TOWARDS HISTORICAL ROOTS OF NECESSARY CONDITIONS OF OPTIMALITY: REGULA OF PEANO

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Dedicated to professor Stefan Rolewicz on his 75-th birthday

Abstract. At the end of 19th century Peano discerned vector spaces, differentiability, convex sets, limits of families of sets, tangent cones, and many other concepts, in a modern perfect form. He applied these notions to solve numerous problems. The theorem on necessary conditions of optimality (*Regula*) is one of these. The formal language of logic that he developed, enabled him to perceive mathematics with great precision and depth. Actually he built mathematics axiomatically based exclusively on logical and set-theoretic primitive terms and properties, which was a revolutionary turning point in the development of mathematics.

1. Introduction

The aim of this paper is to trace back the evolution of mathematical concepts in the work of Giuseppe Peano (1858-1932) that are constituents of *Regula*, that is, Peano's theorem on necessary conditions of optimality.

A well-known necessary conditions of maximality of a function at a point, is formulated in terms of derivative of the function and of tangent cone of the constraint at that point. Consider a real-valued function $f : X \to \mathbb{R}$, where X is a Euclidean affine space, and a subset A of X.

Regula (of Optimality) *If f is differentiable at $x \in A$ and $f(x) = \max\{f(y) : y \in A\}$, then*

$$\langle Df(x), y - x\rangle \leq 0 \text{ for every } y \in \operatorname{Tang}(A, x). \tag{1.1}$$

The *derivative* $Df(x)$ is defined to be *the* vector $Df(x)$ such that

$$\lim_{y\to x} \frac{f(y) - f(x) - \langle Df(x), y - x\rangle}{|y - x|} = 0. \tag{1.2}$$

The *affine tangent cone* $\operatorname{Tang}(A, x)$ of A at x (for arbitrary $x \in X$) is given by

$$\operatorname{Tang}(A, x) := \operatorname{Ls}_{\lambda\to+\infty} (x + \lambda(A - x)), \tag{1.3}$$

where the *upper limit* $\operatorname{Ls}_{\lambda\to+\infty} A_\lambda$ of sets A_λ (as λ tends to $+\infty$) is defined by

$$\operatorname{Ls}_{\lambda\to+\infty} A_\lambda := \{y \in X : \liminf_{\lambda\to+\infty} \operatorname{dist}(y, A_\lambda) = 0\}. \tag{1.4}$$

Of course, dist is the distance in (1.3), $A_\lambda := x+\lambda(A-x) := \{x+\lambda(a-x) : a \in A\}$.

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It is generally admitted among those who study optimization, that modern definition of differentiability was introduced by Fréchet [22, (1911)], of tangent cone by Bouligand [13, (1932)], and of limit of sets by Painlevé [85, (1905) p. 8] [1]. So we were very surprised to discover that *Regula* was already known by Peano in 1887. Indeed, in order to formulate it, one needed to possess the notions of differentiability and of affine tangent cone, hence also that of limit of sets.

But our surprise was even greater, because not only all these notions were familiar to Peano at the end of 19-th century, but they were formulated in a rigorous, mature way of today mathematics, in contrast with the approximated imprecise style that dominated in mathematical writings in those times, and often persisted during several next decades.

Impressed by the so early emergence of these notions, we started to peruse the work of Peano in order to understand the evolution of the ideas that lead to *Regula*.

All the citations of Peano's activity, related to the concepts involved in the optimality conditions, prevalently concern the span of time between the first appearance of *Regula* in "**Applicazioni Geometriche**" [50, (1887)] and its ultimate form in "**Formulario Mathematico**"[2] [64, (1908) p. 335], where *Regula* is stated exactly as above.

Tracing back the development and applications of differentiability, tangency, limit and other concepts, in the work of Peano over the years, we see evolution and enrichment of their facets. Peano built mathematics axiomatically, based exclusively on logical and set-theoretic primitive terms and properties. This was a revolutionary turning point in the development of mathematics. The reduction of every mathematical object to the founding concept of "set" (*genus supremum*) of Cantor, enabled the emergence of new concepts related to properties of sets, unconceivable otherwise. Early and illuminating examples of the fecundity of Cantor's views are in the books *Fondamenti per la teorica delle funzioni di variabili reali* of Dini [19, (1878)], *Calcolo differenziale e integrale* of Genocchi and Peano [25, (1884)], and the second edition of *Cours d'Analyse* of Jordan [38, (1893-96)].

To appreciate the novelty of Cantor's approach to mathematics, we should remember the opposition of some luminaries of mathematics that existed at the beginning of the twentieth century. For example, in the address to the Congress of Mathematicians in Rome [70, p. 182] in 1908, Poincaré said

> Quel que soit le remède adopté [contre le "cantorisme"], nous pouvons nous promettre la joie du médecin appelé à suivre un beau cas pathologique.

This paper is not a definitive word on historical roots of conditions of optimality. For instance, a confront of (1.1) with virtual work principle has still to be

[1] A Painlevé's student, Zoretti (1880-1948), attests in [87, (1912) p. 145] that Painlevé introduced both upper and lower limit of a family of sets. Following Zoretti, Hausdorff in [34, (1927) p.280] and Kuratowski in [41, (1928) p. 169] reiterate this attribution to Painlevé. More clearly, Zoretti calls "*set-limit*" of a sequence of sets the today's upper limit; while by "*point-limit*" he means "point belonging to the lower limit" of the sequence. Painlevé uses the notion of "*set-limit*" from 1902 in [85, (1905) p. 8]; on the other hand, one finds in [86, (1909) p. 8] a Zoretti's use of "*point-limit*". Both "*set-limit*" and "*point-limit*" are present in [87, (1912) p. 145].

[2] The previous four editions of *Formulario mathematico* are *Formulaire Mathématique* tome 1 (1895), tome 2 (1899), tome 3 (1901) and tome 4 (1903). The first half of the fifth edition was printed in 1905; the other half earlier in 1908. The "Index and Vocabulary" to *Formulario mathematico* of 1908 was published separately in 1906.

addressed[3]. We have found no evidence of this relationship in the work of Peano, but it is plausible that he was aware of it (remark that *Regula* is placed in *Formulario Mathematico* within the context of mechanics).

This article concerns several historical aspects. From a methodological point of view, we are focused on primary sources, and not on secondary founts, that is, on mathematical facts, and not on opinions or interpretations of other scholars of history of mathematics. On the other hand, we will avoid to mention, if not necessary, historical facts that are well-known among those who study optimization (see, for example, Rockafellar and Wets [72], Borwein and Lewis [11], Aubin [6], Aubin and Frankowska [7], Hirriart-Urruty and Lemaréchal [35], Pallaschke and Rolewicz [47]).

2. Affine and vector spaces

Applicazioni Geometriche is based on the extension theory of Grassmann [27, (1844)], presented in detail in "Calcolo Geometrico" [51, (1888)], where, forgoing the philosophical aura founding the work of Grassmann, Peano introduces the modern notion of vector space.

In Grassmann's work, points and vectors coexist distinctly in a common structure, together with other objects, like exterior products of points and vectors (see Greco [29]). This subtle distinction was very demanding in comparison with today habits of mathematicians. Peano maintains the distinction. For instance, a difference $y - x$ is a vector if both y and x are either points or vectors; otherwise, it is a point (if x is a vector) or a point of mass -1 (if y is vector).

Moreover, Peano follows Grassmann in construction of metric concepts from the scalar product of vectors (introduced by Grassmann in [26]). Following Grassmann and Hamilton, he conceives the gradient of a function as a vector, differently from a common habit (of using the norm of the gradient) that prevailed at the pre-vectorial epoch [4].

In several papers Peano applies the geometric calculus of Grassmann, for instance, to define area of a surface (see [50, (1887) p. 164] and [54]) and to give in [62, (1898)] an axiomatic refoundation (today standard) of Euclidean geometry, based on the primitive notions of point, vector and scalar product.

Peano's approach to the definition of linear map was slightly different from (but equivalent to) that commonly adopted nowadays. Peano says that a map g between spaces is *linear* if is additive and bounded, that is, $g(x + y) = g(x) + g(y)$ for all x and y, and if $\sup\{|g(x)| : |x| < 1\}$ is finite. The reader has certainly observed

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[3]If a force acting on a material point in equilibrium x, has a potential f, then the corresponding virtual work states

$$\delta L := \langle Df(x), \delta x\rangle \le 0 \text{ for each } \delta x,$$

where δx is a virtual displacement of that point with respect to an *ideal constraint* A (either bilateral or unilateral) independent of the time (see, for example, Stefan Banach [8, (1951)]).

[4]These observations are relevant for the understanding of Peano's interpretation of the formula $\langle Df(x), y - x\rangle \le 0$ that appears in *Regula*.

that the today condition of *homogeneity* is substituted by that of *boundedness* [5]. For Peano, the interest of employing boundedness in the definition, was to obtain simultaneously a concept of *norm* (*module* in his terminology) on spaces of linear maps.

The norm was useful in his study of systems of linear differential equations [52, (1888)]; to give a formula for a solution in terms of resolvent, he defines the exponential of matrix and proves its convergence using the norm (see also [59, (1894)] and the English translation of [52, (1888)] in [9] of G. Birkhoff).

As other new theories, the theory of vector spaces was contested by many prominent mathematicians. Even those (few) who adopted the vector approach, were not always entirely acquainted with its achievements. To perceive the atmosphere of that time, we give an excerpt from the introduction of Goursat to book *Leçons de géométrie vectorielle* [12, (1924)] of Bouligand:

> Si le calcul vectoriel a été un peu lent à pénétrer en France, il est bien certain que la multiplicité des notations et l'abus du symbolisme ont justifié en partie la défiance de nos étudiants. Or, dans le livre de M. Bouligand, le symbolisme est réduit au minimum, et l'auteur n'hésite pas à revenir aux procédés habituels du calcul quand les méthodes lui paraissent plus directes. [...]
>
> M. Bouligand a devisé son ouvrage en trois parties, consacrées respectivement aux opérations vectorielles en géométrie linéaire, en géométrie métrique et aux opérations infinitésimales.

A decisive role in the dissemination of vector spaces had a book *Space-Time-Matter* [79, (1918)] of Weyl (for details see Zaddach [83]).

3. Differentiability

In *Applicazioni Geometriche* (p. 131) Peano says that a vector $\mathbf{u}$ is a *derivative* at a point x of a real-valued function f defined on a finite-dimensional Euclidean affine space X, if there exists a vector $\varepsilon(y)$ such that

$$f(y) - f(x) = \langle y - x, \mathbf{u} + \varepsilon(y)\rangle \text{ with } \lim_{y\to x} \varepsilon(y) = 0. \tag{3.1}$$

The reader recognizes in (3.1) the Taylor formula of order 1 and, on the other hand, the characterization of derivability which Carathéodory gives in [16, p. 119]: "f is derivable at x if there is a function φ continuous at x such that $f(y) - f(x) = \langle y - x, \varphi(y)\rangle$ for every y" [6].

In *Formulario Mathematico* of 1908 (p. 334 and 330) the derivative $\mathbf{u}$ is denoted by $Df(x)$ and is defined by (1.2) and, more generally, one finds a definition of differential of map between finite dimensional Euclidean vector spaces, namely if

[5] In other moments (for example, in fourth edition of *Formulaire Mathématique* [63, (1903) p. 203]) Peano adopts a different (but equivalent) definition of linearity, replacing *boundedness* with *continuity*. All these variants are related to the following fundamental lemma (see a proof in *Formulario Mathematico* [64, pp. 117-118], where Peano quotes Darboux [18, footnote of p. 56]): "*For an additive function $f : \mathbb{R}^n \to \mathbb{R}$ are equivalent: (1) homogeneity, (2) continuity and (3) boundedness on bounded sets*"

[6] Remarkably, this Carathéodory reformulation "leads to some sharp, concise proofs of important theorems: chain rule, inverse function theorem, ..." (see Kuhn [40]) and "makes perfect sense in general linear topological spaces" (see Acosta-Delgado [1]).

$f : \mathbb{R}^m \to \mathbb{R}^n$ then a *derivative* of f at x is *the* linear map $Df(x) : \mathbb{R}^m \to \mathbb{R}^n$ (called nowadays the *Fréchet derivative* of f at x) such that

$$\lim_{y\to x} \frac{f(y) - f(x) - Df(x)(y-x)}{|y-x|} = 0. \tag{3.2}$$

In giving this definition, Peano refers to the second edition of *Ausdehnungslehre* of 1962 of Grassmann [27, v. 2 p. 295] and to an article *De determinantibus functionalibus* 1841 of Jacobi [36, v. 3, p. 421]. Actually the citation of Jacobi refers to the concept of *Jacobian*, that is, derivative defined in terms of partial derivatives.

With respect to Peano's quotation of Grassmann, our verification of the source leads the following facts. For a map $f : \mathbb{R}^m \to \mathbb{R}^n$, Grassmann defines a *differential* $df(x)$ at x, as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$, by

$$df(x)(v) := \lim_{q\to 0} \frac{f(x + q\,v) - f(x)}{q}, \tag{3.3}$$

which is radially continuous in variable x for each fixed $v \in V$.[7] Grassmann proves that if f is everywhere differentiable in this sense, then f is radially continuous, the differential is linear in v, and that (3.3) becomes the partial derivative when v is an element of the canonical base. Moreover, he claims that the chain rule holds. In contrast to the comments [28, p. 398] of Kannenberg (author of English translation [28] of *Ausdehnungslehre* 1962), this claim is however false, as can be seen from Acker-Dickstein's example [2, Ex. 2.5, p. 26] [8].

In various moments of his activity, Peano studied the concept of derivative. His contributions are very rich and diversified, and concern

(1) strict derivative (see [48], [49] and [57])[9],
(2) Taylor formula with infinitesimal remainder (called *Peano remainder*) (see [25, p. XIX] or [74, (1893), pp. 90-91]),
(3) asymptotic development and Taylor formula [56] [10],
(4) derivation of measures [50, (1887) p. 169] (see also [30]), and
(5) mean value theorem.

Here is an excerpt concerning the latter. In *Calcolo Geometrico* [51, (1888)] Peano gives a *mean value theorem* for vector-valued functions f of one variable [11],

[7]A map g is *radially continuous at x* if for every vector v the map $h \longmapsto g(x+hv)$ is continuous at 0. Grassmann adopts the term *continuous* to denote "radially continuous".

[8]A counterexample to Grassmann's claim is given by the functions f, and g of two real variables defined by $g(x,y) := (x, y^2)$, $f(x,y) := \frac{x^3 y}{x^4+y^2}$ and $f(0,0) := 0$. These functions was used in [2] to invalidate a similar claim for Gateaux differentiability.

[9]In [48] he observes the equivalence between continuity of derivative f' of f at x and "strict derivability of f at x" (that is, $\lim_{a,b\to x} \frac{f(b)-f(a)}{b-a} = f'(x)$). In [49] he notices that the uniform convergence of the difference quotient function $\frac{f(x+h)-f(x)}{h}$ in variable x to $f'(x)$ (as h tends to 0) amounts to continuity of derivative f' in variable x. As observed Mawhin in [45, p. 430], Peano formulates in [49] an approximation property of primitives equivalent to Kurzweil integrability of all the functions having a primitive.

[10]If f is a function, and $P(h) = a_0 + a_1 h + \cdots + a_n h^n$ is a polynomial function such that $f(x+h) - P(h) = h^n \eta(h)$ where $\eta(h)$ tends to 0 with h, then the *Peano derivative of order n* is $n!a_n$ (see, for example, Weil [78], Svetic and Volkmer [75]). Peano gives an example of function that is discontinuous in every neighborhood of x, and for which the Peano derivatives of all order exist.

[11]Peano gives a proof (by scalarization) in [61, p. 975].

that is, if f has an $(n+1)$-derivative $f^{(n+1)}$ on $[t, t+h]$, then there exists an element k of the closed convex hull of the image of interval $[t, t+h]$ by $f^{(n+1)}$ such that

$$f(t+h) = f(t) + hf'(t) + \cdots + \frac{h^n}{n!} f^{(n)}(t) + \frac{h^{n+1}}{(n+1)!} k. \tag{3.4}$$

Here is another surprise in front of the first appearance of modern notions of *convex set* [12] and *convex hull*, as we thought that it was Minkowski who introduced these concepts for the first time in [46, (1896)].

Among the first who studied the modern notion of differentiability of functions of several variables, were Stolz [74, (1893), §.IV 8 p. 130], Pierpont [69, (1905) p. 269], W.H. Young [82, (1910) p. 21]. Besides, Maurice Fréchet in [22, (1911)][13] defines the *differential* (of a function of two variables) as the linear part of the approximation

$$f(a+h, b+k) - f(a, b) = hp + kq + h\rho(h, k) + k\sigma(h, k), \tag{3.5}$$

where ρ and σ tend to 0 with h and k.

Peano's definition liberates the concept of derivative from the coordinate system and from partial derivatives. The definitions (3.1)-(3.2) are coordinate-free in contrast with the predominant habit of the epoch.

We have found no clear evidence in the mathematical literature of an acknowledgement of Peano's definition of derivative, with the exception of a paper [81, (1921)] of Wilkosz, where Peano is cited jointly with Stolz.

4. Limits of variable sets

In *Applicazioni Geometriche* (p. 30) Peano introduces a notion of limit of straight lines, planes, circles and spheres (that depend on parameter). He considers these objects as sets, which leads him to extend the definition of limit to variable figure (in particular, curves and surfaces).

A *variable figure* (or *set*) is a family, indexed by the reals, of subsets A_λ of an affine Euclidean space X. Peano defines in *Applicazioni Geometriche* [50, (1887) p. 302] the *lower limit of a variable figure* by

$$\mathrm{Li}_{\lambda\to+\infty} A_\lambda := \{y \in X : \lim_{\lambda\to+\infty} \mathrm{d}(y, A_\lambda) = 0\}. \tag{4.1}$$

In the edition of *Formulario Mathematico* [64, (1908) p. 237] we find the lower limit together with a definition of *upper limit of a variable figure*:

$$\mathrm{Ls}_{\lambda\to+\infty} A_\lambda := \{y \in X : \liminf_{\lambda\to+\infty} \mathrm{d}(y, A_\lambda) = 0\}, \tag{4.2}$$

which we have already seen (1.4). Besides, he writes down (p. 413) the upper limit as

$$\mathrm{Ls}_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \mathrm{cl} \bigcup_{k\geq n} A_k, \tag{4.3}$$

[12] Peano employs the concept of convex set for the first time to axiomatic foundation of geometry [53, p. 90 Axioma XVII]; more precisely, his *Axiom XVII of continuity* states: *Let A be a convex set of points, and let x, y points such that $x \in A$ and $y \notin A$. Then there exists a point $w \in xy$ (the open segment between x and y) such that $xw \subset A$ and $wy \cap A = \varnothing$.*

[13] A month after the publication of [22] in which he presented the concept of differentiability, Fréchet publishes a second Note [23] in order to recognize the priority of Young.

where the *closure* $\mathrm{cl}\,A$ of a set A, is defined (p. 177) by[14]

$$\mathrm{cl}\,A := \{y \in X : \mathrm{d}(y, A) = 0\}. \tag{4.4}$$

In several papers, Peano analyzes the meanings that are given in mathematics to the word *limit* (see, for example, [60, (1894)]): least upper bound, greatest lower bound of sets, (usual) limit and adherence of sequences and functions.

Peano conceives the "upper limit of variable sets" as a natural extension of the *adherence* of functions. He attributes to Cauchy the introduction of adherence, see [60, (1894) p. 37] where he says:

> Selon la définition de la limite, aujourd'hui adoptée dans tous les traités, toute fonction a une limite seule, ou n'a pas de limite. [...]
>
> Cette idée plus générale de la limite [*the adherence*] est clairement énoncée par Cauchy; on lit en effet dans son Cours d'Analyse algébrique, 1821, p. 13: ≪Quelquefois...une expression converge à-la-fois vers plusieurs limites différentes les unes des autres≫, et à la page 14 il trouve que les valeurs limites de $\sin \frac{1}{x}$, pour $x = 0$, constituent l'intervalle de -1 à $+1$. Les auteurs qui ont suivi Cauchy, en cherchant de préciser sa définition un peu vague, se sont mis dans un cas particulier.

Peano studies the notion of "lower limit of variable sets", in particular, in a celebrated article on existence of solutions of a system of ordinary differential equations [55, (1888)]. Peano carries on the proof of existence in a framework of logical and set-theoretic ideography, thanks to which he is able to detect the *axiom of choice*[15].

The awareness of the problem of "limits of variable sets" was present on the threshold of the 20-th Century (for example see Manheim [43]).[16] The book of Kuratowski [42, p. 241] [17] has definitely propagated the concept of limit of variable sets.

Among the first mathematicians who studied the limits of variable sets are Burali-Forti [14, (1895)] [18], Zoretti [85, (1905) p. 8] and [86, (1909)], Janiszewski

[14]Peano defined closure, interior and boundary earlier in [50, (1887) pp. 152-158]; these notions were introduced by Jordan in [38]. Peano relates the closure with the concept of *closed set* of Cantor: *the closure of A is the least closed set including A*.

[15]Peano proves the existence of a solution with the aid of approximated solutions. In order to obtain a solution, he is confronted with a problem of non-emptiness of the lower limit of a sequence of subsets of a Euclidean space. To this end, he needs to select an element from every set of the sequence. At that point he realizes that he would need to make *infinite arbitrary choices*, which, starting from the paper of Zermelo [84] of 1904 is called *Axiom of choice*. He avoids to apply a new axiom, which is not present in mathematical literature and, consequently, the tradition does not grant it. Instead, using the lexicographic order, he is able to construct a particular element of every set, because the sets of the sequence are compact.

[16]In [10, (1903)] (see also Manheim [43, p. 114]) Borel suggests a "promising" notion of limit of straight lines and of planes, that is, 16 years after the introduction of the limit of arbitrary sets in *Applicazioni Geometriche*.

[17]Kuratowski, by his work, consacrates the use of upper and lower limits in mathematics, that are called today *upper and lower Kuratowski limits*.

[18]Burali-Forti studies only lower limits.

[37, (1911)], Hausdorff [33, (1914) p. 234] [19], Vietoris [77, 1922][20], Vasilesco [76, (1925)], Cassina [17, (1926-27)] and Kuratowski [41, (1928)].

5. Tangent cones

In *Applicazioni Geometriche* (pp. 58, 116) Peano gives a metric definition of tangent straight line and tangent plane, then reaches, in a natural way, a unifying notion: that of *affine tangent cone*:

$$\operatorname{tang}(A,x) := \operatorname{Li}_{\lambda\to+\infty}(x+\lambda(A-x)). \tag{5.1}$$

Later, in *Formulario Mathematico* (p. 331), he introduces another type of tangent cone, namely

$$\operatorname{Tang}(A,x) := \operatorname{Ls}_{\lambda\to+\infty}(x+\lambda(A-x)) \tag{5.2}$$

To distinguish the two notions above, we shall call the first *lower affine tangent cone* and the second *upper affine tangent cone* [21]. Peano lists several properties of the upper tangent cone [22]. If A is a subset of a Euclidean affine space X, then

(1) If $x \notin \operatorname{cl} A$ then $\operatorname{Tang}(A,x) = \varnothing$;
(2) If x is isolated in A then $\operatorname{Tang}(A,x) = \{x\}$;
(3) If $x \in \operatorname{cl}(A\setminus\{x\})$ then $\operatorname{Tang}(A,x) \neq \varnothing$;
(4) If $x \in \operatorname{int} A$ then $\operatorname{Tang}(A,x) = X$;
(5) If $y \in \operatorname{Tang}(A,x)\setminus\{x\}$ then $x+\mathbb{R}_+(y-x) \subset \operatorname{Tang}(A,x)$;
(6) If $A \subset B$ then $\operatorname{Tang}(A,x) \subset \operatorname{Tang}(B,x)$;
(7) $\operatorname{Tang}(A\cup B,x) = \operatorname{Tang}(A,x)\cup\operatorname{Tang}(B,x)$;
(8) $\operatorname{Tang}(\operatorname{Tang}(A,x),x) = \operatorname{Tang}(A,x)$.[23]

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[19] In the celebrated *Grundzüge der Mengelehre* (1914) Hausdorff studies both upper and lower limits. Moreover, he defines a metric on the set of bounded subsets of a metric space X (*Hausdorff distance*) and proves that the related convergence of bounded subsets $\{A_\lambda\}_\lambda$ to A (as $\lambda\to+\infty$) is equivalent to $\operatorname{Ls}_{\lambda\to+\infty} A_\lambda \subset A \subset \operatorname{Li}_{\lambda\to+\infty} A_\lambda$, if X is compact.

[20] Kuratowski limits, Hausdorff distance and Vietoris's topology [71, p. 1234] are milestones in the search of notions of limit of variable sets.

[21] Of course, $\operatorname{tang}(A,x)$ is defined with the aid of lower limit, while $\operatorname{Tang}(A,x)$ with the aid of the upper limit of the same homothetic sets. Hence, $\operatorname{tang}(A,x) \subset \operatorname{Tang}(A,x)$. For the covenience of the reader, in order to compare the two definitions, we give their alternative descriptions in terms of limits of sequences:

$$\operatorname{tang}(A,x) = x + \left\{ v : \exists\,\{x_n\}_n \subset A \text{ such that } x = \lim_{n\to\infty} x_n \text{ and } v = \lim_{n\to\infty}\frac{x_n - x}{1/n}\right\}; \tag{5.3}$$

$$\operatorname{Tang}(A,x) = x + \left\{ v : \exists\,\{\lambda_n\}_n \to 0^+,\ \exists\,\{x_n\}_n \subset A \text{ such that } x = \lim_{n\to\infty} x_n \text{ and } v = \lim_{n\to\infty}\frac{x_n - x}{\lambda_n}\right\}.$$

The second formula is standard, while we have never seen in the literature the first one (5.3). We have not found in Peano's papers any example of set A for which the cones above are different. Here is another, perhaps most intuitive, formula for the "lower" affine tangent cone:

$$\operatorname{tang}(A,x) = x + \{v : \exists\gamma : [0,1]\to A \text{ such that } x = \gamma(0), \gamma'(0) \text{ exists and } v = \gamma'(0)\}. \tag{5.4}$$

Notice that $\operatorname{tang}(A,x) = \operatorname{Tang}(A,x)$ in case of differential manifold A (at x).

[22] It was Peano who introduced the notions of the closure $\operatorname{cl} A$, the interior $\operatorname{int} A$ and the exterior $\operatorname{ext} A$ (of A) in [50, pp. 152–158].

[23] One find in fourth edition of *Formulaire Mathématique* [63, (1903) p. 296] the same definition of upper affine tangent cone and, besides, the eight properties (1)–(8). Besides, one find both lower and upper limit of variable sets [63, (1903) p. 289].

We have not found in any of five editions of *Formulario Mathematico* (= collection of logical and set-teoretical formulas) any other property on tangent cones. Today other fundamental properties are well-known: (1) $x \in \operatorname{cl} A \iff \operatorname{Tang}(A,x) \neq \emptyset \iff x \in \operatorname{Tang}(A,x)$; (2) $x \in \operatorname{cl}(A \setminus$

As usual, after abstract investigation of a notion, Peano considers significant special cases; he calculates the upper affine tangent cone in several basic figures (closed ball, curves and surfaces parametrized in a regular way).

Various types of tangent cones have been studied in the literature. Their definitions depend on variants of limiting process. The most known contribution to the investigation of tangent cones is due to Bouligand [13, (1932) p. 60]. One can find a mention about other contributors in a paper [24, (1937) p. 241] of Fréchet:

> Cette théorie des "contingents et paratingents" dont l'utilité a été signalée d'abord par M. Beppo Levi, puis par M. Severi, mais dont on doit à M. Bouligand et à ses élèves d'avoir entrepris l'étude systématique.

The diffusion of the concept of tangent cone was due mainly to Stanislaw Saks [73, (1937) pp. 262–263], who adopted the definition of Bouligand, and to Federer [20, (1959) p. 433], who introduced it in a modern vector version: if $x \in A$ then define the *upper vector tangent cone*:

$$\begin{aligned}&\operatorname{Tan}(A,x) := \{0\} \cup \\ &\left\{u \neq 0 : \forall \varepsilon > 0,\ \exists y \in A,\ 0 < |y-x| < \varepsilon \text{ and } \left|\frac{y-x}{|y-x|} - \frac{u}{|u|}\right| < \varepsilon\right\}.\end{aligned} \tag{5.5}$$

Federer does not give any reference of the origin of the definition (5.5)[24]. Notice that

$$\operatorname{Tang}(A,x) = x + \operatorname{Tan}(A,x). \tag{5.6}$$

Neither Whitney cites nobody in [80, (1972) chap. 7], where he introduces six variants of vector tangent cone among which, one recognizes the upper vector tangent cone discussed above (5.5).

We can say that, as far as tangent cones are concerned, main references are, respectively, Bouligand in optimization theory, Ferderer in geometric measure theory and calculus of variations and Whitney in differential geometry. A rare direct reference to Peano's definition is that of Guido Ascoli [25] [4, (1953)], who writes about Peano's work in [5, (1955) pp. 26-27]:

> [...] il merito maggiore [...] specialmente delle *Applicazioni* [*Geometriche*], non sta tanto nel metodo usato, quanto nel contenuto; ché vi sono profusi, in forma così semplice da parere definitiva, idee e risultati divenuti poi classici, come quelli sulla misura degli insiemi, sulla rettificazione delle curve, sulla definizione dell'area di una superfice, sull'integrazione di campo, sulle funzioni additive d'insieme; ed altri che sono tuttora poco noti o poco studiati. Ci basti indicare tra questi il concetto di limite di una figura variabile, destinato a ricomparire, con altro nome di autore, quarant'anni dopo presso la scuola di "geometria infinitesimale diretta" del Bouligand, e l'originalissima definizione di "figura tangente ad

---

$\{x\}) \iff \operatorname{Tang}(A,x) \setminus \{x\} \neq \emptyset$; (3) $\operatorname{Tang}(A,x) = \operatorname{Tang}(A \cap B, x)$, if $x \in \operatorname{int} B$; (4) $\operatorname{Tang}(A,x) = \operatorname{Tang}\big(\operatorname{cl}(A), x\big)$; finally, (5) $\operatorname{Tang}(A,x)$ is closed (because it is an upper limit of variable sets).

[24] The book of Saks [73] is among bibliographic references.

[25] One should not confound Guido Ascoli(1887-1957) with Giulio Ascoli(1843-1896), the latter known because of the Ascoli-Arzelà theorem. Guido proved the geometric version [3] of the Hahn-Banach theorem for separable normed spaces; a year later, Mazur proved it for arbitrary normed spaces [44].

> un insieme in un punto", che ha fornito a chi scrive, or è qualche anno la chiave di una difficile questione asintotica.

The *contingent cone* of Bouligand [13, (1932)] is defined by Saks in [73, (1937) p. 262] as follows: if x is an accumulation point of A, then the *contingent cone* of A at x is given by

$$\mathrm{Cont}(A, x) := \{l : l \text{ tangent half-line to } A \text{ at } x\} \tag{5.7}$$

where a *half-line* l issued from x is said to be *tangent to* A at x if there exist a sequence $\{x_n\}_n \subset A$ and a sequence of half-lines $\{l_n\}_n$ issued from x such that $x \neq x_n \in l_n, x = \lim x_n$ and the angle between l_n and l tends to 0.

Peano's upper affine tangent cone, Federer's upper vector tangent cone and Bouligand's contingent cone describe the same intuitive concept in terms, respectively, of points of affine space (via *blow-up*), of vectors (via *directions* of tangent half-lines) and of half-lines (via *limits* of half-lines)[26]. Finally, observe that the tangent cone is built on the notion of distance by Peano, of norm by Federer and of angle (consequently, of scalar product) by Bouligand.

6. Maxima and minima

In *Applicazioni Geometriche* (pp. 143-144) Peano analyzes the variation of a real-valued function around a point in a particular direction $\mathbf{p}$ in terms of the scalar product of the derivative at that point with $\mathbf{p}$.

Theorem 6.1. *Let f be a real-valued function such that $Df(\bar{x}) \neq 0$. Let $\mathbf{p}$ be a unit vector and $\{x_n\}_n$ be a sequence so that*

$$\bar{x} = \lim_{n\to\infty} x_n \qquad \textit{and} \qquad \mathbf{p} = \lim_{n\to\infty} \frac{x_n - \bar{x}}{\|x_n - \bar{x}\|}. \tag{6.1}$$

$$\textit{If} \quad \langle Df(\bar{x}), \mathbf{p}\rangle > 0, \textit{ then } f(x_n) > f(x_0) \textit{ for almost all } n; \tag{6.2}$$

$$\textit{if} \quad \langle Df(\bar{x}), \mathbf{p}\rangle < 0 \textit{ then } f(x_n) < f(x_0) \textit{ for almost all } n. \quad \square \tag{6.3}$$

Peano specifies that $\{x_n\}_n$ in Theorem 6.1 can be taken either arbitrarily or constrained by some conditions, for example, lying on a line or on a surface.

If $\{x_n\}_n$ is included in a set A, then $\mathbf{p}$ is one of the directions (unitary vectors) of the upper vector tangent cone of A at x. By taking all such sequences, we get all the directions of the upper vector tangent cone (5.5) of A at x. Hence, by relating (6.1) to upper vector tangent cone, Theorem 6.1 implies

Theorem 6.2 (**Regula o**f Maximality)**.** *If f is differentiable at $x \in A$ and $f(x) = \max\{f(y) : y \in A\}$, then$\langle Df(x), y - x\rangle \leq 0$ for every $y \in \mathrm{Tang}(A, x)$.*

Theorem 6.3 (**Regula o**f Minimality)**.** *If f is differentiable at $x \in A$ and $f(x) = \min\{f(y) : y \in A\}$, then$\langle Df(x), y - x\rangle \geq 0$ for every $y \in \mathrm{Tang}(A, x)$*

---

[26]Bouligand, in spite of his knowledge of vector spaces (see, for example, [12, (1924)] and his introduction to the French translation of [79, (1918)] of Weyl, does not use vectors while defining the contingent cone. In the preface to [21, 1957] Bouligand appraises Peano's *Calcolo Geometrico* [51, (1888)]: ≪Pour être moins incomplet, il faudrait encore citer l'exposé repris par Peano en 1886 [sic!] *du calcul extensif* de Grassmann, l'article fondamental malgré sa brièveté paru en 1900 [sic!] dans l'Enseignement Mathématique au sujet des relations d'équivalence, rédigé par Burali-Forti (Sur quelques notions dérivées de la notion d'égalité et leurs applications dans la sciences≫. Burali-Forti was Peano's assistant and friend; the article quoted is [15, (1899)].

One finds both Theorems 6.2 and 6.3 in *Formulario Mathematico* (p. 335). It is worthwhile to note that Peano's use of *Regula* exhibits the normality of gradient with respect to the constraint.

Optimization problems were among principal interests of Peano. His research with regard to these problems was intense, continual and influential. The precision with which Peano studied maxima and minima was notorious.

Hancock, student of Weierstrass, is author of a booklet: *Lectures on the theory of maxima and minima of functions of several variables. Weierstrass' theory* (1903). In the second edition of this book he says [32, (1917) pp. iv-v]:

> In the preface to the German translation by Bohlmann and Schepp of Peano's of *Calcolo differenziale e principii di calcolo integrale*, Professor A. Mayer [editor of Math. Annalen together with Felix Klein] writes that this book of Peano not only is a model of precise presentation and rigorous deduction, whose propitious influence has been unmistakably felt upon almost every calculus that has appeared (in Germany) since that time (1884), but by calling attention to old and deeply rooted errors, it has given an impulse to new and fruitful development.
>
> The important objection contained in this book [*Calcolo differenziale e principii di calcolo integrale*] (Nos. 133-136) showed unquestionably that the entire former theory of maxima and minima needed a thorough renovation; and in the main Peano's book is the original source of the beautiful and to a great degree fundamental works of Scheeffer, Stolz, Victor v. Dantscher, and others, who have developed new and strenuous theories for *extreme* values of functions. Speaking for the Germans, Professor A. Mayer, in the introduction to the above-mentioned book, declares that there has been a long-felt need of a work which, for the first time, not only is free from mistakes and inaccuracies that have been so long in vogue but which, besides, so incisively penetrates an important field that hitherto has been considered quite elementary.

7. Appendix

All articles of Peano are collected in *Opera Omnia* [68], an optical support edited by S. Roero. Selected works of Peano were assembled and commented in *Opere scelte* [65] by Cassina, a pupil of Peano. A few have English translations in *Selected Works* [67]. Regrettably, even fewer Peano's articles have a public URL and are freely downloadable.

One finds the following articles of Peano

in *Opere scelte*, vol. 1:	[48, (1884)], [49, (1884)], [52, (1888)], [54, (1890)], [55, (1890)], [56, (1891)], [57, (1892)], [59, (1894)], [60, (1894)],
in *Opere scelte*, vol. 2:	[53, (1889)],
in *Opere scelte*, vol. 3:	[61, (1896)], [62, (1898)],
in *Selected Works*:	[50, (1887) pp. 152–160, 185–7], [51, (1887) pp. 1–32], [54, (1890)], [59, (1894)], [61, (1896)].

Before the bibliography we attach several pages of *Applicazioni Geometriche* (AG) and of *Formulario Mathematico* (FM) corresponding to Regula, limits of variable sets, derivative and tangent cones.

For reader's convenience, we provide a chronological list of some mathematicians mentioned in the paper, together with biographical sources.

Every `html` file listed below can be attained at University of St Andrews's webpage `http://www-history.mcs.st-and.ac.uk/history/`

JACOBI, Carl (1804-1851), see `Biographies/Jacobi.html`
HAMILTON, William R. (1805-1865), see `Biographies/Hamilton.html`
GRASSMANN, Hermann (1809-1877), see `Biographies/Grassmann.html`
WEIERSTRASS, Karl (1815-1897), see `Biographies/Weierstrass.html`
GENOCCHI, Angelo (1817-1889), see `Biographies/Genocchi.html`
JORDAN, Camille (1838-1922), see `Biographies/Jordan.html`
MAYER, Adolph (1839-1907), see `Biographies/Mayer.html`
DARBOUX, Gaston (1842-1917), see `Darboux.html` at University of St Andrews
STOLZ, Otto (1842-1905), see `Biographies/Stolz.html`
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BURALI-FORTI, Cesare (1861-1931), see `Biographies/Burali-Forti.html`
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HANCOCK, Harris (1867-1944) `http://www.bgsu.edu/departments/math/Ohio-section/bicen/hancock.html`
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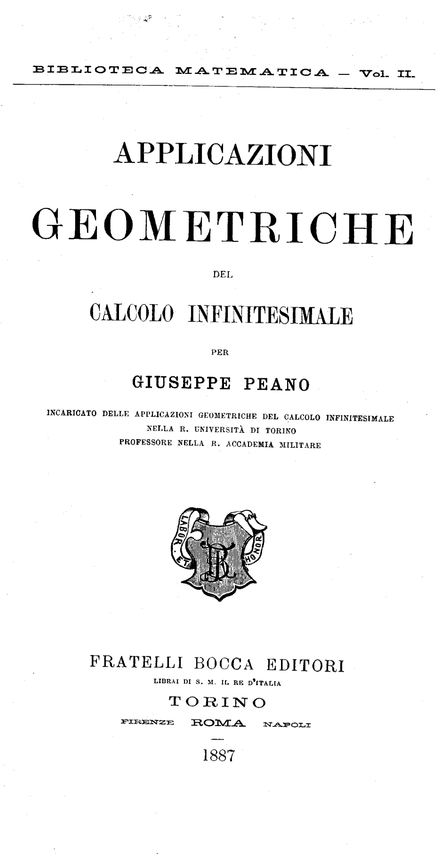
BIBLIOTECA MATEMATICA — Vol. II.

APPLICAZIONI

GEOMETRICHE

DEL

CALCOLO INFINITESIMALE

PER

GIUSEPPE PEANO

INCARICATO DELLE APPLICAZIONI GEOMETRICHE DEL CALCOLO INFINITESIMALE
NELLA R. UNIVERSITÀ DI TORINO
PROFESSORE NELLA R. ACCADEMIA MILITARE

FRATELLI BOCCA EDITORI
LIBRAI DI S. M. IL RE D'ITALIA
TORINO
FIRENZE ROMA NAPOLI

1887

FIGURE 1. Applicazioni geometriche

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2. Diremo che U ha per derivata il segmento **u**, corrispondentemente ad una data posizione di P, e scriveremo $\mathbf{u} \equiv \frac{d\mathrm{U}}{d\mathrm{P}}$ se, attribuendo al punto una nuova posizione P′, la differenza $\Delta\mathrm{U}$ dei due valori assunti da U si può mettere sotto la forma

$$\Delta\mathrm{U} = \overline{\mathrm{PP}'} \times \overline{(\mathbf{u} + \epsilon)}\,,$$

— 132 —

ove $\bar{\epsilon}$ è un segmento che ha per limite zero col tendere di P′ a P. Il prodotto del membro di destra è un prodotto geometrico; quindi se fosse $\epsilon \equiv 0$, ovvero se si trascura ϵ che ha per limite zero, $\Delta\mathrm{U}$ si riduce al solo termine $\overline{\mathrm{PP}'} \times \mathbf{u}$, e quindi l'incremento della funzione U è eguale alla grandezza di **u** moltiplicata per la proiezione di PP′ sulla direzione di **u**.

Eseguendo il prodotto indicato, si ha

$$\Delta\mathrm{U} = \overline{\mathrm{PP}'} \times \mathbf{u} + \alpha\,,$$

ove si è posto $\alpha = \overline{\mathrm{PP}'} \times \bar{\epsilon}$, e quindi, se U ha per derivata **u**, il numero α è infinitesimo d'ordine superiore al primo, ove si prenda la lunghezza di PP′ per infinitesimo principale. Reciprocamente, se $\Delta\mathrm{U}$ si può mettere sotto la forma $\overline{\mathrm{PP}'} \times \mathbf{u} + \alpha$, ove α sia infinitesimo d'ordine superiore al primo, U ha per derivata **u**; invero, se si determina il segmento $\bar{\epsilon}$ che abbia la stessa direzione di PP′ e tale che $\alpha = \overline{\mathrm{PP}'} \times \bar{\epsilon} = gr\,\mathrm{PP}' \times gr\,\epsilon$, si avrà $\Delta\mathrm{U} = \overline{\mathrm{PP}'} \times \overline{(\mathbf{u} + \epsilon)}$, e poichè $gr\epsilon = \frac{\alpha}{gr\mathrm{PP}'}$ ha per limite zero, anche ϵ è infinitesimo, e quindi U ha per derivata **u**.

FIGURE 2. AG (pp. 131–132): derivative

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11. Sia U una funzione numerica della posizione del punto P, avente derivata **u**. Data al punto la nuova posizione P′, sarà

$$\Delta \mathrm{U} = \mathrm{PP}' \times (\mathbf{u} + \epsilon) \qquad (\mathit{lim}\, \epsilon = 0).$$

— 144 —

Si dividano ambo i membri per la grandezza di PP′. Sarà

$$\frac{\Delta \mathrm{U}}{gr\mathrm{PP}'} = \frac{\mathrm{PP}'}{gr\mathrm{PP}'} \times (\mathbf{u} + \epsilon)\,,$$

ed il segmento $\frac{\mathrm{PP}'}{gr\mathrm{PP}'}$ è diretto secondo PP′ ed eguale all'unità di misura. Si faccia ora tendere P′ verso P, in modo che $\frac{\mathrm{PP}'}{gr\mathrm{PP}'}$ tenda ad un limite **p**, il quale sarà pure un segmento eguale all'unità di misura; sarà $\mathit{lim}\frac{\Delta \mathrm{U}}{gr\mathrm{PP}'} = \mathbf{p} \times \mathbf{u}$. Ora, supposto **u** non nullo, se l'angolo $\widehat{\mathbf{p}, \mathbf{u}}$ è acuto, sarà $\mathbf{p} \times \mathbf{u} > 0$, e poichè $gr\,\mathrm{PP}'$ è un numero positivo, sarà anche, per posizioni di P′ sufficientemente prossime a P, $\Delta \mathrm{U} > 0$. Se invece l'angolo $\widehat{\mathbf{p}, \mathbf{u}}$ è ottuso, sarà $\mathbf{p} \times \mathbf{u} < 0$, e, per posizioni di P′ sufficientemente prossime a P, sarà $\Delta \mathrm{U} < 0$. Noi diremo che il punto P si sposta nella direzione e senso del segmento **p**, eguale all'unità di misura, se, essendo P′ una sua nuova posizione, $\mathit{lim}\frac{\mathrm{PP}'}{gr\mathrm{PP}'} = \mathbf{p}$. Quindi, da quanto si disse, si conchiude che:

Se U è funzione numerica del punto P, avente derivata **u** non nulla, se il punto P si sposta nella direzione e senso del segmento **p** che fa un angolo acuto con **u**, U cresce; se invece la direzione e senso in cui si sposta P fa un angolo ottuso con **u**, U diminuisce.

Questa proposizione serve nella ricerca dei massimi e minimi valori che assume U, qualora P varii o liberamente nello spazio, ovvero sia obbligato a condizioni restrittive, come a descrivere una linea od una superficie data. Se U diventa massima per una posizione speciale di P, è necessario che spostando P in ogni direzione e senso compatibile colle condizioni imposte, U diminuisca; e se U è minima per la posizione considerata di P, spostando P in ogni direzione e senso possibile, U deve crescere.

FIGURE 3. AG (pp. 143–144): Regula

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§ 1. Limiti di figure variabili.

1. Intenderemo sempre per *figura,* o *campo di punti,* ogni insieme di punti. Diremo *distanza d'un punto* A *da una figura* F il limite inferiore delle distanze dal punto A da tutti i punti della figura F. Essa può anche essere la minima distanza del punto A dai punti della figura, e questo avverrà certamente se i punti della figura formano un campo chiuso. La distanza del punto A dalla figura F sarà nulla quando A appartiene alla figura data, ovvero al campo limite di essa.

Una figura si può considerare o come fissa o come variabile. Diremo *limite d'una figura variabile* F il luogo dei punti le cui distanze dalla figura F hanno per limite zero.

Già si è definito (Cap. I, 2) il limite d'una retta e d'un piano; è facile lo scorgere che le definizioni allora date coincidono coll'attuale, ove la retta ed il piano si considerino come figure. Si è pure definito il limite d'un cerchio o d'una sfera variabili; si può dimostrare che anche quelle definizioni concordano coll'attuale. Invero

FIGURE 4. AG (p. 302): lower limit of variable sets

3. Come applicazione delle cose precedenti, tratteremo da un nuovo punto di vista le tangenti a curve ed i piani tangenti a superficie.

Sia F una linea od una superficie, e P_0 un punto di essa. Si immagini la figura omotetica della F, con centro di omotetia in P_0 e con rapporto di omotetia r. Col crescere indefinitamente di r, questa figura omotetica tende nei casi più comuni verso un limite. Esso è in generale la tangente alla linea, od il piano tangente alla superficie F; ma può anche essere, in casi speciali, una figura più complicata, cui potremo dare in ogni caso il nome di *figura tangente* alla F nel suo punto P_0.

FIGURE 5. AG (p. 305): lower affine tangent cone

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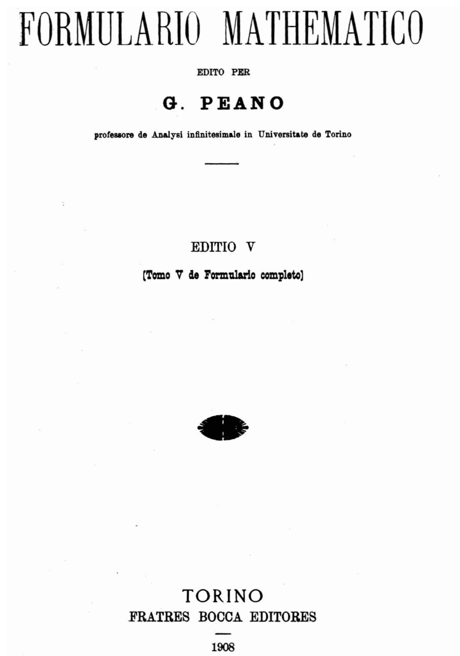

FORMULARIO MATHEMATICO

EDITO PER

G. PEANO

professore de Analysi infinitesimale in Universitate de Torino

EDITIO V

[Tomo V de Formulario completo]

TORINO

FRATRES BOCCA EDITORES

1908

FIGURE 6. Formulario Mathematico

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`http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=02120001&seq=1`

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[35] J.-B. Hirriart-Urruty and C. Lemaréchal: *Convex Analysis and Minimization Algorithms.* Springer-Verlag, Berlin 1996.

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`http://quod.lib.umich.edu/cgi/t/text/text-idx?c=umhistmath&idno=ABR8803`

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[38] C. Jordan: *Cours d'Analyse de l'École Polytechnique*. 2nd edition, Gauthier-Villars, Paris 1893-96.

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`http://www.jstor.org`

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lim vct

✻ 70. (v $\cdot$|Cx) P51
(p |Cx) ——

Nos extende Df de limite ad vectores et ad punctos.

✻ 71. $k\varepsilon$ Cls'(q $\cup$ Cx $\cup$ p $\cup$ v) . $x\varepsilon\, \Delta k$. $f\varepsilon$ (Cls'p)fk .$\supset$.

·1 $\lim(f,k,x) = \mathrm{p} \frown \bar{a}\mathfrak{z}\{\lim[\,\mathrm{d}(a, fz)\,|z, k, x] = 0\,\}$ Df

Si f es classe de punctos, vel figura, functione de numeros reale, vel complexo, vel puncto, vel vectore, pertinente ad aliquo classe k, proximo ad x, tunc limite de figura variabile f, in classe k, pro valore x, es omni puncto a tale que limite de distantia de a ad figura fz, quando z varia, in classe k, et tende ad x, vale zero.

Ex. § rectaT, planO, ...

·2 $a\varepsilon$p . $u\varepsilon$ (v$-\iota$0)fk . $\lim(u,k,x)\ \varepsilon$ v$-\iota$0 .$\supset$.
$\lim[\mathrm{recta}(a,ux)|x, k, x] = \mathrm{recta}[a, \lim(u,k,x)]$

[Hp . $v = \lim(u,k,x)$.$\supset$. $\lim[\mathrm{recta}(a,ux)|x,k,x] =$
$\mathrm{p}\frown \mathfrak{z}\mathfrak{z}\{\lim\{\mathrm{d}[z,\mathrm{recta}(a,ux)]\,|x,k,x\} = 0) =$
$\mathrm{p}\frown \mathfrak{z}\mathfrak{z}\{\lim[\,(\sqrt{\{(z-a)^2-[(z-a)\times \mathrm{U}u]^2\}})/(\mathrm{mod}\,ux)|x,k,x] = 0\} =$
$\mathrm{p}\frown \mathfrak{z}\mathfrak{z}[\,\sqrt{\{(z-a)^2-[(z-a)\times \mathrm{U}v]^2\}}/(\mathrm{mod}\,v) = 0] = \mathrm{p}\frown \mathfrak{z}\mathfrak{z}\{\mathrm{d}[z, \mathrm{recta}(a,v)] = 0$
$= \mathrm{recta}(a,v)$]

Limite de recta passante per punto fixo et parallelo ad vectore variabile, es recta passante per puncto fixo et parallelo ad limite de vectore variabile. Nos suppone que vectore variabile habe valore non nullo, in campo determinato, et que suo limite es determinato et non nullo.

In modo simile, nos determina limite de alio figura; usu de vectores, vel de coordinatas, reduce illos ad limite de numero.

·3 $a\varepsilon$p . $l\varepsilon$ v$-\iota$0 . $u\varepsilon$ (v$-$ql)fk . $\lim(u,k,x)\ \varepsilon$ v$-$ql .$\supset$.
$\lim[\mathrm{plan}(a, l, ux)|x, k, x] = \mathrm{plan}[a, l, \lim(u,k,x)]$

·4 $\mathrm{Lm}(f,k,x) = \mathrm{p}\frown a\mathfrak{z}\{0\varepsilon\ \mathrm{Lm}[\mathrm{d}(a, fz)|z, k, x]\}$ Df

Generalizatione de Df·1. Ex. §Tang.

·5 $a\varepsilon$Cls'p . $x\varepsilon$p .$\supset$. $\lim[\mathrm{d}(y, a)\ |\ y, \mathrm{p}, x] = \mathrm{d}(x, a)$

✻ 72. $k\varepsilon$ Cls'q . $x\varepsilon\, \delta k$. $r\varepsilon$ 1···3 . $u\varepsilon\, \varphi^r$fk .$\supset$.
$\lim(u,k,x) = \iota\,\varphi^r \frown b\mathfrak{z}\{c\varepsilon\, \mathrm{p}^{4-r} .\supset_c. \lim[(uxac)|x,k,x] = bac\}$ Df

Df de limite de forma geometrico, analogo ad Df·§α 9·5.

FIGURE 7. FM (p. 237): lower (n. 71.1) and upper (71.4) limit of sets

[41] K. Kuratowski: Sur les décompositions semi-continues d'espaces métriques compacts. *Fund. Math.* **11**:167-85, 1928.

[42] K. Kuratowski: *Topologie*, vol. I. 4-th edition, Monografie Matematyczne, Warszawa 1948. `http://matwbn.icm.edu.pl/kstresc.php?tom=20&wyd=10&jez=`

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[45] J. Mawhin: *Analyse. Fondaments, techniques, évolution.* De Boeck, Bruxelles 1997.

[46] H. Minkowski. *Geometrie der Zahlen.* Teubner, Leipzig 1896. `http://gallica.bnf.fr/ark:/12148/bpt6k99643x` `http://gallica.bnf.fr/ark:/12148/bpt6k3102f`

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primo variabile. Nos suppone existentia de $\mathrm{D}_2 f(a,y)$, pro aliquo valore a de primo variabile, et pro omni valore de secundo variabile y; tunc seque existentia de $\mathrm{D}_2 f(x,y)$, et de $\mathrm{D}_2\mathrm{D}_1 f(x,y)$ pro omni valore de x et de y.

·5 $h,k \varepsilon \mathrm{q} \;.\supset.\; (h\mathrm{D}_1+k\mathrm{D}_2)f(a,b) = h\mathrm{D}_1 f(a,b)+k\mathrm{D}_2 f(a,b)$ Df

·6 $n \varepsilon \mathrm{N}_1 : r,s \varepsilon \mathrm{N}_0 \,.\, r+s \overline{\leqq} n \;.\supset_{r,s}.\; \mathrm{D}_1{}^r\mathrm{D}_2{}^s f \;\varepsilon\; \mathrm{qF}(u{:}v)\mathrm{cont} \,.\, h \varepsilon u-a$ $.\, k \varepsilon v-b \;.\supset.\; f(a+h,b+k) = \Sigma[(h\mathrm{D}_1+k\mathrm{D}_2)^r f(a,b)/r! \,|r,\, 0\cdots(n-1)] +$ $(h\mathrm{D}_1+k\mathrm{D}_2)^n f(a+zh,b+zk)/n! \,|z\text{‘}\theta$

Si n es numero naturale, et pro omni dyade de numeros r et s, de summa non superiore ad n, semper existe derivata de ordine r pro primo variabile de derivata de ordine s pro secundo variabile, et es continuo, et si nos sume duo quantitate h et k, in modo que $a+h$ es u, et $b+k$ es v, tunc $f(a+h,b+k)$ es evolubile secundo potestates de h et de k, plus termine complementare. Operatione $(h\mathrm{D}_1+k\mathrm{D}_2)^r$ resulta definito per Prop. ·5, et pote es evoluto ut potestate de binomio.

✱ 67. Derivata de functione de numero complexo.

$m,n \varepsilon \mathrm{N}_1 \,.\, u \varepsilon\, \mathrm{Cls}'\mathrm{Cx}n \,.\, f \varepsilon\, \mathrm{Cx}m\, \mathrm{F}\, u \,.\, x \varepsilon\, u \frown \delta u \;.\supset.\; \mathrm{D}fx =$
$\imath\, (\mathrm{Cx}m \;\mathrm{f}\; \mathrm{Cx}n)\mathrm{lin} \frown g \ni [\lim\{[fy-fx-g(y-x)]/\mathrm{mod}(y-x)|y,\, u,x\} = 0]$ Df

Derivata de functione reale de variabile reale (pag. 275), et de numero complexo, vectore, puncto, functione de variabile reale, es, per definitione, limite de ratione de incremento de functione ad incremento de variabile:

$$\mathrm{D}fx = \lim(fy-fx)/(y-x).$$

Si nos habe functione reale aut complexo de variabile complexo, non habe sensu ratione de duo numero complexo, et definitione præcedente non es applicabile. Sed pro numeros reale, derivata satisfac conditione:

$$\lim[fy-fx-(y-x)\times\mathrm{D}fx]/\mathrm{mod}(y-x) = 0,$$

que nos sume per definitione de derivata, si variabile independente es complexo.

Derivata de f, numero complexo de ordine m, functione de complexo de ordine n, in campo u, pro valore x prope u, es illo transformatio lineare g, tale que incremento de functio $fy-fx$ minus g de incremento de variabile $y-x$ diviso per $\mathrm{mod}(y-x)$, tende ad 0, quando y, in u, tende ad x.

Derivata es trasformatione que habe ut elementos derivatas partiale de coordinatas de functione pro coordinatas de variabile.

Derivata de functione de numero complexo es consideratò per:
Jacobi a.1841, opera t.3 p.421. Grassmann a.1862, t. 2, p.295.
Vide in Geometria, derivata de potentiale P71.

Figure 8. FM (p. 330): Derivative (n. 67)

[47] D. Pallaschke and S. Rolewicz: *Foundations of Mathematical Optimization. Convex Analysis without Linearity.* Kluwer, Dordrecht 1997.
[48] G. Peano: [Extrait d'une lettre]. *Nouvelles Annales de Mathématiques*, **3**:45–47, 1884.
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[52] G. Peano: Intégration par séries des équations différentielles linéaires. *Mathematische Annalen*, **32**:450–456, 1888. http://gdzdoc.sub.uni-goettingen.de/sub/digbib/loader?ht=VIEW&did=D29534

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✱ 68. Tang (FIGURA TANGENTE).

$u \varepsilon \mathrm{Cls'p} \,.\, x \varepsilon \mathrm{p} \,.\!\supset.$

·0 $\mathrm{Tang}(u,x) = \mathrm{Lm}\{[x+h(u-x)] \,|h,\, \mathrm{Q},\, \infty\}$
$\qquad = \mathrm{Lm}[\mathrm{Homot}(x,h)u \,|h,\, \mathrm{Q},\, \infty]$ Df

Si u es figura, et x puncto, tunc $\mathrm{Tang}(u,x)$, lege « figura tangente ad u in puncto x », indica limite de figura homothetico de u cum centro de homothetia in x, quando ratione de homothetia h cresce ad infinito.

·1 $x \,\varepsilon\, \mathrm{ex}u \,.\!\supset.\, \mathrm{Tang}(u,x) = \wedge$

Si x es puncto externo ad u, tunc figura tangente ad u in puncto es classe nullo.

·2 $x \varepsilon\, u - \delta u \,.\!\supset.\, \mathrm{Tang}(u,x) = \iota x$

Si x es puncto isolato de u, figura tangente contine solo puncto x.

·3 $x \varepsilon\, \delta u \,.\!\supset.\, \exists\, \mathrm{Tang}(u,x)$

Si x es puncto prope alios u, tunc semper existe punctos de figura tangente ad u in puncto x.

·4 $x \varepsilon\, \mathrm{in}u \,.\!\supset.\, \mathrm{Tang}(u,x) = \mathrm{p}$

Si x es interno ad u, figura tangente es toto spatio.

·5 $y \varepsilon\, \mathrm{Tang}(u,x) - \iota x \,.\!\supset.\, x + \mathrm{Q}(y-x) \supset \mathrm{Tang}(u,x)$

Si in figura tangente ad u, in puncto x, nos sume aliquo puncto y, differente de x, toto radio de origine x, et que i trans y, pertine ad figura tangente.

·6 $v \varepsilon \mathrm{Cls'p} \,.\, v \supset u \,.\!\supset.\, \mathrm{Tang}(v,x) \supset \mathrm{Tang}(u,x)$

Si figura v continere in u, et figura tangente ad v in puncto x continere in figura tangente ad u.

·7 $v \varepsilon \mathrm{Cls'p} \,.\!\supset.\, \mathrm{Tang}(u \cup v, x) = \mathrm{Tang}(u,x) \cup \mathrm{Tang}(v,x)$

Operatione « Tang » es distributivo ad « $\cup$ ».

·8 $\mathrm{Tang}[\mathrm{Tang}(u,x),x] = \mathrm{Tang}(u,x)$

✱ 69. $a,p \,\varepsilon \mathrm{p} \,.\, r \varepsilon \mathrm{Q} \,.\, \mathrm{d}(p,a) = r \,.\!\supset.$

·1 $\mathrm{Tang}\{\mathrm{p} \cap x \ni [\mathrm{d}(x,a) = r],\, p\} = \mathrm{plan}[p,\, \mathrm{I}(p-a)]$

Nos considera loco de punctos que dista ab puncto dato a per quantitate dato r, id es superficie de sphæra de centro a et de radio r. Tunc figura tangente ad superficie dicto, in suo puncto p, es plano per puncto p, et normale ad vectore $p-a$.

FIGURE 9. FM (p. 331): Upper tangent affine cone (n.68)

[53] G. Peano: *I principii di geometria logicamente esposti.* Fratelli Bocca Editori, Torino 1889. `http://quod.lib.umich.edu/cgi/t/text/text-idx?c=umhistmath&idno=ABV4128`

[54] G. Peano: Sulla definizione dell'area di una superficie. *Rend. Acc. Lincei*, **6**:54–57, 1890.

[55] G. Peano. Démonstration de l'intégrabilité des équations différentielles ordinaires. *Mathematische Annalen*, **37**:182–228, 1890. `http://gdzdoc.sub.uni-goettingen.de/sub/digbib/loader?ht=VIEW&did=D27538`

[56] G. Peano: Sulla formula di Taylor. *Atti R. Accad. Scienze Torino*, **27**:40–46, 1891.

[57] G. Peano: Sur la définition de la dérivée. *Mathesis*, **2** :12–14, 1892.

[58] G. Peano: *Lezioni di analisi infintesimale.* 2 vol., Candeletti, Torino 1893.

[59] G. Peano: Sur les systèmes linéaires. *Monatshefte für Mathematik und Physik*, **5**:136, 1894.

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·2 $\mathrm{Tang}\{\mathrm{p} \frown x\ni[\mathrm{d}(x,a) \leqq r], p\} = \mathrm{plan}[p, \mathrm{I}(p-a)] + \mathrm{Q}_0(a-p)$

Figura tangente ad solido sphærico in puncto de superficie es semi-spatio limitato per plano præcedente.

·3 $a,x\,\varepsilon\,\mathrm{p} \,.\, m,n\,\varepsilon\,\mathrm{q} \,.\, m\mathrm{U}(x-a) + n\mathrm{U}(x-b) \neq 0 \,.\supset.$
$\mathrm{Tang}\{\mathrm{p} \frown y\ni[m\mathrm{d}(y,a) + n\mathrm{d}(y,b) = m\mathrm{d}(x,a)+n\mathrm{d}(x,b)]\} =$
$\mathrm{p} \frown z\ni\{(z-x) \times [m\mathrm{U}(x-a) + n\mathrm{U}(x-b)] = 0\}$

Tangente ad ovales de Descartes.
Descartes, *La Geometrie*, Leyde a. 1637, Œuvres t. 6, p. 429.

·4 $u\varepsilon\,\mathrm{Intv} \,.\, p\varepsilon\,(\mathrm{pF}u)\mathrm{cont} \,.\, x\varepsilon\,\mathrm{in}\,u \,.\, px \,-\varepsilon\, p`(u-\iota x) \,.\, \mathrm{D}px\,\varepsilon\,\mathrm{v}-\iota 0$
$.\supset.\ \mathrm{Tang}(p`u, px) = \mathrm{rectaT}(p, x)$

Puncto mobile p es functio definito et continuo in intervallo u, x es interno ad isto intervallo: puncto px non es multiplo de curva $p`u$, et habe derivata non nullo; tunc figura tangente (Tang) ad curva $p`u$ es recta tangente (rectaT), ante considerato.

✻ 70. PLANO TANGENTE AD SUPERFICIE.

·1 $u,v\varepsilon\,\mathrm{Intv} \,.\, p\varepsilon\,\mathrm{pF}(u{:}v) \,.\, x\varepsilon\,\mathrm{in}\,u \,.\, y\varepsilon\,\mathrm{in}\,v \,.\, p(x,y) \,-\varepsilon\, p`(u{:}v)-\iota(x,y).$
$\mathrm{D}_1p, \mathrm{D}_2p\ \varepsilon\ [\mathrm{vF}(u{:}v)]\mathrm{cont} \,.\, \mathrm{D}_1p(x,y) \neq 0 \,.\, \mathrm{D}_2p(x,y) \,-\varepsilon\, \mathrm{q}\,\mathrm{D}_1p(x,y)$
$.\supset.\ \mathrm{Tang}[p`(u{:}v), p(x,y)] = \mathrm{plan}[p(x,y), \mathrm{D}_1p(x,y), \mathrm{D}_2p(x,y)]$

Dem. $\mathrm{Hp} \,.\, u',v'\varepsilon\,\mathrm{Intv} \,.\, u' \supset u \,.\, v' \supset v \,.\, x\varepsilon\,\mathrm{in}\,u' \,.\, y\varepsilon\,\mathrm{in}\,v' \,.\supset:$
$a\varepsilon\,\mathrm{Tang}[p`(u{:}v), p(x,y)] \,.=.\, a\varepsilon\,\mathrm{Tang}[p`(u'{:}v'), p(x,y)]$
$.=.\ \lim \mathrm{dist}\{a, p(x,y) + h[p(u',v')-p(x,y)]|h, \mathrm{Q}, 0\} = 0$
$.=.\ \lim \mathrm{dist}[a, p(x,y) + h(u'-x)\,\mathrm{D}_1p(u',v') + h(v'-y)\mathrm{D}_2p(u',v')] = 0$
$.=.\ \lim \mathrm{dist}\{a, \mathrm{plan}[p(x,y), \mathrm{D}_1p(u',v'), \mathrm{D}_2p(u',v')]\} = 0$
$.=.\ \mathrm{dist}\{a, \mathrm{plan}[p(x,y), \mathrm{D}_1p(x,y), \mathrm{D}_2p(x,y)\} = 0$
$.=.\ a\varepsilon\,\mathrm{plan}[p(x,y), \mathrm{D}_1p(x,y), \mathrm{D}_2p(x,y)]$

Es dato duo intervallo u et v, et puncto mobile p, functione de duo variabile, in dicto intervallos. Tunc puncto p describe « superficie », et duo variabile es dicto « coordinatas curvilineo de puncto in superficie ».

Nos sume valore x interno ad u, et y interno ad v, et suppone que $p(x,y)$ es puncto simplice de superficie descripto per puncto p; tunc si puncto habe derivatas pro ambo variabile, continuo, non nullo et non parallelo, figura tangente ad superficie loco de punctos p, in puncto considerato, es plano per puncto et que contine derivatas de puncto pro ambo variabile. Ce plano vocare « plano tangente ».

FIGURE 10. FM (p. 332): Calculus of tangent lines and planes

[60] G. Peano: Sur la définition de la limite d'une fonction. Exercice de la logique mathématique. *Amer. J. Math.*, **17**:37–68, 1894.
[61] G. Peano: Saggio di calcolo geometrico. *Atti R. Accad. Scienze Torino*, **31**:952–975, 1895-96.
[62] G. Peano. Analisi della teoria dei vettori. *Atti R. Acc. Scienze Torino*, **33**:513–534, 1897-98.
[63] G. Peano: *Formulaire Mathématique (tome IV)*. Fratelli Bocca, Torino 1903.
[64] G. Peano: *Formulario Mathematico (Editio V)*. Fratelli Bocca, Torino 1908.
[65] G. Peano: *Opere scelte*. Edizioni Cremonese, Roma 1957-9.
[66] G. Peano: *Formulario Mathematico*. Edizioni Cremonese, Roma 1960 (reprint of [64])
[67] G. Peano: *Selected works* (H.C. Kennedy). University Toronto Press, 1973
[68] G. Peano: *Opera Omnia*. 3 CD (S. Boero, ed.). Dipartimento di Matematica, Università, Torino.
[69] J. Pierpont: *The theory of functions of real variables*, vol. I. Ginn and Co., Boston 1905.

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Si, in superficie es descripto curva que habe tangente in suo puncto p, tunc tangente ad curva jace in plano tangente ad superficie. Seque de Prop. 68·6.

Plure Auctore sume ce proprietate ut definitione. « Plano tangente ad superficie in suo puncto p » ès definito ut « plano que contine recta tangente in p ad omni curva, descripto in superficie, et que i trans p ».

Tunc, si per puncto p, in superficie dato, nos duce linea sine tangente (ut spira mirabile in suo polo, loxodromia in suos polo, etc.), plano tangente contine tangente ad linea, que non habe tangente; quod es contradictorio.

Aliquo Auctore corrige definitione præcedente, et voca plano tangente « plano que contine tangente ad dicto curvas, que habe tangente ». Tunc omni plano es tangente ad superficie, que contine nullo linea cum tangente. Es tale superficie genito per rotatione de curva $y = x \sin 1/x$, circa oy, in puncto o.

Vide alio Df de plano tangente in Formul. t.4 p.295.

·2 $k \varepsilon \mathrm{Intv} \,.\, a \varepsilon \mathrm{pF}k \,.\, u \varepsilon (\mathrm{v} - \iota 0)\mathrm{F}k \,.\, x \varepsilon k \,.\, y \varepsilon \mathrm{q} \,.\, \mathrm{D}ax, \mathrm{D}ux \,\varepsilon \mathrm{v} \,.$
$\mathrm{D}ax + y\mathrm{D}ux - \varepsilon \,\mathrm{q}\, ux \;.\supset.\; \mathrm{Tang}[\bigcup \mathrm{recta}(ax,ux) \,|x‘k,\, ax + yux]$
$= \mathrm{plan}(ax,\, ux,\, \mathrm{D}ax + y\mathrm{D}ux)$

Dem. $\mathrm{plan}[\bigcup \mathrm{recta}(ax,ux)|x‘k,\, au + yux]$
$= \mathrm{plan}[(ax + yux)|(x;y)‘(k{:}\mathrm{q}),\, ax + yux]$
$= \mathrm{plan}\{ax + yux,\, \mathrm{D}_1[(ax + yux)|(x,y)](x,y),\, \mathrm{D}_2[(ax + yux)|(x,y)](x,y)\}$
$= \mathrm{plan}(ax + yux,\, \mathrm{D}ax + y\mathrm{D}ux,\, ux)$
$= \mathrm{plan}(ax, ux,\, \mathrm{D}ax + y\mathrm{D}ux)$

Nos considera puncto a et vectore non nullo u, ambo functione dato in intervallo k.

Es dato x in intervallo k, et quantitate reale y. Nos suppone existentia de derivatas de a et de u, et que vectore $\mathrm{D}ax + y\mathrm{D}ux$ non es parallelo ad ux. Superficie loco de rectas per ax et parallelo ad ux, ubi x varia in intervallo k, es expresso per $\bigcup \mathrm{recta}(ax,ux)|x‘k$.

Theorema dice que figura tangente ad ce superficie in suo puncto $ax + yux$ es plano per ax, et parallelo ad vectores ux et $\mathrm{D}ax + y\mathrm{D}ux$.

Superficie loco de recta mobile $=$ F. surface reglée $=$ D. Regelfläche $=$ I. superficie rigata.
F. règle $=\|$ D. Regel $\subset$ L. regula.
I. riga $\subset$ Germanico : riga $\supset$ D. Reihe, A. row.
Ce superficie habe in Germania nomen Latino, et in Italia nomen Germanico.

FIGURE 11. FM: Calculus of a tangent plane

[70] H Poincaré: L'avenir des mathématiques. *Atti del IV Congresso Internazionale dei Matematici* (Roma, 6-11 Aprile 1908), **I**:167-182, Acc. Lincei, Roma 1909.
[71] H. Reitberger: VLeopold Vietoris (1891-2002).*Notices AMS* **49**:1232–6, 2002 `http://www.ams.org/notices/200210/fea-vietoris.pdf`
[72] R. T. Rockafellar and R. Wets: *Variational Analysis.* Springer-Verlag, 1997
[73] S. Saks: *Theory of the Integral.* Hafner, New York 1937. `http://matwbn.icm.edu.pl/kstresc.php?tom=7&wyd=10&jez=`
[74] O. Stolz: *Grundzüge der Differenzial- und Integralrechnung.* Teubner, Leipzig 1893.
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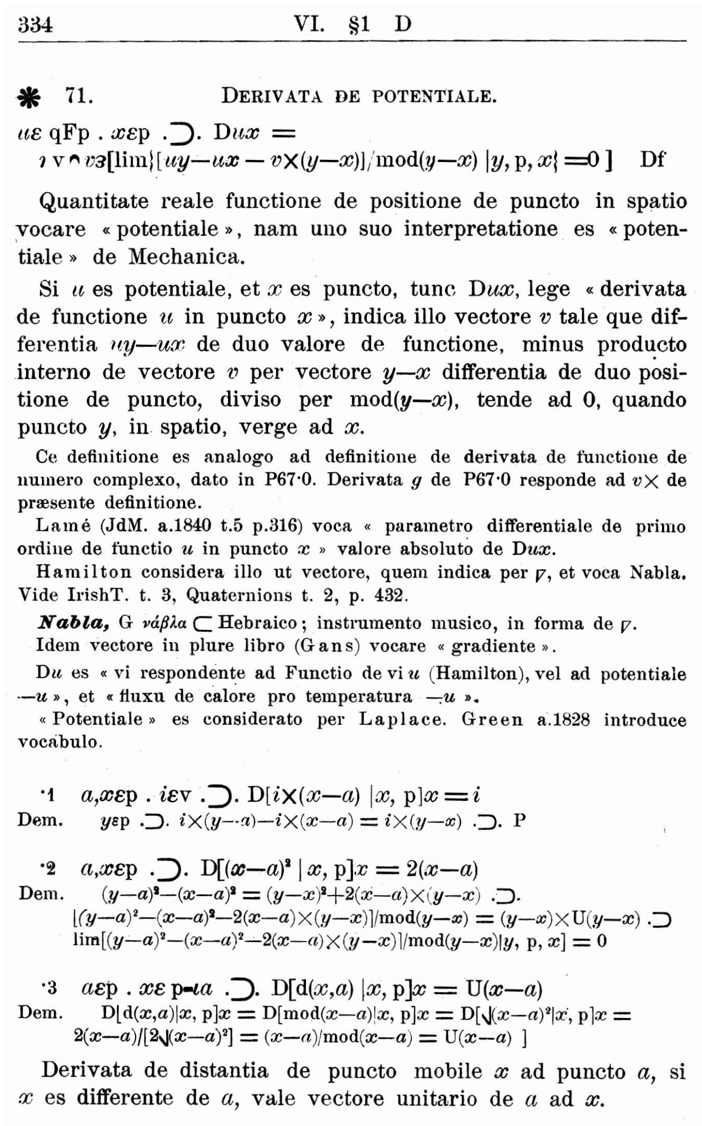

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✱ 71. DERIVATA DE POTENTIALE.

$u \varepsilon \mathrm{qFp} \,.\, x \varepsilon \mathrm{p} \,.\supset.\, \mathrm{D}ux =$
$\quad \imath\, \mathrm{v} \frown v \ni [\lim\{[uy - ux - v \times (y-x)]/\mathrm{mod}(y-x) \,|y, \mathrm{p}, x\} = 0\,]$ Df

Quantitate reale functione de positione de puncto in spatio vocare « potentiale », nam uno suo interpretatione es « potentiale » de Mechanica.

Si u es potentiale, et x es puncto, tunc $\mathrm{D}ux$, lege « derivata de functione u in puncto x », indica illo vectore v tale que differentia $uy-ux$ de duo valore de functione, minus producto interno de vectore v per vectore $y-x$ differentia de duo positione de puncto, diviso per $\mathrm{mod}(y-x)$, tende ad 0, quando puncto y, in spatio, verge ad x.

Ce definitione es analogo ad definitione de derivata de functione de numero complexo, dato in P67·0. Derivata g de P67·0 responde ad $v\times$ de præsente definitione.

Lamé (JdM. a.1840 t.5 p.316) voca « parametro differentiale de primo ordine de functio u in puncto x » valore absoluto de $\mathrm{D}ux$.

Hamilton considera illo ut vectore, quem indica per ∇, et voca Nabla. Vide IrishT. t. 3, Quaternions t. 2, p. 432.

Nabla, G νάβλα $\subset$ Hebraico; instrumento musico, in forma de ∇.

Idem vectore in plure libro (Gans) vocare « gradiente ».

Du es « vi respondente ad Functio de vi u (Hamilton), vel ad potentiale $-u$ », et « fluxu de calore pro temperatura $-u$ ».

« Potentiale » es considerato per Laplace. Green a.1828 introduce vocabulo.

·1 $a, x \varepsilon \mathrm{p} \,.\, i \varepsilon \mathrm{v} \,.\supset.\, \mathrm{D}[i \times (x-a) \,|x, \mathrm{p}]x = i$

Dem. $y \varepsilon \mathrm{p} \,.\supset.\, i\times(y-a) - i\times(x-a) = i\times(y-x) \,.\supset.\,$ P

·2 $a, x \varepsilon \mathrm{p} \,.\supset.\, \mathrm{D}[(x-a)^2 \,|x, \mathrm{p}]x = 2(x-a)$

Dem. $(y-a)^2 - (x-a)^2 = (y-x)^2 + 2(x-a)\times(y-x) \,.\supset.$
$[(y-a)^2 - (x-a)^2 - 2(x-a)\times(y-x)]/\mathrm{mod}(y-x) = (y-x)\times \mathrm{U}(y-x) \,.\supset$
$\lim[(y-a)^2 - (x-a)^2 - 2(x-a)\times(y-x)]/\mathrm{mod}(y-x)|y, \mathrm{p}, x] = 0$

·3 $a \varepsilon \mathrm{p} \,.\, x \varepsilon \mathrm{p} - \iota a \,.\supset.\, \mathrm{D}[\mathrm{d}(x,a) \,|x, \mathrm{p}]x = \mathrm{U}(x-a)$

Dem. $\mathrm{D}[\mathrm{d}(x,a)|x, \mathrm{p}]x = \mathrm{D}[\mathrm{mod}(x-a)|x, \mathrm{p}]x = \mathrm{D}[\sqrt{(x-a)^2}|x, \mathrm{p}]x =$
$2(x-a)/[2\sqrt{(x-a)^2}] = (x-a)/\mathrm{mod}(x-a) = \mathrm{U}(x-a)$]

Derivata de distantia de puncto mobile x ad puncto a, si x es differente de a, vale vectore unitario de a ad x.

FIGURE 12. FM (p. 334): Derivative and potential

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·4 $a\varepsilon\,\mathrm{p}^2\cup\mathrm{p}^3 \,.\, x\varepsilon\,\mathrm{p}-a \;.\supset.\; \mathrm{D}[\mathrm{d}(x,a)|x,\mathrm{p}]x = \mathrm{U}[x-(\mathrm{proj}\ a)x]$

Si a indica recta vel plano, et x es puncto ex a, tunc derivata de distantia de x ad figura a es vectore unitario secundo projectione super a de x ad x.

·5 $k\varepsilon\,\mathrm{Cls'p} \,.\, k=\lambda k \,.\, x\varepsilon\,\mathrm{p}-k \,.\, a\varepsilon k \,.\, \mathrm{d}(x,a)=\mathrm{d}(x,k) : y\varepsilon\, k-\iota a \;.\supset_y.$
$\mathrm{d}(x,a) < \mathrm{d}(x,y) \;:\supset.\; \mathrm{D}[\mathrm{d}(x,k)|x,\mathrm{p}]x = \mathrm{U}(x-a)$

Derivata de distantia de puncto x ab figura k es vectore unitario in directione de distantia, in hypothesi scripto.

·6 $a\varepsilon\mathrm{p} \,.\, i\varepsilon\,\mathrm{v}-\iota 0 \,.\, x\,\varepsilon\,\mathrm{p}-(a+\mathrm{q}i) \;.\supset.\; \mathrm{D}[\mathrm{ang}(x-a,i)\,|x,\mathrm{p}]x =$
$\mathrm{U}\{[\mathrm{cmp}\perp(x-a)]i\}/\mathrm{d}(x,a)$

✱ 72.

·1 $u\varepsilon\,\mathrm{qFp} \,.\, x\varepsilon\mathrm{p} \,.\, ux = \max u\text{'}\mathrm{p} \,.\, \mathrm{D}ux\,\varepsilon\mathrm{v} \;.\supset.\; \mathrm{D}ux = 0$

·11 (min | max) P·1

Si potentiale u es maximo pro aliquo puncto x, pro que existe derivata de u, tunc ce vectore derivata vale 0.

Idem pro minimo.

·2 $u\varepsilon\,\mathrm{qFp} \,.\, k\varepsilon\,\mathrm{Cls'p} \,.\, x\varepsilon k \,.\, ux = \min u\text{'}k \,.\, \mathrm{D}ux\,\varepsilon\mathrm{v} \,.$
$y\varepsilon\,\mathrm{Tang}(k,x) \;.\supset.\; (\mathrm{D}ux)\times(y-x) \geqq 0$

Si nos considera solo punctos de aliquo figura k, et si potentiale sume pro puncto x, valore minimo inter valores respondente ad punctos de figura k, et derivata de u in x es vectore determinato, tunc producto de ce vectore per omni variatione $y-x$ de puncto in figura tangente ad k in x, es positivo aut nullo.

·3 $u\varepsilon\,\mathrm{qFp} \,.\, k\varepsilon\,\mathrm{Cls'p} \,.\, x\varepsilon k \,.\, ux = \max u\text{'}k \,.\, \mathrm{D}ux\,\varepsilon\mathrm{v} \,.$
$y\varepsilon\,\mathrm{Tang}(k,x) \;.\supset.\; (\mathrm{D}ux)\times(y-x) \leqq 0$

Regula simile subsiste pro maximo.

·4 $u\varepsilon\,\mathrm{qFp} \,.\, x\varepsilon\mathrm{p} \,.\, \mathrm{D}ux\,\varepsilon\,\mathrm{v}-\iota 0 \;.\supset.$
$\mathrm{Tang}[\mathrm{p}\cap y\ni(uy=ux), x] = \mathrm{plan}(x, \mathrm{ID}ux)$

Figura loco de punctos y, que satisfac conditione $uy = ux$, vocare «superficie æquipotentiale» que i trans x.

FIGURE 13. FM (p. 335): Regula n. 72.2 (min) and n. 72.3(max)

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Si nos considera solo punctos in plano dato, in loco de superficie, occurre « linea æquipotentiale ».

Si derivata de potentiale u, in puncto x, es vectore non nullo, tunc figura tangente ad superficie æquipotentiale que transi per x, es plano per x, et normale ad vectore Dux. Id es, Dux es vectore « normale » ad superficie æquipotentiale.

Applicationes.

1. Normale ad loco de punctos x que redde constante $\mathrm{d}(x,a)+\mathrm{d}(x,b)$, ubi a et b es puncto dato (ellipsi de foco a et b), es directo secundo $\mathrm{U}(x-a)+\mathrm{U}(x-b)$, vel secundo bisectrice de radios focale. (Apollonio).

2. Si es constante summa de distantias de x ad plure puncto fixo $a_1,\dots$ normale es directo secundo vectore $\mathrm{U}(x-a_1)+\dots$
(Leibniz, Math.S a.1693 t.6 p.233).

3. Si es constante functione $f(r_1, r_2)$, ubi $r_1 = \mathrm{d}(x, a_1)$, $r_2 = \mathrm{d}(x,a_2)$, vectore $\mathrm{D}_1 f(r_1,r_2)\mathrm{U}(x-a_1) + \mathrm{D}_2 f(r_1,r_2)\mathrm{U}(x-a_2)$ es normale ad loco.
(Poinsot a.1806, p.206).

4. Puncto que redde minimo summa de distantias ab plure puncto dato es in æquilibrio sub actione de fortias æquale inter se, et directo ad punctos dato. (Steiner t. 2, p.95).

Vide demonstratione et alio applicationes de propositiones præcedente in meo libro a. 1887, p.131-151.

Vide etiam Hurwitz MA. t. 22, p.231, Wetzig JfM. t. 62, p. 346, Baker AmerJ. t. 4 p.327, Sturm JfM. t. 96 p.36, t. 97 p.49.

✻ 73. RELATIONE INTER POTENTIALE ET ENERGIA.

$u\varepsilon\,\mathrm{qFp}\,.\,p\varepsilon\,\mathrm{pFq}\,.\,\mathrm{D}^2 p = -\mathrm{D}up\ .\supset.\ \{[(\mathrm{D}pt)^2/2+upt]\ |t, \mathrm{q}\}\ \varepsilon\ \mathrm{const}$

Dem. $\mathrm{D}[(\mathrm{D}p)^2/2+up] = \mathrm{D}p\times\mathrm{D}^2p+\mathrm{D}up\times\mathrm{D}p = \mathrm{D}p\times(\mathrm{D}^2p+\mathrm{D}up) = 0$

Si u es quantitate functione de positione de puncto, vel potentiale, et si p es puncto mobile, vel puncto materiale cum massa 1, et si acceleratione de puncto vale vi respondente ad potentiale u, tunc summa de energia cum potentiale, dum varia tempore, es constante.

·1 $u\varepsilon\,\mathrm{qFp}\,.\,p\varepsilon\,\mathrm{pFq}\,.\,(\mathrm{D}^2p + \mathrm{D}up)\times\mathrm{D}p = (\iota 0\mathrm{:q})\ .\supset.$ Ths

Idem fi, si puncto p move se, in modo que suo velocitate es semper normale ad $\mathrm{D}^2p + \mathrm{D}up$, id es si vi D^2p que move illo es summa de $-\mathrm{D}u$, vi de potentiale u, plus vi normale ad trajectoria de puncto. Ce casu se præsenta, si puncto es mobile super linea dato, aut super superficie dato, sine attrito.

FIGURE 14. FM (p. 336): Applications

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