SEQUENCES FREED FROM ORDER

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Abstract. Traditional definition of convergence of a sequence uses the order on the set of its indices, but the only structure, needed on that set to characterize convergence, is the cofinite topology. The only aspect of a sequence from convergence point of view is that of the filter it generates. Sequential filters, the filters generated by sequences, are precisely the images of cofinite filters of countably infinite sets. One cannot totally substitute sequences by the corresponding filters, because sequences serve to list, but often it is useful to replace them by *quences*, that is, maps from countably infinite sets.

1. On definitions of sequence and subsequence

A sequence on a set X is usually defined as a map from the ordered set of natural numbers $\mathbb{N}$ to X and denoted by $(x_n)_{n\in\mathbb{N}}$. Of course, a sequence can be, and typically is, specified as the ordered list of its terms x_0, x_1, $x_2, \ldots$

Example 1.1. Consider the following sequences

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots \tfrac{1}{n}, \ldots \tag{1.1}$$

$$1, \tfrac{1}{2}, 1, \tfrac{1}{2}, 1, \tfrac{1}{2}, \ldots \tag{1.2}$$

$$1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \ldots, \underbrace{\tfrac{1}{n}, \ldots, \tfrac{1}{n}}_{n \text{ times}}, \ldots \tag{1.3}$$

$$1, 1, \tfrac{1}{2}, 1, \tfrac{1}{2}, \tfrac{1}{3}, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4} \ldots, \underbrace{1, \ldots, \tfrac{1}{n}}_{n \text{ terms}}, \ldots \tag{1.4}$$

on the set $\mathbb{R}$ (of real numbers). The sequence (1.1) is one-to-one, while (1.3) is finite-to-one. The remaining ones are not finite-to-one. The sequence (1.2) has finite range and the preimage of every element of the range is infinite. The range of (1.4) is $\{\frac{1}{n} : n \in \mathbb{N}_1\}$ and the preimage of an element $\frac{1}{n}$ of the range is infinite.

The standard definition of sequence has some inconveniences. Following Peano [5], let

$$\mathbb{N}_k := \{n \in \mathbb{N} : n \geq k\}. \tag{1.5}$$

Example 1.2. Consider the sequence (1.1). In fitting it to the formal definition, it would be natural to set $x_n = \frac{1}{n}$, but then one cannot use the

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whole $\mathbb{N}$ as the set of indices, because $0 \in \mathbb{N}$. So either we define this sequence as $\left(\frac{1}{n}\right)_{n\in\mathbb{N}_1}$, which does not completely agree with the definition above, or as $\left(\frac{1}{n+1}\right)_{n\in\mathbb{N}}$, which is a bit cumbersome.

Traditionally, a sequence $(y_k)_{k\in\mathbb{N}}$ is called a *subsequence* of a sequence $(x_n)_{n\in\mathbb{N}}$ if there exists a strictly increasing map $h : \mathbb{N} \to \mathbb{N}$ such that $y_k = x_{h(k)}$ for each $k \in \mathbb{N}$. On setting $n_k := h(k)$, we get $y_k = x_{n_k}$, that is a usual notation for a subsequence. On defining $g(k) := y_k$ and $f(n) := x_n$, we get the following diagram.

$$\begin{tikzcd} \mathbb{N} \arrow[d, "h"'] \arrow[dr, "g"] & \\ & X \\ \mathbb{N} \arrow[ur, "f"'] & \end{tikzcd}$$

Example 1.3. In Example 1.1, (1.1) is a subsequence of (1.3), which is a subsequence of (1.4). On the other hand, (1.2) is not a subsequence of (1.3), but is easily seen to be a subsequence of (1.4).

To mitigate nuisances described in Example 1.2, Greco gave in [4] the following

Definition 1.1. *A* sequence *on X is a map from an arbitrary infinite subset N of $\mathbb{N}$ to X.*

We shall adopt Definition 1.1 in the sequel. Except for the need of writing down a specific sequence, it is irrelevant which particular infinite subset N of $\mathbb{N}$ is chosen to index a sequence. Hence, instead of writing $(x_n)_{n\in N}$, for general considerations it is sufficient to write $(x_n)_n$, with the understanding that the index set is infinite.

If $\varphi : \mathbb{N} \to N$ is an increasing bijection and $(x_n)_{n\in N}$ is a sequence on X, then $\left(x_{\varphi(n)}\right)_{n\in\mathbb{N}}$ is a sequence in the traditional sense. In Greco's framework,

Definition 1.2. *A sequence $(x_n)_{n\in N_1}$ is called a* subsequence *of a sequence $(x_n)_{n\in N_0}$ whenever $N_0 \supset N_1$.*

Of course, the injection from N_1 into N_0 is a strictly increasing map (with respect to the order induced from $\mathbb{N}$). Therefore, if $\varphi : \mathbb{N} \to N_0$ is an increasing bijection and $(x_n)_{n\in N_0}$ and $\psi : \mathbb{N} \to N_1$ is an increasing bijection and $(x_n)_{n\in N_1}$, then $(x_n)_{n\in N_1}$ is a subsequence of $(x_n)_{n\in N_0}$ if and only if $\left(x_{\psi(n)}\right)_{n\in\mathbb{N}}$ is a subsequence of $\left(x_{\varphi(n)}\right)_{n\in\mathbb{N}}$ in the traditional sense; if h is as in the traditional definition, then $\psi^{-1} \circ h \circ \varphi$ is the injection from N_1 into N_0.

Therefore, the adopted approach to sequences and subsequences provides a framework that is equivalent to the traditional one, but simplifies the description on reducing diagrams to set inclusions. Although more versatile than the traditional ones, Greco's definitions have also some inconveniences.

For example, in general there is no common subsequence of a countable sequence of sequences, each of which is a subsequence of the preceding one.

Let $(N_k)_{k\in K}$ be a sequence of infinite subsets of $\mathbb{N}$ such that $N_k \supset N_l$ if $k < l$. Accordingly, $(x_n)_{n\in N_l}$ constitutes a subsequence of $(x_n)_{n\in N_k}$. A diagonal procedure uses a sequence $(x_n)_{n\in N_\infty}$ where $N_\infty = \{n_k : k \in K\}$ is an increasing selection of elements of $(N_k)_{k\in K}$, that is, $n_k \in N_k$ and $n_k < n_l$ if $k < l$. It turns out that, in general, $(x_n)_{n\in N_\infty}$ cannot be represented as a common sequence of $(x_n)_{n\in N_k}$ for $k \in \mathbb{N}$. In fact, $(x_n)_{n\in N_\infty}$ is *almost* a subsequence of $(x_n)_{n\in N_k}$ for $k \in \mathbb{N}$, but not a subsequence.

Example 1.4. If $N_k := \mathbb{N}_k$ as in (1.5) and $x_n := n$, then there is no common subsequence, neither in the adopted nor in the traditional sense, of $(x_n)_{n\in N_k}$ for each $k \in \mathbb{N}$. In fact, if $(x_n)_{n\in N}$ were such a subsequence and $n_0 \in N \cap N_0$ then $n_0 \notin N_{n_0+1}$.

How to get rid of the inconvenience of non-existence of a common subsequence in the diagonal procedure, without loosing the benefits (like that of listing) of sequences?

2. Almost inclusion

As we have seen, the problem of the classical diagonal procedure consists in the fact that the set of indices of the "common" subsequence is not included in the corresponding sets of indices, but "almost included".

A set X is said to be *almost included* in a set Y if $X \setminus Y$ is finite,

$$X \subset_0 Y.$$

Two sets X and Y are called *almost equal* if $X \subset_0 Y$ and $Y \subset_0 X$, that is if their symmetric difference $(Y \setminus X) \cup (X \setminus Y)$ is finite.

In the same vein, a sequence $(y_k)_{k\in K}$ is an *almost subsequence* of a sequence $(x_n)_{n\in N}$ if there exists a strictly increasing map $h \in \mathbb{N}^K$ such that $h(K) \subset_0 N$ and $y_k = x_{h(k)}$ for each $k \in K$ such that $h(k) \in N$.

It can be easily proved that

Proposition 2.1. *If $(N_k)_k$ is a sequence of infinite subsets of $\mathbb{N}$ such that $N_l \subset_0 N_k$ for $k < l$, then there is an infinite subset N_∞ of $\mathbb{N}$ such that*

$$N_\infty \subset_0 N_k$$

for each k.

Therefore, we have succeeded to rigorously formalize the diagonal procedure of Example 1.4.

Corollary 2.2. *If $(f_k)_k$ is a sequence of sequences such that f_k is an almost subsequence of f_l for $k < l$, then there is a sequence f_∞ that is an almost subsequence of f_k for every k.*

3. Convergence of sequences

A sequence $(x_n)_n$ on a topological space X *converges* to an element x of X if for every neighborhood V of x there is m such that $x_n \in V$ for each $n \geq m$.

Let us rephrase this standard definition in terms of almost inclusion.

Proposition 3.1. *A sequence $f \in X^N$ converges to x if and only if N is almost included in $f^{-1}(V)$ for every neighborhood V of x.*

Notice that this proposition makes no reference to the order on the set of indices. If we pass now to the complements of the preimages of neighborhoods, we recover the following characterization.

Proposition 3.2. *A sequence $(x_n)_n$ converges to x if and only if*

$$\{n : x_n \notin V\}$$

is finite for every neighborhood V of x.

Actually, arbitrary permutations of indices preserve convergence.

Example 3.3. Let $(x_n)_{n \in N}$ and $(y_k)_{k \in K}$ be sequences on a topological space. If $(x_n)_n$ converges to x and $h : N \to K$ is a bijection such that $y_{h(n)} = x_n$ for each n, then $(y_k)_{k \in K}$ converges to x.

If we analyze Proposition 3.2, we realize that the only condition for a sequence $f \in X^N$ to converge to x is that the preimage $f^{-1}(V)$ of every neighborhood of x has finite complement in N, in other words, is *cofinite.*

4. Cofiniteness

Since cofinite sets play an essential role in convergence of sequences, we shall investigate them in detail.

Definition 4.1. *A subset M of X is* cofinite *if $X \setminus M$ is finite. Let $(X)_0$ denote the set of all cofinite subsets of X.*

Therefore,

Proposition 4.2. *A set M is a cofinite subset of X if and only if X is almost included in M.*

Of course, if X is finite, each subset of X is cofinite; in particular, $\varnothing \in (X)_0$. If, however, X is infinite, then

$$\varnothing \notin (X)_0\,,$$
$$M \supset N \in (X)_0 \Longrightarrow M \in (X)_0\,,$$
$$N_0, N_1 \in (X)_0 \Longrightarrow N_0 \cap N_1 \in (X)_0\,.$$

If X is an infinite set and $\infty \notin X$, then the *cofinite topology* of X at ∞ is defined as the topology on $X \cup \{\infty\}$, for which each $x \in X$ is isolated and V is a neighborhood of ∞ if $\infty \in V$ and $X \setminus V$ is finite.

Recall that h is a *partial map* from X to Y (in symbols, $h : X \rightarrowtail Y$) if there is a subset of X denoted by $\operatorname{dom} h$ such that $h : \operatorname{dom} h \to Y$.

Definition 4.3. *We say that* $h : X \rightarrowtail Y$ *is* cofinitely continuous *if* $h^{-1}(M)$ *is cofinite in* X *for each cofinite subset* M *of* Y.

It follows that the domain of a cofinitely continuous partial map is cofinite, because $\operatorname{dom} h = h^{-1}(Y)$. Of course, $h : X \rightarrowtail Y$ is cofinitely continuous if and only if $h_\infty : \operatorname{dom} h \cup \{\infty_X\} \to Y \cup \{\infty_Y\}$, where $\infty_X \notin X$ and $\infty_Y \notin Y$, defined by

$$h_\infty(x) := \begin{cases} h(x), & \text{if } x \in \operatorname{dom} h, \\ \infty_Y, & \text{if } x = \infty_X, \end{cases}$$

is continuous from the cofinite topology at ∞_X of $\operatorname{dom} h \cup \{\infty\}$ to the cofinite topology at ∞_Y of Y.

It is straightforward that a partial map between infinite sets is cofinitely continuous if and only if the preimages of finite sets are finite and the domain is cofinite.

5. Quences

As the order on the set of indices of a sequence is irrelevant from the convergence point of view, we introduce a more general concept of quence.

Definition 5.1. *A* quence *on a set* X *is a map from a countably infinite set to* X.

Every sequence is a quence. On the other hand, if $f : N \to X$ is a quence on X and $h : \mathbb{N} \to N$ is a bijection, then $f \circ h$ is a sequence. Accordingly, a quence is an abstraction of a sequence (in the adopted sense).

Definition 5.2. *If* X *is a topological space, then a quence* $f : N \to X$ converges to x *(or* x *is a* limit of f*) if* $f^{-1}(V)$ *is cofinite for every neighborhood* V *of* x.

We denote by $\lim_X f$ (or simply $\lim f$) the set all the limits of f in X.

In other words, $f \in X^N$ converges to x whenever $\widehat{f}$, defined by $\widehat{f}(n) := f(n)$ for each $n \in N$ and $\widehat{f}(\infty) := x$, is continuous from the cofinite topology of N at ∞ to X.

In order to carry this observation to its logical conclusion, we need to extend a concept of subsequence to that of subquence. It would be not enough to mimic the defining process of Greco's subsequence by taking a subset of the set of indices.

Definition 5.3. *A quence* $g : B \to X$ *is called a* subquence *of a quence* $f : A \to X$, *in symbols,*

$$g \succ f,$$

if there exists a cofinitely continuous partial map h *from* B *to* A *such that* $g = f \circ h$ *on* $\operatorname{dom} h$. *Two quences* f *and* g *are called* equivalent, *in symbols,* $f \approx g$, *if* $g \succ f$ *and* $f \succ g$.

Of course, a subsequence of a sequence is also its subquence, but not conversely. It follows immediately from this description that if $g \succ f$ then $\lim f \subset \lim g$.

To see to what extend the notion of subquence transcends that of subsequence, consider

Proposition 5.1. *If $f \in X^{\mathbb{N}}$, then $g \in X^{\mathbb{N}}$ is a subquence of f if and only if there exists $h : \mathbb{N} \rightarrowtail \mathbb{N}$ be such that $g = f \circ h$ on $\operatorname{dom} h$ and $\lim_{n\to\infty} h(n) = \infty$.*

Notice, that the so defined h need not be increasing, even in the broad sense, nor $\operatorname{dom} h$ need be the whole of $\mathbb{N}$.

Example 5.4. Sequence (1.1) and (1.3) from Example 1.1, are equivalent quences. We have observed that (1.1) is a subsequence, hence a subquence, of (1.3). On the other hand, the map $h : \mathbb{N} \to \mathbb{N}$ defined by $h(n) := \frac{1}{n}$ is cofinitely continuous and its composition with (1.1) yields (1.3), that is, (1.3) is a subquence of (1.1).

On the other hand, each partial map with cofinite domain h such that the composition of (1.3) with h coincides with (1.4) on its domain, has infinite preimages of singleton. Therefore, (1.4) is not equivalent to (1.3).

Proposition 5.5. *A quence $g : M \to X$ is a subquence of a quence $f : N \to X$ if and only if for each $A \in (N)_0$ there exists $B \in (M)_0$ such that $g(B) \subset f(A)$.*

Proof. Let h be as in Definition 5.3. If A is a cofinite subset of N, then $B := h^{-1}(A)$ is cofinite in M. As $g = f \circ h$ on $\operatorname{dom} h$,

$$g(B) = g\left(h^{-1}(A)\right) = f\left(h\left(h^{-1}(A)\right)\right) \subset f(A),$$

which proves the condition.

Conversely, suppose that the condition holds. Up to a bijection, $N = \mathbb{N}$. There exists a sequence of cofinite subsets $(M_n)_{n\in\mathbb{N}}$ of M such that $g(M_n) \subset f(\mathbb{N}_n)$. By an immediate induction we can assume that $M_n \supsetneq M_{n+1}$ for all n and $\bigcap_{n\in\mathbb{N}} M_n = \varnothing$. For each $m \in M_n$, let

$$h_n(m) := \min\{k \in \mathbb{N}_n : g(m) = f(k)\},$$

and let

$$h(m) := h_n(m) \text{ if } m \in M_n \setminus M_{n+1}.$$

Then, for each $n \in \mathbb{N}$ the set $h^{-1}(\mathbb{N}_n) \supset M_n$, hence is cofinite, and thus h is cofinitely continuous. On the other hand, $g(m) = f(h(m))$ for each $m \in M_0$. ∎

On the other hand,

Proposition 5.6. *For each subquence g of a sequence f there is subquence s of g that is a subsequence of f.*

Proof. Let $f \in X^N$ (where N is an infinite subset of $\mathbb{N}$) and let $g \in X^B$ be a subquence of f. Let $h : B \rightarrowtail N$ be a cofinitely continuous partial map such that $g = f \circ h$ on $\operatorname{dom} h$. Accordingly $h(B)$ is infinite and thus a map $s : h(B) \to X$ such that $s(n) = f(n)$ for each $n \in h(B)$, is a subsequence of f. For each $n \in h(B)$, let $j(n)$ be any element of $h^{-1}(n)$. Then $j : h(\operatorname{dom} h) \to B$ is cofinitely continuous and $s = g \circ j$, hence s is a subquence of g. ∎

Proposition 5.7. *If a quence converges to x then its every subquence converges to x.*

Proof. Let $f : A \to X$ be a quence and $x \in \lim f$, that is, $f^{-1}(V)$ is cofinite in A for each neighborhood V of x. If $g : B \to X$ is a subquence of f, that is, there is a cofinitely continuous map $h : B \rightarrowtail A$ such that $g = f \circ h$ on $\operatorname{dom} h$, then

$$g^{-1}(V) = (f \circ h)^{-1}(V) = h^{-1}\left(f^{-1}(V)\right)$$

is cofinite, so that $x \in \lim g$. ∎

By Proposition 5.7, if a sequence converges to x then its every subquence, hence its every subsequence, converges to x.

6. Families of sets

If $\mathcal{A}$ and $\mathcal{D}$ are families of subsets of X, then $\mathcal{A}$ is *coarser* than $\mathcal{D}$ ($\mathcal{D}$ is *finer* than $\mathcal{A}$)

$$\mathcal{A} \leq \mathcal{D} \tag{6.1}$$

if for each $A \in \mathcal{A}$ there is $D \in \mathcal{D}$ such that $D \subset A$. A family $\mathcal{A}$ is said to be *isotone* if $D \supset A \in \mathcal{A}$ implies that $D \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{D}$ are isotone, then (6.1) if and only if $\mathcal{A} \subset \mathcal{D}$. If $\mathcal{A} \leq \mathcal{D}$ and $\mathcal{D} \leq \mathcal{A}$ then we say that $\mathcal{A}$ and $\mathcal{D}$ are *equivalent* and write $\mathcal{A} \approx \mathcal{D}$.

A family $\mathcal{F}$ of subsets of X is called a *filter* on X if

$$(F_0 \in \mathcal{F}) \wedge (F_1 \in \mathcal{F}) \iff F_0 \cap F_1 \in \mathcal{F}, \tag{6.2}$$

where $\wedge$ stands for the conjunction. It follows from the definition that if $\mathcal{F}$ is a filter then $G \supset F \in \mathcal{F}$ implies that $G \in \mathcal{F}$. A filter $\mathcal{F}$ is called *degenerate* if $\varnothing \in \mathcal{F}$ and *nondegenerate* otherwise. It is immediate that the degenerate filter on X is equal to the power set 2^X of X. A family $\mathcal{B} \subset \mathcal{F}$ is called a *base* of $\mathcal{F}$ (or $\mathcal{F}$ is called *generated* by $\mathcal{B}$) if $\mathcal{F} \leq \mathcal{B}$.

Notice that if N is infinite, then the family $(N)_0$ (of cofinite subsets of N) is a nondegenerate filter. Observe as well, that the family of neighborhoods of a fixed point of a topological space is a nondegenerate filter.

If $\mathcal{A}$ is a family of subsets of X and $\mathcal{B}$ is a family of subsets of Y, then we denote

$$f[\mathcal{A}] := \{f(A) : A \in \mathcal{A}\}, \tag{6.3}$$

$$f^{-1}[\mathcal{B}] := \{f^{-1}(B) : B \in \mathcal{B}\}. \tag{6.4}$$

Observe that if $\mathcal{F}$ is a filter on X and $f : X \to Y$, then $f[\mathcal{F}]$ is a filter base on Y.

We can now rephrase Proposition 5.5.

Corollary 6.1. *Let X, Y be infinite. A partial map $f : X \rightarrowtail Y$ is cofinitely continuous if and only if $f[(X)_0] \geq (Y)_0$, equivalently, $(X)_0 \geq f^{-1}[(Y)_0]$.*

Since for every infinite set N, the family $(N)_0$ is a filter on N, the family $f[(N)_0]$ is a filter base on X for each quence $f \in X^N$. In these terms, Proposition 5.5 becomes

Proposition 6.2. *A quence $g \in X^M$ is a subquence of a quence $f \in X^N$ if and only if*

$$f[(N)_0] \leq g[(M)_0]. \tag{6.5}$$

Let us observe that (6.5) amounts to

$$\left(g^{-1} \circ f\right)[(N)_0] \leq (M)_0. \tag{6.6}$$

If $\mathcal{V}_X(x)$ stands for the neighborhood filter of x on X, then

$$x \in \lim_X f$$

for $f \in X^N$, is equivalent to

$$\mathcal{V}_X(x) \leq f[(N)_0]. \tag{6.7}$$

A filter $\mathcal{F}$ on X is called *sequential* ([1]) if it is generated by a quence, equivalently, by a sequence, that is, if there is a quence $f \in X^N$ such that $f[(N)_0]$ is a base of $\mathcal{F}$. Therefore a quence converges to a point x if the corresponding sequential filter is finer than the neighborhood filter of x.

Sure enough, equivalent quences generate the same (sequential) filter.

A filter $\mathcal{F}$ on X is called *free* if $\bigcap_{F \in \mathcal{F}} F = \varnothing$, and is called *principal* provided that $\bigcap_{F \in \mathcal{F}} F \in \mathcal{F}$.

Example 6.3. In Example 1.1, $\left\{\left\{\frac{1}{n} : n \geq m\right\} : m \in \mathbb{N}_1\right\}$ is a base for the filters generated by (1.1) and by (1.3). Therefore, these two sequences generate the same filter, which is, by the way, free and finer than that generated by (1.4), because (1.1) (and (1.3)) are subsequences of (1.4).

The filter generated by (1.2) is $\left\{F \subset \mathbb{R} : \{1, \frac{1}{2}\} \subset F\right\}$. The filter generated by (1.4) is $\left\{F \subset \mathbb{R} : \{\frac{1}{n} : n \in \mathbb{N}_1\} \subset F\right\}$. Both these filters are principal, the former being finer than the latter.

The filter $\mathcal{V}(x)$ of neighborhoods of a point x in the real line with its usual topology is neither free (for $x \in \bigcap_{V \in \mathcal{V}(x)} V$) nor principal (for $\bigcap_{V \in \mathcal{V}(x)} V = \{x\} \notin \mathcal{V}(x)$).

It is known [2] that

[1]Such filters where called *elementary filters* in [3]. They are often called *Fréchet filters*. This term, however, is better suited to the intersections of sequential filters, because a topology is Fréchet if and only if each neighborhood filter is such an intersection (see, e.g., [1]).

Theorem 6.4. *For every filter $\mathcal{F}$ on X, there exists a unique pair of (possibly degenerate) filters $\mathcal{F}^{\circ}$ and $\mathcal{F}^{\bullet}$ such that $\mathcal{F}^{\circ}$ is free, $\mathcal{F}^{\bullet}$ is principal, and*

$$\mathcal{F} = \mathcal{F}^{\circ} \wedge \mathcal{F}^{\bullet} \text{ and } \mathcal{F}^{\circ} \vee \mathcal{F}^{\bullet} = 2^{X}. \tag{6.8}$$

Here, the supremum $\mathcal{G} \vee \mathcal{H}$ is the (possibly degenerate) filter

$$\{G \cap H : G \in \mathcal{G}, H \in \mathcal{H}\}$$

and the infimum $\mathcal{G} \wedge \mathcal{H} := \mathcal{G} \cap \mathcal{H}$ consists of sets that belong to both $\mathcal{G}$ and $\mathcal{H}$.

7. Sequential filters

At the end, from the convergence point of view, a sequence amounts to the filter it generates. If $(x_n)_n$ is a sequence on X, then

$$\{\{x_n : n \geq k\} : k \in \mathbb{N}\}$$

is a filter base. Hence, each sequential filter contains a countable (possibly finite) set.

Which filters on a given set are sequential? By the very definition, these filters are of the form $f\left[(N)_0\right]$ where with $f : N \to X$. But let us give some criteria that enables one to directly recognize sequential filters without a recourse to the definition.

A principal filter $\mathcal{F}$ is sequential if and only if $F_\infty := \bigcap_{F \in \mathcal{F}} F$ is countable. Indeed, if the condition holds, then we can arrange F_∞ in a sequence, like in (1.2) or in (1.4). A free filter $\mathcal{F}$ is sequential if there is a countably infinite set $F_0 \in \mathcal{F}$ and then any bijection $f : N \to F_0$ defines a base $f\left[(N)_0\right]$ of $\mathcal{F}$.

In fact,

Theorem 7.1. *A filter $\mathcal{F}$ is sequential if and only if there is a countable set $F_0 \in \mathcal{F}$ and $F_0 \subset_0 F$ for each $F \in \mathcal{F}$.*

Proof. Indeed, let $\mathcal{F}$ be sequential, that is, generated by $f\left[(N)_0\right]$, where N is countable. Thus $F_0 := f(N)$ is countable, and $f(N_0) \in \mathcal{F}$. Moreover, for each $F \in \mathcal{F}$ there is $A \subset N$ such that $N \setminus A$ is finite and $f(A) \subset F$. Therefore,

$$F_0 \setminus F \subset f(N) \setminus f(A) \subset f(N \setminus A)$$

is finite.

Conversely, if the condition holds, then either $F_\infty = \varnothing$, that is each cofinite subset of F_0 belongs to $\mathcal{F}$, or $\mathcal{F}^{\bullet} := \{F \subset X : F_\infty \subset F\}$ and $\mathcal{F}^{\circ}$ is generated by $\{F \setminus F_\infty : F \in \mathcal{F}\}$. The latter filter is free and fulfills the condition. Therefore, $\mathcal{F}^{\bullet}$ and $\mathcal{F}^{\circ}$ are sequential and fulfill (6.8) and the sequence generating $\mathcal{F} = \mathcal{F}^{\circ} \wedge \mathcal{F}^{\bullet}$ can be constructed by alternating the terms of the sequences corresponding to $\mathcal{F}^{\circ}$ and to $\mathcal{F}^{\bullet}$. ∎

Corollary 7.2. *A filter is sequential if and only if it contains a countable set and admits a countable base consisting of almost equal sets.*

The countable base in the corollary above can be finite, and in this case, there is a base consisting of a singleton.

Similar characterizations were formulated in [3].

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