MODIFIED DUALITY AND PRODUCT THEOREMS FOR SUBCLASSES OF SEQUENTIAL SPACES

S. DOLECKI AND F. MYNARD

1. Introduction

The primary motivation for this note is to complete a family of product theorems for various subclasses of sequential topological spaces. We apply the method of *modified duality* developed in [2, 7, 6, 9], and recently extended and simplified in [3], where it is shown that all but the last results of Table 1 below are instances of this unifying principle. In this note, we show that the method can be used to complete the table.

Since modified duality takes place in the category of convergence spaces, this is the general context of this work, even if we are primarily interested in topological corollaries. We use notations and terminology from the recent book [3], where the general theory of modified duality is presented in details, so that we omit many definitions and results and simply refer to [3]. Let us recall notions necessary to discuss the class of problems we are to consider.

If $\mathbb{D}$ denotes a class of filters, then $\mathbb{D}X$ stands for the set of filters on a space X that are in $\mathbb{D}$. For instance, $\mathbb{F}_\kappa$ denotes the class of filters of character less than $\aleph_\kappa$, that is, that admit a filter-base of cardinality less than $\aleph_\kappa$. In particular, $\mathbb{F}_0$ is the class of filters with a finite filter-base, that is, principal filters, and $\mathbb{F}_1$ is the class of countably based filters. Accordingly, $\mathbb{F}_0X$ and $\mathbb{F}_1X$ are the sets of principal and of countably-based filters on X respectively. $\mathbb{F}$ denotes the class of all filters, and $\mathbb{U}$ that of ultrafilters. To a sequence $\{x_n\}_n$ on a set X we associate the filter $(x_n)_n^{\uparrow}$ generated by the filter-base $\{\{x_n : n \geq k\} : k \in \omega\}$. The class of such *sequential filters* is denoted by $\mathbb{E}$.

Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of X *mesh*, in symbol $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \varnothing$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The *adherence of a filter* $\mathcal{H}$ in a (convergence or topological) space ξ is

$$\mathrm{adh}_\xi \mathcal{H} := \bigcup_{\mathbb{F}\ni\mathcal{G}\#\mathcal{H}} \lim\nolimits_\xi \mathcal{G},$$

while, given a class $\mathbb{J}$ of filters, the adherence in its $\mathbb{J}$-based modification $\mathrm{B}_{\mathbb{J}}\,\xi$ ([1]) is

[1] Given a class $\mathbb{J}$ of filters, we say that a convergence is $\mathbb{J}$-*based* if whenever $x \in \lim \mathcal{F}$, there exists $\mathcal{G} \in \mathbb{J}$ with $\mathcal{G} \leq \mathcal{F}$ and $x \in \lim \mathcal{G}$. Let $\mathrm{B}_{\mathbb{J}}$ be defined on objects of **Conv** by

$$\lim\nolimits_{\mathrm{B}_{\mathbb{J}}\,\xi} \mathcal{F} := \bigcup_{\mathbb{J}\ni\mathcal{G}\leq\mathcal{F}} \lim\nolimits_\xi \mathcal{G}.$$

It is easily seen (e.g., [1], [3, Corollary XIV.4.4]) that when $\mathbb{J}$ is an $\mathbb{F}_0$-composable class of filters, then $\mathrm{B}_{\mathbb{J}}$ is a (concrete) coreflector. If moreover $\mathbb{J}$ is composable, then $\mathrm{B}_{\mathbb{J}}$ commutes with finite product.

$$\operatorname{adh}_{\mathrm{B}_{\mathbb{J}}\,\xi}\mathcal{H} := \bigcup_{\mathbb{J}\ni\mathcal{J}\#\mathcal{H}} \lim_{\xi}\mathcal{J}.$$

In particular, for $\mathbb{J} = \mathbb{F}_1$ we abbreviate

$$\mathrm{I}_1 := \mathrm{B}_{\mathbb{F}_1},$$

to denote the coreflector on *first-countable* convergences, which commutes with finite products.

Of course, $\operatorname{adh}_{\mathrm{I}_1\,\xi}\mathcal{H} \subset \operatorname{adh}_{\xi}\mathcal{H}$ for every filter $\mathcal{H}$. If the reverse inclusion

$$\operatorname{adh}\mathcal{H} \subset \operatorname{adh}_{\mathrm{I}_1\,\xi}\mathcal{H} \tag{1.1}$$

holds for principal filters, X is called *Fréchet-Urysohn*, or *Fréchet* in short. A space is *strongly Fréchet* if this condition holds for every countably-based filter $\mathcal{H}$. We can reinterpret these conditions in terms of neighborhood filter. We associate to a class $\mathbb{D}$ of filters, the class of filters $\mathbb{D}^{\triangle}$. A filter $\mathcal{F}$ belongs to $\mathbb{D}^{\triangle}$ of for every $\mathcal{D} \in \mathbb{D}$ with $\mathcal{D}\#\mathcal{F}$, there is $\mathcal{L} \in \mathbb{F}_1$ with $\mathcal{L} \geq \mathcal{F}$ and $\mathcal{L}\#\mathcal{D}$.

Note that $\mathbb{F}_1 \subseteq \mathbb{D}^{\triangle}$ regardless of what the class $\mathbb{D}$ is, for we can take $\mathcal{F} = \mathcal{L}$ if $\mathcal{F} \in \mathbb{F}_1$.

It is easily checked that the neighborhood filters of a topological space are in $\mathbb{D}^{\triangle}$ if and only if (1.1) for all $\mathcal{H} \in \mathbb{D}$. In particular, a topological space is Fréchet if and only if all of its neighborhood filters are in $\mathbb{F}_0^{\triangle}$, strongly Fréchet if and only if they are in $\mathbb{F}_1^{\triangle}$, and *bisequential* (in the sense of [5]) if and only if they are in $\mathbb{F}^{\triangle}$. A topological space is *productively Fréchet* [4] if (1.1) holds for every $\mathcal{H} \in \mathbb{F}_1^{\triangle}$, equivalently, if every neighborhood filter is in $\mathbb{F}_1^{\triangle\triangle} := \left(\mathbb{F}_1^{\triangle}\right)^{\triangle}$. A topological space is *finitely generated* if every neighborhood filter is principal, and *prime* if it has at most one non-isolated point. A topological space X is *sequential* if every sequentially closed subset is closed. Finally *strongly sequential* topological spaces were introduced (and characterized internally) in [6] as those whose product with every metrizable space is sequential. Table 1 below gathers a family of results all but the last one of which are known to follow from the same abstract principle of modified duality [3, Section XVII.7]. The italicized conditions should be thought of as preceded by "equivalently".

Most are known results with reference in the last column. In the case of the fourth row, it is likely to be known but we could not attribute it. In contrast, the last row is an entirely new result, settling a problem that naturally arises when examining this table.

Note that the names *productively* Fréchet and *productively* sequential are justified by the fact that a finite product of productively Fréchet/sequential spaces is productively Fréchet/sequential, as can easily be seen via their characterizations in terms of product.

2. Basic ingredients

As stated before, we use definitions and notations from [3], restricting ourselves to Kent spaces, so that topologies form a surjectively finally dense subcategory of the category **Conv** of convergence spaces and continuous maps.

X is	iff for all Y	$X\times Y$ is	Ref.
Fréchet	finitely generated *finitely generated prime*	Fréchet	[7]
strongly Fréchet	bisequential *metrizable prime*	strongly Fréchet *Fréchet*	[5]
productively Fréchet	strongly Fréchet *strongly Fréchet prime*	strongly Fréchet *Fréchet*	[4]
sequential	finitely generated *finitely generated prime*	sequential	folklore?
strongly sequential	first-countable *metrizable prime*	strongly sequential *sequential*	[6]
productively sequential	strongly sequential	strongly sequential *sequential*	new

TABLE 1

All *functors* considered here are, like the topological modifier T, concrete endofunctors of **Conv**. In other words, a functor F is characterized by its action on objects in the following way: it associates to each convergence ξ another convergence $F\xi$ with the same underlying set, in such a way that $f:|F\xi|\to|F\tau|$ is continuous whenever $f:|\xi|\to|\tau|$ is continuous. In particular, if $\xi\geq\tau$ then $F\xi\geq F\tau$. Among such functors, *reflectors* are those that are *idempotent* (i.e., $FF\xi=F\xi$) and *contractive* (i.e., $F\xi\leq\xi$), and *coreflectors* are those that are idempotent and *expansive* (i.e., $\xi\leq F\xi$).

If F is a (concrete) functor, let ([2])

$$\mathrm{Epi}_F^{\sigma}\,\xi:=i^{-}[F[\xi,\sigma],\sigma]$$

and if R is a reflector, let

$$\mathrm{Epi}_F^{R}\,\xi:=\bigvee_{\sigma=R\sigma}\mathrm{Epi}_F^{\sigma}\,\xi.$$

Let Dis denote the *discretization functor*, which associates to each convergence ξ the discrete convergence on $|\xi|$.

Lemma 1. *Let F be a functor, R a reflector, and let ξ and σ be two convergences.*

(1) $x\in\lim_{\mathrm{Epi}_F^{\sigma}\xi}\mathcal{F}$ *if and only if for every* $f\in C(\xi,\sigma)$ *and every* $\mathcal{G}\in\mathbb{F}(C(\xi,\sigma))$,

$$f\in\lim_{F[\xi,\sigma]}\mathcal{G}\Longrightarrow f(x)\in\lim_{\sigma}\langle\mathcal{F},\mathcal{G}\rangle\,.$$

(2)

$$\mathrm{Epi}_{\mathrm{Dis}}^{\sigma}=R_{\sigma}\ \textit{and}\ \mathrm{Epi}_{\mathrm{Dis}}^{R}=R.$$

Proposition 2. [7] [3, Proposition XVII.2.1] *If F is a functor, so is Epi_F^{σ}, and thus, so is Epi_F^{R}, for each reflector R.*

Here is the fundamental theorem of modified duality ([3, Corollary XVII.2.6]):

[2]Here, $f^{-}\tau$ denotes the initial convergence associated to a map $f:X\to|\tau|$. On the other hand, $[\xi,\sigma]$ denotes the *natural convergence* on the set $C(\xi,\sigma)$ of continuous maps from ξ to σ, that is, the coarsest convergence on $C(\xi,\sigma)$ making the evaluation map jointly continuous

Theorem 3. *Let R be a reflector and let F be a functor. Assume $\theta \geq R\xi$, and let $\mathbf{D}$ be an initially dense subclass of $\mathbf{R}$. The following are equivalent:*

(1) $\theta \geq \mathrm{Epi}_F^R \xi$;
(2) $\theta \times F\tau \geq \mathrm{Epi}_F^R(\xi \times \tau)$ *for every* τ;
(3) $\theta \times F\tau \geq R(\xi \times \tau)$ *for every* τ;
(4) $F[\xi, \sigma] \trianglerighteq [\theta, \sigma]$ *for every* $\sigma = R\sigma$;
(5) $F[\xi, \sigma] \trianglerighteq [\theta, \sigma]$ *for every* $\sigma \in \mathbf{D}$.

In the case where F is a coreflector, additional conditions can be obtained:

Corollary 4. [3, Corollary XVII.2.10] *Let R be a reflector and let F be a productive coreflector. Assume $\theta \geq R\xi$, and let $\mathbf{D}$ be an initially dense subclass of $\mathbf{R}$ and $\mathbf{W}$ is a finally dense subclass of $\mathbf{F}$. The following are equivalent:*

(1) $\theta \geq \mathrm{Epi}_F^R \xi$;
(2) $\theta \times F\tau \geq \mathrm{Epi}_F^R(\xi \times \tau)$ *for every* τ;
(3) $\theta \times \tau \geq R(\xi \times \tau)$ *for every* $\tau \in \mathbf{W}$;
(4) $F[\xi, \sigma] \trianglerighteq [\theta, \sigma]$ *for every* $\sigma \leq \mathrm{Epi}_F^R \sigma$;
(5) $F[\xi, \sigma] \trianglerighteq [\theta, \sigma]$ *for every* $\sigma = R\sigma$;
(6) $F[\xi, \sigma] \trianglerighteq [\theta, \sigma]$ *for every* $\sigma \in \mathbf{D}$.

In particular, when F is a coreflector, we have (e.g., [7, (4.28)])

$$\tau = F\tau \Longrightarrow R(\xi \times \tau) = R(\mathrm{Epi}_F^R \xi \times \tau). \tag{2.1}$$

Note that rows 1, 2 (for $R = \mathrm{S}_0$) and 4, 5 (for $R = \mathrm{T}$) of Table 1 are simple instances of the following consequence of Corollary 4:

Corollary 5. [3, Proposition XVII.7.1] *Let R be a reflector and let $\mathbf{W}$ be a finally dense subclass of $\mathbf{B}_{\mathbb{D}}$. Let $\mathbb{D}$ be a composable class of filters containing $\mathbb{F}_0$ and let $\mathbb{J}$ be a class of filters closed under finite product such that $\mathbb{D} \subseteq \mathbb{J}$. The following are equivalent:*

(1) $\xi \geq \mathrm{Epi}_{\mathrm{B}_{\mathbb{D}}}^R \mathrm{B}_{\mathbb{J}} \xi$;
(2) $\xi \times \tau \geq \mathrm{Epi}_{\mathrm{B}_{\mathbb{D}}}^R \mathrm{B}_{\mathbb{J}}(\xi \times \tau)$ *for every $\mathbb{D}$-based convergence* τ;
(3) $\xi \times \tau \geq R\,\mathrm{B}_{\mathbb{J}}(\xi \times \tau)$ *for every* $\tau \in \mathbf{W}$.

That row 3 falls under the same umbrella is slightly more sophisticated (because the class $\mathbb{F}_1^{\triangle}$ of strongly Fréchet filters is not composable) and is proved in [3, Corollary XVII.7.4].

We will need the following simple observations.

Lemma 6. *Let J be a reflector and let E be a coreflector. If $\sigma = EJ\tau$ for $\tau = E\tau$ then $\sigma = EJ\sigma$.*

Proof. Indeed,

$$EJ\sigma = EJEJ\tau \geq EJJ\tau = EJ\tau = \sigma$$

and

$$\sigma = EJ\tau = EJE\tau \geq EJEJ\tau = EJ\sigma.$$

□

Lemma 7. *Let R be a reflector, E be a coreflector that commutes with finite product, and let $\xi = E\xi$. A convergence θ on the same underlying set as ξ satisfies*

$$\theta \times \mathrm{Epi}_E^R \sigma \geq R(\xi \times \sigma) \tag{2.2}$$

for all $\sigma = E\sigma$ if and only if it satisfies (2.2) for all $\sigma = E\,\mathrm{Epi}_E^R \sigma$.

Proof. Assume that (2.2) for all $\sigma = E\,\mathrm{Epi}_E^R\,\sigma$ and let $\tau = E\tau$. Then $R(\xi\times\tau) = R(\xi\times\mathrm{Epi}_E^R\,\tau)$ by (2.1) and thus

$$R(\xi\times\tau)\leq R(\xi\times E\,\mathrm{Epi}_E^R\,\tau).$$

Let

$$\sigma := E\,\mathrm{Epi}_E^R\,\tau.$$

Then $\sigma = E\,\mathrm{Epi}_E^R\,\sigma$ by Lemma 6. Thus (2.2) applies, to the effect that

$$R(\xi\times\tau)\leq\theta\times\mathrm{Epi}_E^R\,E\,\mathrm{Epi}_E^R\,\tau.$$

Since $\mathrm{Epi}_E^R\,E\,\mathrm{Epi}_E^R\leq\mathrm{Epi}_E^R\,E$, we obtain

$$R(\xi\times\tau)\leq\theta\times\mathrm{Epi}_E^R\,\tau.$$

□

It turns out that a topology ξ is sequential if and only if (e.g., [1])

$$\xi\geq\mathrm{T}\,\mathrm{I}_1\,\xi,$$

and we take this functorial inequality as the definition of a *sequential convergence*. A convergence ξ is *strongly sequential* if ([3])

$$\xi\geq\mathrm{Epi}_{\mathrm{I}_1}^{\mathrm{T}}\,\mathrm{I}_1\,\xi.$$

3. PRODUCTIVELY SEQUENTIAL SPACES

Recall that a topological space is *productively sequential* if its product with every strongly sequential convergence is sequential.

Lemma 8. *If ξ is productively sequential then ξ is strongly sequential and*

$$\xi\times\mathrm{Epi}_{\mathrm{I}_1}^{\mathrm{T}}\,\sigma\geq\mathrm{T}(\mathrm{I}_1\,\xi\times\sigma) \tag{3.1}$$

for every $\sigma = \mathrm{I}_1\,\sigma$.

Proof. By definition, if $\xi\times\tau$ is sequential for every strongly sequential convergence, then in particular, $\xi\times\tau$ is sequential for every first countable τ, so that ξ is strongly sequential.

In view of the Lemma 7 (for $R = \mathrm{T}$ and $E = \mathrm{I}_1$), we only need to show (3.1) for $\sigma = \mathrm{I}_1\,\mathrm{Epi}_{\mathrm{I}_1}^{\mathrm{T}}\,\sigma$. Let $\tau := \mathrm{Epi}_{\mathrm{I}_1}^{\mathrm{T}}\,\sigma$. As $\sigma\geq\tau$, $\tau\geq\mathrm{Epi}_{\mathrm{I}_1}^{\mathrm{T}}\,\mathrm{I}_1\,\tau$, that is, τ is strongly sequential. As ξ is productively sequential, $\xi\times\tau$ is sequential, that is,

$$\xi\times\tau\geq\mathrm{T}\,\mathrm{I}_1(\xi\times\tau) = \mathrm{T}(\mathrm{I}_1\,\xi\times\mathrm{I}_1\,\tau).$$

Moreover $\mathrm{I}_1\,\tau = \mathrm{I}_1\,\mathrm{Epi}_{\mathrm{I}_1}^{\mathrm{T}}\,\sigma = \sigma$, so that

$$\xi\times\mathrm{Epi}_{\mathrm{I}_1}^{\mathrm{T}}\,\sigma\geq\mathrm{T}(\mathrm{I}_1\,\xi\times\sigma).$$

□

---

[3] strongly sequential spaces were characterized in [6]. For instance, a Hausdorff regular topology ξ is strongly sequential if and only if it is sequential and

$$\mathrm{adh}_\xi\,\mathcal{H}\neq\varnothing\Longrightarrow\mathrm{adh}_{\mathrm{B}_{\mathbb{E}}}\,\mathcal{H}\neq\varnothing$$

for every countably based filter $\mathcal{H}$.

As $\mathrm{S}_1 \geq \mathrm{Epi}^{\mathrm{T}}_{\mathrm{I}_1}$, Lemma 8 implies in particular that if ξ is productively sequential, then

$$\forall \sigma = \mathrm{I}_1\, \sigma,\, \xi \times \mathrm{S}_1\, \sigma \geq T(\mathrm{I}_1\, \xi \times \sigma). \tag{3.2}$$

Convergences ξ satisfying this functorial inequality are characterized via [7, Theorem 6.3]. To state this characterization, let us define [7] a modifier $Q_{\mathbb{D},\mathbb{J}}$ that depends on two classes $\mathbb{D}$ and $\mathbb{J}$ of filters in the following way: $x \in \lim_{Q_{\mathbb{D},\mathbb{J}}\xi} \mathcal{F}$ if $x \in \lim_\xi \mathcal{F}$ and for every $V \in \mathcal{N}_\xi(x)$ there is $\mathcal{C}_V \in \mathbb{D}$, such that $\mathcal{C}_V \leq \mathcal{F}$ and $\mathcal{C}_V$ is ξ-$\mathbb{J}$-compact at V ([4]).

With this definition in mind, it is a particular case of [7, Theorem 6.3] that (3.2) is equivalent to

$$\xi \geq \mathrm{S}\, Q_{\mathbb{F}_1,\mathbb{F}_1}\, \mathrm{Epi}^{\mathrm{T}}_{\mathrm{I}_1}\, \mathrm{I}_1\, \xi. \tag{3.3}$$

Recall from [5] that a *q-sequence* in a topological space X is a decreasing sequence $(A_n)_n$ of subsets of X such that there is a countably compact set K with $(A_n)_n \geq \mathcal{N}(K)$ and $\bigcap_{n\in\mathbb{N}} A_n \supseteq K$, and that a topological space is *bi-quasi-k* if every convergent ultrafilter contains a q-sequence.

Lemma 9. *If ξ be a regular topology. Then ξ is bi-quasi-k if and only if*

$$\xi \geq \mathrm{S}\, Q_{\mathbb{F}_1,\mathbb{F}_1}\xi.$$

Proof. Let ξ be a quasi-bi-k topology and let $x \in \lim_\xi \mathcal{U}$ where $\mathcal{U}$ is an ultrafilter. Then $\mathcal{U}$ contains a q-sequence $(A_n)_n$, and $\mathcal{U} \geq \mathcal{N}(x)$. As ξ is regular, $\mathcal{N}(x)$ has a filter-base of closed sets. Let V be a closed neighborhood of x. As $V \in \mathcal{U} \geq (A_n)_n \geq \mathcal{N}(K)$, we have $V \in \mathcal{N}(K)^\#$ so that $V \cap K \neq \varnothing$ because V is closed. Then

$$(A_n \cap V)_{n\in\mathbb{N}} \subseteq \mathcal{U}$$

is countably compact at V. Indeed, if $\mathcal{D} \in \mathbb{F}_1$ and $\mathcal{D}\#(A_n \cap V)_{n\in\mathbb{N}}$ then $(\mathcal{D} \vee V)\#\mathcal{N}(K)$ so that ([5]) $\mathrm{adh}^\natural(\mathcal{D} \vee V)\#K$ and

$$\mathrm{adh}(\mathcal{D} \vee V) = \mathrm{adh}(\mathrm{adh}^\natural(\mathcal{D} \vee V)) \cap (V \cap K) \neq \varnothing$$

because V is closed.

Conversely, assume $\xi \geq \mathrm{S}\, Q_{\mathbb{F}_1,\mathbb{F}_1}\xi$ and let $x \in \lim_\xi \mathcal{U}$ for some ultrafilter $\mathcal{U}$. For each $V \in \mathcal{N}(x)$, which we can pick closed by regularity, there is a countably based filter $\mathcal{C}_V \leq \mathcal{U}$ that is countably compact at V. Then each $\mathrm{cl}^\natural\, \mathcal{C}_V$ is a q-sequence in $\mathcal{U}$, with

$$K = V \cap \mathrm{adh}\, \mathcal{C}_V = V \cap \mathrm{adh}(\mathrm{cl}^\natural\, \mathcal{C}_V) :$$

Since $\mathcal{C}_V$ is countably based and countably compact at V, $K \neq \varnothing$. Moreover, K is countably compact.

Indeed, if $\mathcal{D}\#K$ and $\mathcal{D}$ is countably based, so is $\mathcal{D} \vee K$. Moreover, $(\mathcal{D} \vee K)\#\, \mathrm{cl}^\natural\, \mathcal{C}_V$, so that $\mathcal{N}(\mathcal{D} \vee K)\#\mathcal{C}_V$ and $\mathrm{adh}\, \mathcal{N}(\mathcal{D} \vee K) \cap V \neq \varnothing$. By regularity,

$$\varnothing \neq \mathrm{adh}\, \mathcal{N}(\mathcal{D} \vee K) \subseteq \mathrm{adh}\, \mathcal{D} \vee K \subseteq K,$$

because K is closed, and $\mathrm{adh}(\mathcal{D} \vee K) \subseteq \mathrm{adh}\, \mathcal{D}$. Hence $\mathrm{adh}\, \mathcal{D} \cap K \neq \varnothing$.

---

[4]that is, that is, for every $\mathcal{J} \in \mathbb{J}$,

$$\mathcal{J}\#\mathcal{C}_V \Longrightarrow \mathrm{adh}_\xi\, \mathcal{J} \cap V \neq \emptyset.$$

[5]If $a : 2^X \to 2^X$ and $\mathcal{A} \subset 2^X$ then $a^\natural(\mathcal{A}) := \{a(A) : A \in \mathcal{A}\}^\uparrow$.

By definition, $K \subseteq \bigcap(\mathrm{cl}^\natural \mathcal{C}_V)$. Finally, if U is an open set containing K, then $U \in \mathrm{cl}^\natural \mathcal{C}_V$ for otherwise, $U^c \in (\mathrm{cl}^\natural \mathcal{C}_V)^\#$ and $\mathcal{H} = \mathrm{cl}^\natural \mathcal{C}_V \vee U^c$ is a countably based filter meshing with $\mathrm{cl}^\natural \mathcal{C}_V$ which is countably compact at V. Therefore $\mathrm{adh}\,\mathcal{H} \cap V \neq \emptyset$. But $\mathrm{adh}\,\mathcal{H} \subseteq \mathrm{adh}(\mathrm{cl}^\natural \mathcal{C}_V) \cap U^c$ which is disjoint from V. □

Since $\mathrm{Epi}^{\mathrm{T}}_{\mathrm{I}_1} \mathrm{I}_1 \xi \geq \xi$ for every topology ξ, we conclude that a topology satisfying (3.3) is bi-quasi-k. Moreover, $SQ_{\mathbb{F}_1,\mathbb{F}_1}\xi \geq \xi$ for every topology ξ, so that a topology satisfying (3.3) is also strongly sequential. Conversely, if ξ is a regular bi-quasi-k topology then by Lemma 9, $\xi \geq SQ_{\mathbb{F}_1,\mathbb{F}_1}\xi$. Moreover, $\xi \geq \mathrm{Epi}^{\mathrm{T}}_{\mathrm{I}_1} \mathrm{I}_1 \xi$ because ξ is strongly sequential. Thus ξ satisfies (3.3). We can now conclude:

Theorem 10. *Let ξ be a regular topology. The following are equivalent:*
(1) *ξ satisfies (3.3);*
(2) *ξ is bi-quasi-k and strongly sequential;*
(3) *ξ is bi-quasi-k and sequential;*
(4) *$\xi \times \tau$ is strongly sequential for every strongly sequential topology τ;*
(5) *ξ is productively sequential.*

Proof. We have observed (1) $\iff$ (2) and (2) $\Longrightarrow$ (3) by definition. (3) $\Longrightarrow$ (2) follows from [8, Proposition 5]. We have already observed that (5) $\Longrightarrow$ (1), (3) $\Longrightarrow$ (4) is [10, Theorem 12-(2)i], and (4) $\Longrightarrow$ (5) is obvious. □

In view of the equivalence of (4) and (5) above, we immediately have:

Corollary 11. *A product of two productively sequential topological spaces is productively sequential.*

More information on product properties of bi-quasi-k sequential topological spaces can be found in [10].

References

[1] S. Dolecki. Convergence-theoretic methods in quotient quest. *Topology Appl.*, **73**:1–21, 1996.
[2] S. Dolecki and F. Mynard. Convergence-theoretic mechanisms behind product theorems. *Topology Appl.*, **104**:67–99, 2000.
[3] S. Dolecki and F. Mynard. *Convergence Foundations of Topolgy.* World Scientific, 2016.
[4] F. Jordan and F. Mynard. Productively Fréchet spaces. *Czech. Math. J.*, 54(129):981–990, 2004.
[5] E. Michael. A quintuple quotient quest. *Gen. Topology Appl.*, **2**:91–138, 1972.
[6] F. Mynard. Strongly sequential spaces. *Comment. Math. Univ. Carolinae*, **41**:143–153, 2000.
[7] F. Mynard. Coreflectively modified continuous duality applied to classical product theorems. *Appl. Gen. Top.*, 2 (2):119–154, 2002.
[8] F. Mynard. More on strongly sequential spaces. *Comment. Math. Univ. Carolinae*, 43(3):525–530, 2002.
[9] F. Mynard. Coreflectively modified duality. *Rocky Mountain J. of Math.*, 34(2):733–758, 2004.
[10] Y. Tanaka. Tanaka spaces and products of sequential spaces. *Comm. Math. Univ. Carol.*, 48(3):529–540, 2007.