed convergence spaces for a projector is projective. Dually, we call *coprojector* an idempotent and expansive ($\xi \leq M\xi$) modifier, and the class of fixed convergence spaces for a coprojector is coprojective. For instance, topological spaces

[10]If $\mathcal{A} \subset 2^X$, then $\mathcal{A}^{\uparrow} = \{B \subset X : \exists A \in \mathcal{A}, A \subset B\}$.

[11]Where $f(\mathcal{F})$ denotes the filter $\{f(F) : F \in \mathcal{F}\}^{\uparrow}$.

can be viewed as particular convergence spaces. The class of topological spaces is projective in **Conv**. Explicitly, the *topological projection* $T\xi$ of a convergence ξ is formed by the *open* subsets of ξ, that is, those that are element of every filter that ξ-converges to one of their points. On the other hand, sequentially based, and more generally $\mathbb{D}$-based, convergences form a coprojective subclass of **Conv**. The *sequentially based coprojection* $\mathrm{Seq}\xi$ of ξ is defined by

$$\lim_{\mathrm{Seq}\xi}\mathcal{F}=\bigcup_{(x_n)_{n\in\omega}\leq\mathcal{F}}\lim_{\xi}(x_n)_{n\in\omega},$$

where a sequence $(x_n)_{n\in\omega}$ is identified with the filter $\{\{x_k : k\geq n\} : n\in\omega\}^{\uparrow}$. More generally, to a class $\mathbb{D}$ of filters that does not depend on the convergence ([12]), S. Dolecki associated in [7] two fundamental modifiers of the category of convergence spaces; a projector $\mathrm{Adh}_{\mathbb{D}}$:

$$\lim_{\mathrm{Adh}_{\mathbb{D}}\xi}\mathcal{F}=\bigcap_{\mathbb{D}\ni\mathcal{D}\#\mathcal{F}}\mathrm{adh}_{\xi}\mathcal{D}, \tag{6}$$

where the adherence of a filter $\mathcal{D}$ is given by

$$\mathrm{adh}_{\xi}\mathcal{D}=\bigcup_{\mathcal{H}\#\mathcal{D}}\lim_{\xi}\mathcal{H};$$

and a coprojector $\mathrm{Base}_{\mathbb{D}}$ where

$$\lim_{\mathrm{Base}_{\mathbb{D}}\xi}\mathcal{F}=\bigcup_{\mathbb{D}\ni\mathcal{D}\leq\mathcal{F}}\lim_{\xi}\mathcal{D}. \tag{7}$$

Note that if F is a modifier (a functor) then $\mathbf{F}^{\leq I}=\{\xi : F\xi\leq\xi\}$ is coprojective (coreflective) and $\mathbf{F}^{\geq I}=\{\xi : F\xi\geq\xi\}$ is projective (reflective). If $\mathbf{C}$ is a class of convergences, we will often talk of a $\mathbf{C}$-*convergence* for an element of $\mathbf{C}$. If $\mathbf{C}$ is (co)projective, we use the convention that the same non-bold letter C stands for the corresponding (co)projector. If a (co)projector F appears as the primary object, we use the same bold letter to denote the corresponding (co)projective class.

Example 40 (projectors).
(1) *Let $\mathbb{D}$ be the class $\mathbb{F}_0$ of principal filters. Then the projective class associated with* $\mathrm{Adh}_{\mathbb{D}}$ *is that of* pretopologies [5], *also called Čech closure spaces after* [4]. *Therefore, we will often use P for the projector* $\mathrm{Adh}_{\mathbb{F}_0}$.
(2) *Let $\mathbb{D}$ be the class $\mathbb{F}_1$ of countably based filters. Then the projective class associated with* $\mathrm{Adh}_{\mathbb{D}}$ *is that of* paratopologies *introduced in* [7]. *Therefore, we will often use P_ω, as in* [7], *for the projector* $\mathrm{Adh}_{\mathbb{F}_1}$.
(3) *Let $\mathbb{D}$ be the class $\mathbb{F}$ of all filters. Then the projective class associated with* $\mathrm{Adh}_{\mathbb{D}}$ *is that of* pseudotopologies [5]. *We will often use S for the projector* $\mathrm{Adh}_{\mathbb{F}}$. *It is easy to see that $S=\mathrm{Adh}_{\mathbb{F}}=\mathrm{Adh}_{\mathbb{U}}$ where $\mathbb{U}$ is the class of ultrafilters.*

Example 41 (coprojectors).
(1) *Let $\mathbb{D}$ be the class $\mathbb{E}$ of filters generated by sequences. Then the coprojective class associated with* $\mathrm{Base}_{\mathbb{D}}$ *is that of sequentially based convergences, and the corresponding coprojector is denoted* Seq.

[12] In the general scheme, the class $\mathbb{D}$ may depend on the convergence in the sense that the $\mathbb{D}$-filters on $(|\xi|,\xi)$ may be different from the $\mathbb{D}$-filters on $(|\xi|,\tau)$. See [7], [10] for specific conditions on $\mathbb{D}$ to make $\mathrm{Adh}_{\mathbb{D}}$ a projector and to make $\mathrm{Base}_{\mathbb{D}}$ a coprojector.

(2) *Let $\mathbb{D}$ be the class $\mathbb{F}_1$ of countably based filters. Then the coprojective class associated with* $\mathrm{Base}_{\mathbb{D}}$ *is that of* first-countable *convergences.*
(3) *Let $\mathbb{D}$ be the class $\mathbb{F}_0$ of principal filters. Then the coprojective class associated with* $\mathrm{Base}_{\mathbb{D}}$ *is that of* finitely generated *convergences in the sense of* [18]*. Finitely generated pretopological spaces can be identified with (possibly infinite) directed graphs.*

A modifier M is a *functor* ([13]) if the continuity of $f : \xi \to \tau$ implies that of $f : M\xi \to M\tau$.

If $f : X \to \tau$ there exists the coarsest convergence $f^{-}\tau$ on X making f continuous. Dually, if $f : \xi \to Y$, there exists the finest convergence $f\xi$ on Y making f continuous. In this notation,

$$f : \xi \to \tau \text{ is continuous } \iff \xi \geq f^{-}\tau \iff f\xi \geq \tau. \tag{8}$$

Therefore, given a modifier M of **Conv**,

$$M \text{ is a functor } \iff \forall_{f:\xi\to Y}\ f(M\xi) \geq M(f\xi) \tag{9}$$
$$\iff \forall_{f:X\to\tau}\ M(f^{-}\tau) \geq f^{-}(M\tau). \tag{10}$$

If for every $f : X \to \tau$ and every $\mathcal{D} \in \mathbb{D}(\tau)$ the filter $f^{-}\mathcal{D}$ is a $\mathbb{D}$-filter on X, then $\mathrm{Adh}_{\mathbb{D}}$ is a functor and is then called a *reflector.* If for every $f : \xi \to Y$ and every $\mathbb{D}$-filter $\mathcal{D}$ on $|\xi|$ the filter $f(\mathcal{D})$ is a $\mathbb{D}$-filter, then $\mathrm{Base}_{\mathbb{D}}$ is a functor and is then called a *coreflector.* Hence, P, P_ω, S (and also T) are reflectors and Seq, $\mathrm{Base}_{\mathbb{F}_0}$, $\mathrm{Base}_{\mathbb{F}_1}$ are coreflectors.

Recall that a topological space X is

- *sequential* if every sequentially closed subset is closed;
- *Fréchet* if whenever $x \in X$, $A \subset X$ and $x \in \mathrm{cl}A$, there exists a sequence $(x_n)_{n\in\omega}$ on A such that $x \in \lim(x_n)_{n\in\omega}$;
- *strongly Fréchet* if whenever $x \in \bigcap_{n\in\omega} \mathrm{cl}A_n$ for a decreasing sequence of subsets A_n of X, there exists $x_n \in A_n$ such that $x \in \lim(x_n)_{n\in\omega}$;
- *bisequential* if every convergent ultrafilter contains a countably based filter that converges to the same point;
- *weakly bisequential* [17] if whenever $x \in \mathrm{adh}\mathcal{F}$ where $\mathcal{F}$ is a countably deep filter ([14]), there exists a countably based filter $\mathcal{H}\#\mathcal{F}$ such that $x \in \lim\mathcal{H}$.

Hence, a topology ξ is sequential if and only if ξ and $\mathrm{Seq}\xi$ have the same closed sets, that is, $T\xi = T\mathrm{Seq}\xi$, and since $\xi = T\xi \leq T\mathrm{Seq}\xi$ for every topology, if and only if

$$\xi \geq T\mathrm{Seq}\xi. \tag{11}$$

It is easy to see that (11) is equivalent to

$$\xi \geq T\mathrm{Base}_{\mathbb{F}_1}\xi. \tag{12}$$

Moreover, (11) and (12) are meaningful and equivalent for general convergences, and therefore can be used to extend the definition of sequential spaces from topological to convergence spaces.

Similarly, a topology ξ is Fréchet if $\mathrm{adh}_\xi A \subset \mathrm{adh}_{\mathrm{Seq}\xi} A$. This means that $\xi \geq P\mathrm{Seq}\xi$ or equivalently that $\xi \geq P\mathrm{Base}_{\mathbb{F}_1}\xi$.

[13]Of course, our definition of functor is much more restrictive than the accepted definition. Indeed, we restrict ourselves to concrete endofunctors of **Conv**.

[14]A filter $\mathcal{F}$ is *countably deep* if $\bigcap \mathcal{A} \in \mathcal{F}$ whenever $\mathcal{A}$ is a countable subfamily of $\mathcal{F}$.

More generally,

$$\mathcal{J} \in \mathbb{J} \implies \mathrm{adh}_\xi \mathcal{J} \subset \mathrm{adh}_{\mathrm{Base}_\mathbb{D}\xi} \mathcal{J}$$

is equivalent to

$$\xi \geq \mathrm{Adh}_\mathbb{J} \mathrm{Base}_\mathbb{D} \xi.$$

In particular, the functorial inequality

$$\xi \geq \mathrm{Adh}_\mathbb{J} \mathrm{Base}_{\mathbb{F}_1} \xi \tag{13}$$

extends the notions of sequentiality, Fréchetness, strong Fréchetness, weak bisequentiality and bisequentiality from topological to convergence spaces when $\mathbb{J}$ ranges over the classes of principal filters of closed sets, principal filters, countably based filters, countably deep filters and all filters respectively.

Other classical notions can be characterized by functorial inequalities of the form

$$\xi \geq JE\xi \tag{14}$$

where J is a reflector and E is a coreflector (e.g., [7], [10]).

Dually, important properties of convergences can be characterized by functorial inequalities of the type

$$\xi \leq EJ\xi \tag{15}$$

where J is a reflector (or projector) and E is a coreflector. For instance, a convergence space is called *countably Choquet* [2, page 49] or *countably pseudotopological* if a countably based filter converges to x whenever each finer ultrafilter does. Evidently, (X, ξ) is countably pseudotopological if and only if

$$\xi \leq \mathrm{Base}_{\mathbb{F}_1} S\xi. \tag{16}$$

Several other examples are presented in the paper.

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