MULTISEQUENCES

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ABSTRACT. This paper is essentially an intoduction to, and a survey of, multisequences, maps from the sets of maximal elements of some special trees. However it also contains some new concepts and results. Multisequences find application in numerous problems related to sequentiality, like characterizations of sequential and subsequential spaces, preservation of sequentiality and of Fréchetness by various operations, estimates of sequential order of products.

CONTENTS

1. INTRODUCTION

This paper has been conceived as a survey, which would however contain some new concepts and results. Interaction with its first readers convinced me to make of it also an introduction, which would make it easier for those less accustomed with the language and folklore of set theory and general topology.

Multisequences introduced in [13] are designed to characterize iterated sequential adherences. These are maps from the set of maximal elements of some trees. They were inspired by a construction in [18] by Fremlin, but

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were implicitly used before, for example, by Franklin and Rajagopalan in [17]. Kratochvíl used the term *multisequence* in [21][22] to denote a different concept, which I call here *hypersequence*. Multisequences turned out to be instrumental in numerous problems related to sequentiality, like characterizations of sequential and subsequential spaces, preservation of sequentiality and of Fréchetness by various operations, estimates of sequential order of products [12][10][11][15][9].

To be more specific, let τ be a topology on X, and A a subset of X. Then $\mathrm{adh}_{\mathrm{Seq}\,\tau} A$ denotes the sequential adherence of A, that is, the union of the limits of all the sequences with terms in A. Ordinal iterations of the sequential adherence are defined by (transfinite) induction: $\mathrm{adh}^0_{\mathrm{Seq}\,\tau} A = A$ and for every ordinal $\beta > 0$,

$$\mathrm{adh}^{<\beta}_{\mathrm{Seq}\,\tau} A = \bigcup_{\alpha<\beta} \mathrm{adh}^{\alpha}_{\mathrm{Seq}\,\tau} A,$$

and

$$\mathrm{adh}^{\beta}_{\mathrm{Seq}\,\tau} A = \mathrm{adh}_{\mathrm{Seq}\,\tau} \left(\mathrm{adh}^{<\beta}_{\mathrm{Seq}\,\tau} A\right).$$

In particular, in case of an isolated ordinal $\beta > 0$, we have $\mathrm{adh}^{<\beta}_{\mathrm{Seq}\,\tau} A = \mathrm{adh}^{\beta-1}_{\mathrm{Seq}\,\tau} A$.[1]

On rephrasing the definition of sequential adherence, $x \in \mathrm{adh}_{\mathrm{Seq}} A$ if and only if there is a sequence on A, which converges to x. As we shall see, $x \in \mathrm{adh}^{\beta}_{\mathrm{Seq}} A$ if and only if there exists on A a multisequence, which converges to x. A topology is sequential whenever there exists β such that the topological closure cl_τ is equal to $\mathrm{adh}^{\beta}_{\mathrm{Seq}\,\tau}$. Therefore τ is sequential if and only if $x \in \mathrm{cl}_\tau A$ implies the existence of a multisequence on A, which converges to x in τ. This fundamental fact about multisequences leads to numerous applications of the concept.[2]

If $\beta > 0$ then, by definition,

$$x \in \mathrm{adh}^{\beta}_{\mathrm{Seq}\,\tau} A$$

if there exists a sequence on $\mathrm{adh}^{<\beta}_{\mathrm{Seq}\,\tau} A$ that converges to x in τ. It follows that there exist a *non-decreasing* sequence $(\alpha_n)_n$ of ordinals *less than* β, and a sequence $(x_n)_n$ such that $x \in \lim_\tau (x_n)_n$ and $x_n \in \mathrm{adh}^{\alpha_n}_{\mathrm{Seq}\,\tau} A$. If $\beta = 0$, then $x \in A$.

This is a recursive procedure, because each resulting x_n is again an element of an iterated sequential adherence. Therefore, by definition, either

[1] Some authors (for example, [2]) define $\mathrm{adh}^{\beta}_{\mathrm{Seq}\,\tau} A$ in the case of limit ordinal $\beta > 0$ as $\mathrm{adh}^{<\beta}_{\mathrm{Seq}\,\tau} A$. The convention of this paper has the advantage that limit and non limit cases can be treated simultaneously.

[2] For a general convergence space ξ, the *adherence* $\mathrm{adh}_\xi A$ of A is the union of the limits of all the filters, which contain A. Accordingly, the sequential adherence for a topology τ is equal to the adherence with respect to the *sequential modification* $\mathrm{Seq}\,\tau$ of τ, that is, the following convergence: $x \in \lim_{\mathrm{Seq}\,\tau} \mathcal{F}$ if there is a sequence, which converges to x in τ, and is coarser than the filter $\mathcal{F}$.

$\alpha_n = 0$ and thus $x_n \in A$, or there exist a non-decreasing sequence of ordinals $(\alpha_{n,k})_k$ that are less than α_n, and a sequence $(x_{n,k})_k$ such that $x_{n,k} \in \mathrm{adh}^{\alpha_{n,k}}_{\mathrm{Seq}\,\tau} A$ for every $k < \omega$, and $x_n \in \lim_\tau (x_{n,k})_k$.

As each strictly decreasing sequence of ordinals is finite, on continuing this procedure, we come up with finite sequences of finite sequences $(n_1, n_2, \ldots, n_k)$ of integers such that $x_{n_1,n_2,\ldots,n_k} \in \mathrm{adh}^{\alpha_{n_1,n_2,\ldots,n_k}}_{\mathrm{Seq}\,\tau} A$ and so that

$$\beta > \alpha_{n_1} > \alpha_{n_1,n_2} > \ldots > \alpha_{n_1,n_2,\ldots,n_q} = 0$$

for some natural number q.

If now

$$x \in \mathrm{adh}^{\beta}_{\mathrm{Seq}\,\tau} A \setminus \mathrm{adh}^{<\beta}_{\mathrm{Seq}\,\tau} A, \tag{1.1}$$

and a sequence $(x_n)_n$ on $\mathrm{adh}^{<\beta}_{\mathrm{Seq}\,\tau} A$ converges to x in τ, then there exists a sequence $(\alpha_n)_n$ such that $x_n \in \mathrm{adh}^{\alpha_n}_{\mathrm{Seq}\,\tau} A$ and[3]

$$\sup\nolimits_{n<\omega} (\alpha_n + 1)_n = \beta. \tag{1.2}$$

These considerations about iterated sequential adherences lead to the notions of sequential cascade and of multisequence. Multisequences constitute a special case of multifilters [8].

Throughout this paper, I will be often using in definitions and proofs *well-founded induction* (e.g., [19], [29, page 239]). A partially ordered set is *well-founded* if its every non-empty subset admits minimal elements. Well-founded sets generalize well-ordered sets. If X is a non-empty well-founded set, then in particular the set $\mathrm{Min}\, X$ (of minimal elements of X) is non-empty. Therefore we can partition a well-founded set by defining inductively

$$X_0 = \mathrm{Min}\, X,$$

$$X_\beta = \mathrm{Min}(X \setminus \bigcup\nolimits_{\alpha<\beta} X_\alpha) \text{ for } \beta > 0.$$

By the Zermelo theorem on well-ordering, there is the least ordinal γ such that $X_\gamma = \varnothing$, and because X is well-founded, $X = \bigcup_{\alpha<\gamma} X_\alpha$. This defines an ordinal function $h_X : X \to \mathrm{Ord}$, called the *level* of X, by

$$h_X(x) = \beta \Longleftrightarrow x \in X_\beta.$$

The level can be characterized by $h_X(x) = 0$ if and only if $x \in X_0 = \mathrm{Min}\, X$ and otherwise

$$h_X(x) = \sup\{h_X(y) + 1 : y < x\}. \tag{1.3}$$

Let $\Phi : X \to \{0, 1\}$ be a property of elements of X (in other words, x has the property Φ whenever $\Phi(x) = 1$).

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[3]Indeed, the limit in (1.2) is not greater than β, because $\alpha_n < \beta$ for each n. On the other hand, if $\lim_{n<\omega}(\alpha_n + 1) = \gamma < \beta$, then $\{x_n : n < \omega\} \subset \bigcup_{\alpha<\gamma} \mathrm{adh}^{\alpha}_{\mathrm{Seq}\,\tau} A$, hence

$$x \in \mathrm{adh}_{\mathrm{Seq}\,\tau}\{x_n : n < \omega\} \subset \mathrm{adh}_{\mathrm{Seq}\,\tau}(\bigcup_{\alpha<\gamma} \mathrm{adh}^{\alpha}_{\mathrm{Seq}\,\tau} A) = \mathrm{adh}^{\gamma}_{\mathrm{Seq}\,\tau} A,$$

which is contrary to the assumption.

Theorem 1.1 (well-founded induction)**.** [19, Theorem 25'] *If X is well-founded and Φ is a property such that $\Phi(\operatorname{Min} X) = \{1\}$ and for every $x \in X$*

$$\forall_{w<x}\ (\Phi(w) = 1) \Longrightarrow \Phi(x) = 1,$$

then $\Phi(x) = 1$ for every $x \in X$.

The notation of this paper is standard, with the following exceptions: I use f^-, Ω^- rather than f^{-1}, Ω^{-1} to denote the *preimage* of a map f and of a relation Ω; the *set of finite sequences of natural numbers* is denoted by Σ rather than $\omega^{<\omega}$ or $\bigcup_{n\in\omega} \omega^n$; the *empty sequence*, that is, the sequence of length 0 is denoted by o rather than $\varnothing$.

As I write $(n_1, n_2, \dots, n_k)$ to denote a finite sequence (of natural numbers) of length k, I distinguish between a finite sequence (n_k) of length 1 (which consists of an element n_k) and an infinite sequence $(n_k)_k$, the k-th term of which is n_k.

2. Sequential cascades

If A is a subset of a partially ordered set, then $\operatorname{Max} A$ (respectively $\operatorname{Min} A$) denotes the set of maximal (respectively minimal) elements of A, and

$$\operatorname{Ext} A = \operatorname{Min} A \cup \operatorname{Max} A.$$

Recall that a partially ordered set $(T, \leq)$ is a *tree* if for each $t \in T$ the set

$$T^{\downarrow}(t) = \{s \in T : s \leq t\}$$

is well-ordered. A *branch* of a tree T is a maximal chain of T.[4] The *level* $h_T(t)$ of an element t of a tree T is the order type of $\{s \in T : s < t\}$. Of course, $<$ will denote the *strict partial order* associated with $\leq$.[5]

A tree $(T, \leq)$ is called a *sequential cascade* if

$$\operatorname{Min} T \text{ is a singleton,} \tag{2.1}$$

$$\operatorname{Max} A \neq \varnothing \text{ for every non-empty } A \subset T \tag{2.2}$$

$$T^+(t) = \operatorname{Min}\{s \in T : t < s\} \text{ is countably infinite for each } t \notin \operatorname{Max} T. \tag{2.3}$$

A sequential cascade is well-founded, because it is a tree. Therefore it admits the level function, which is equal to 0 at the least element, and fulfills (1.3).

Condition (2.2) stated in other terms is: T is *well-founded* for the inverse order. A tree has no infinite branches if and only if it is well-founded in the inverse order.[6] Hence

Every branch of a sequential cascade is finite.

[4]Every tree T is *well-founded*, that is, if $\varnothing \neq A \subset T$ then $\operatorname{Min} A \neq \varnothing$. In fact, if $t \in A$, then $\{a \in A : a \leq t\}$ has a least element $a(t)$ and thus $a(t) \in \operatorname{Min} A$.

[5]$s < t$ whenever $s \leq t$ and $s \neq t$.

[6]Each branch is finite, because it is well-ordered in both the induced order and its inverse. Conversely, if B were a non-empty subset of a tree such that $\operatorname{Max} B = \varnothing$, then there would exist a strictly increasing infinite sequence (t_n) of elements of B, in contradiction with the fact that branches must be finite.

The level for the inverse order of a sequential cascade will be called the *rank* and will be denoted by r_T. Of course, $r_T(t) = 0$ if and only if $t \in \operatorname{Max} T$, and by (1.3),

$$r_T(t) = \sup\{r_T(s) + 1 : t < s\};$$

for every $t \in T \setminus \operatorname{Max} T$. Therefore, for each $t \in T \setminus \operatorname{Max} T$,

$$r_T(t) = \sup\{r_T(s) + 1 : s \in T^+(t)\}. \tag{2.4}$$

The *rank* of T (denoted by $r(T)$) is the rank of o_T.

Example 2.1. *If a sequential cascade I reduces to a singleton, then it is of rank* 0*, and* $\operatorname{Min} I = \operatorname{Max} I$.

Example 2.2. *If A is a countably infinite set and $o \notin A$ then $T = \{o\} \cup A$ ordered by $o < a$ whenever $a \in A$, is a sequential cascade of rank* 1*. Of course,* $\operatorname{Max} T = A$ *and* $\operatorname{Min} T = \{o\}$.

If the rank of a cascade T is finite, then $r(T) = \max\{h_T(t) : t \in T\}$.[7]

Proposition 2.3. *Every sequential cascade is of countable rank.*

Proof. All the maximal elements of a cascade T are of rank 0. We suppose that for every $s > t$ the rank $r_T(s)$ is countable. Then $r_T(t)$ is countable as the supremum of countably many countable ordinals. By the well-founded induction, $r_T(t)$ is countable for every $t \in T$. ∎

Let T be a sequential cascade, and S_t a sequential cascade such that $\operatorname{Min} S_t = \{t\}$ for every $t \in \operatorname{Max} T$. Assume that $T \cap S_t = \{t\}$ and $S_{t_0} \cap S_{t_1} = \varnothing$ for each t, t_0 and t_1. Then

$$T \cup \bigcup_{t \in \operatorname{Max} T} S_t \tag{2.5}$$

with the partial order extended by $r \leq s$ for $r \in T$ and $s \in S_t$ whenever $r \leq t$ is called a *confluence* of $\{S_t : t \in \operatorname{Max} T\}$ to T, and is denoted by $T \looparrowleft_t S_t$.

Example 2.4. *If I is a sequential cascade of rank* 0*, then there is only one element of* $\operatorname{Max} I$*, and the confluence $I \looparrowleft S$ (of a single cascade S to I) is equal S. On the other hand, if* $\{I_t : t \in \operatorname{Max} T\}$ *is a disjoint family of copies of I, then $T \looparrowleft_t I_t = T$.*

Accordingly, we can consider a confluence of $\{S_t : t \in D\}$ to T, where $D \subset \operatorname{Max} T$, on extending $\{S_t : t \in D\}$ to $\{S_t : t \in \operatorname{Max} T\}$ by declaring S_t a disjoint copy of the cascade I of rank 0 for $t \in \operatorname{Max} T \setminus D$.

[7]Indeed, $r(T) = 0$ whenever $T = \{o\}$, that is, whenever $h_T(o) = 0$. Suppose that the claim holds for the cascades of rank $n-1$, and let T be of rank n. This means that $n = \sup\{r_T(s) : s \in T^+(o_T)\}$, that is, there is $s_0 \in T^+(o_T)$ such that $r_T(s_0) = n-1$ and $r_T(s) \leq n$ for every $s \in T^+(s_0)$. By the inductive assumption, the level of s_0 is equal to $n-1$, and is not less than n for every $s \in T^+(s_0)$.

Example 2.5. *If T is the sequential cascade of rank 1 (Example 2.2) and $\{T_t : t \in \operatorname{Max} T\}$ is a family of disjoint copies of T, then $T \looparrowleft_t T_t$ is a sequential cascade of rank 2.*

A confluence (of sequential cascades) is a sequential cascade.[8]

The confluence is associative, that is, if $R_{t,s}$ is a sequential cascade for every $t \in \operatorname{Max} T$ and each $s \in \operatorname{Max} S_t$, then

$$T \looparrowleft_t (S_t \looparrowleft_s R_{t,s}) = (T \looparrowleft_t S_t) \looparrowleft_s R_{t,s}, \tag{2.6}$$

and thus several consecutive confluences can be written without parentheses.

A subset S of a sequential cascade T is a *subcascade* if

$$o_T \in S, \tag{2.7}$$

$$S^+(t) \text{ is an infinite subset of } T^+(t) \text{ for every } t \in S \setminus \operatorname{Max} T, \tag{2.8}$$

$$t \in S \Rightarrow T^{\downarrow}(t) \subset S. \tag{2.9}$$

The last condition means that S is closed downwards.

It follows from the definition that $\operatorname{Max} S \subset \operatorname{Max} T$.

A subcascade is itself a cascade.[9]

A subcascade S of T is *eventual* if $S^+(t)$ is cofinite in $T^+(t)$ for each $t \in S \setminus \operatorname{Max} T$.

For each $t \in T$, the set $T^{\uparrow}(t) = \{s \in T : t \leq s\}$ with the induced order, is a sequential cascade, and $o_{T^{\uparrow}(t)} = t$ (but it is not a subcascade unless $t = o_T$).

Proposition 2.6. *For every countable ordinal β there exists a sequential cascade of rank β.*

Proof. A straightforward inductive construction consists in constructing a confluence of a cascade T of rank 1 (where $T^+(o)$ is identified with the set of natural numbers) with a sequence of sequential cascades T_n such that the (non-decreasing) sequence $(r(T_n) + 1)_{n<\omega}$ converges to β. ∎

Notice that

$$r(T \looparrowleft_{t \in \operatorname{Max} T} S_t) \leq \sup_{t \in \operatorname{Max} T}(r(S_t) + r(T)). \tag{2.10}$$

Indeed, if T is of rank 1 then $r(T \looparrowleft_{t \in \operatorname{Max} T} S_t) = \sup_{t \in \operatorname{Max} T}(r(S_t) + 1)$. Suppose that the formula holds up to rank β of T. If $r(T) = \beta$, then

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[8] A confluence is obviously a tree; it is also well-founded for the inverse order: if A is a non-empty subset of $T \looparrowleft_t S_t$ and $A \cap T = \varnothing$ then there is $t \in \operatorname{Max} T$ such that $A \cap S_t \neq \varnothing$, hence $\varnothing \neq \operatorname{Max}_{S_t}(A \cap S_t) \subset \operatorname{Max}_{T \looparrowleft_t S_t} A$; finally, each non maximal element of a confluence has a countably infinite set of immediate successors.

[9] Indeed, it is a tree as a subset of a tree, and well-founded (for the inverse order) as a subset of a well-founded (for the inverse order) ordered set; on the other hand, every non maximal element has a countably infinite set of immediate successors; finally, (if S is a subcascade of T, then) there is a unique minimal element of S (actually, $o_S = o_T$), because S is closed downwards.

$T = R \looparrowleft_{r\in \operatorname{Max} R} T^{\uparrow}(r)$ where R is of rank 1, hence $T \looparrowleft_{t\in \operatorname{Max} T} S_t = R \looparrowleft_{r\in \operatorname{Max} R} T^{\uparrow}(r) \looparrowleft_{t\in \operatorname{Max} T} S_t$ and thus, by inductive assumption,

$$\begin{aligned}
r(T \looparrowleft_{t\in \operatorname{Max} T} S_t) &= \sup_{r\in \operatorname{Max} R}(r(T^{\uparrow}(r) \looparrowleft_{t\in \operatorname{Max} T} S_t) + 1) \\
&\leq \sup_{r\in \operatorname{Max} R}\left\{\sup_{t\in \operatorname{Max} T^{\uparrow}(r)}\left[r(S_t) + r(T^{\uparrow}(r))\right] + 1\right\} \\
&\leq \sup_{t\in \operatorname{Max} T}\left\{r(S_t) + \sup_{r\in \operatorname{Max} R}\left[r(T^{\uparrow}(r)) + 1\right]\right\} \\
&= \sup_{t\in \operatorname{Max} T}\left\{r(S_t) + r(T)\right\}.
\end{aligned}$$

3. Embeddings in the sequential tree

I denote by Σ the *sequential tree*, that is, the set of finite sequences of natural numbers. The empty sequence (in other words, the sequence of length 0) is denoted by o. If $s = (n_1, \dots, n_p)$ and $t = (m_1, \dots, m_q)$ are elements of Σ, then the *concatenation* $(n_1, \dots, n_p, m_1, \dots, m_q)$ of s and t is denoted by $s \frown t$. The abbreviation (s, n) for $s \frown (n)$ (where $s \in \Sigma$ and $n < \omega$) is a useful abuse of notation. By definition, $s < t$ if there is a non-empty finite sequence r such that $t = s \frown r$. With so defined partial order Σ becomes a tree.

Recall that a subset S of a partially ordered set V is *closed downwards* if $v < s \in S$ implies $v \in S$ for each $v \in V$. A subset T of Σ is called *full* if $T \cap \Sigma^{+}(s) \neq \varnothing$ implies that $\Sigma^{+}(s) \subset T$ for every $s \in \Sigma$.

Theorem 3.1. *Every sequential cascade is order-isomorphic to a full closed downwards subset of* Σ.

Proof. Let T be a sequential cascade. For every non maximal $t \in T$, let $\blacktriangleleft_t$ be an order of type ω on the set $T^{+}(t)$ of immediate successors of t. We define $f_0(o_T) = o \in \Sigma$. Suppose that for every level $k \leq m$ an injective map f_k has been defined from $\{t \in T : h(t) = k\}$. If $h(t) = m + 1$, then there is the greatest predecessor of t that we call t_-. By the inductive assumption, there exist $n_1, \dots, n_m < \omega$ such that $f_m(t_-) = (n_1, \dots, n_m)$. The set $T^{+}(t_-)$ is ordered by $\blacktriangleleft_{t_-}$, which is of the type ω. Therefore there is $n_{m+1} < \omega$ such that t is the n_{m+1}-th element of $T^{+}(t_-)$ for $\blacktriangleleft_{t_-}$. We define $f_{m+1}(t) = (n_1, \dots, n_m, n_{m+1})$. Then f_{m+1} is injective on $T^{+}(t_-)$, hence on all the elements of level $m + 1$. Therefore $f = \bigcup_{m<\omega} f_m$ is injective, and by construction, $s \leq t$ if and only if $f(s) \subset f(t)$. ∎

A cascade is called *assigned*, if for each non-maximal $t \in T$, an order $\blacktriangleleft_t$ of the type ω is defined on $T^{+}(t)$. Notice that the construction of an embedding of a sequential cascade T in the sequential tree Σ consists in defining an *assignment* $\blacktriangleleft$ on $T \setminus \operatorname{Max} T$.

An assigned cascade T is said to be *monotone* if for every $t \in T \setminus \max T$ the restriction of r_T to $T^{+}(t)$ endowed with the order $\blacktriangleleft_t$ is non-decreasing.

A totally ordered subset of an ordered set is called a *chain*, and a maximal chain is called a *branch*. An *antichain* is a subset of an ordered set such that all its elements are incomparable, and an antichain is called a *section* if it intersects every branch. Each section is a maximal antichain, but the converse need not be true even in a tree.

Example 3.2 (maximal antichain that is not a section)**.** *An element $(n_1, n_2, \ldots, n_m)$ of Σ belongs to A whenever $n_i = 1$ for $i < m$ and $n_m > 1$. The set A is an antichain, for if $(n_1, n_2, \ldots, n_m) \neq (k_1, k_2, \ldots, k_p)$ belong to A, then they are incomparable. This antichain is maximal, for if $t = (n_1, n_2, \ldots, n_m) \notin A$, then either all its components are equal to 1, hence $t \subset (t, 2) \in A$, or there is the least $1 \leq i < m$ such that $n_i > 1$ and thus $A \ni (n_1, \ldots, n_i) \subset t$. Nevertheless, the branch P defined by $o \in P$, and $(n_1, n_2, \ldots, n_m) \in P$ if $n_i = 1$ for $1 \leq i \leq m$, does not intersect A.*

Proposition 3.3. *Each maximal antichain in a cascade is a section.*

Proof. Suppose that T is a cascade, $A \subset T$ is a maximal antichain and P is a branch. Then $\cap P$ is non-empty, because it contains o. Therefore $\mathrm{Max}(T^{\downarrow}(A) \cap P)$ is non-empty, and because P is a chain, it is a singleton that we call p_0. I claim that $p_0 \in A$, hence $p_0 \in P \cap A$. In fact, either $p_0 \in \mathrm{Max}\, P \subset \mathrm{Max}\, T$, thus $p_0 \in A$, because $p_0 \in T^{\downarrow}(A)$, or the immediate successor p_1 of p_0 in P would be incomparable with the elements of A contrary to the maximality of A as an antichain. Indeed, by definition $p_1 \notin T^{\downarrow}(A)$; on the other hand, if there were $a \in A$ such that $a \leq p_1$ then $a \in P$, because P is a maximal chain in a tree, showing that $P \cap A \neq \varnothing$. But in this case, $a = p_0$, which is a contradiction. ∎

We are ready to describe those subsets of the sequential tree, which are sequential cascades with respect to the induced order.

Theorem 3.4. *A closed-downwards subset T of Σ is a fully embedded cascade if and only if $\mathrm{Max}\, T$ is a section of Σ.*

Proof. In fact, if P is a branch in Σ then $o \in T \cap P$, so that $T \cap P$ is a non-void subset of T and hence admits a greatest element t_0 in T, which must belong to $\mathrm{Max}\, T$, because otherwise $P \cap T^{+}(t) \neq \varnothing$.

Conversely, if A is a section of Σ, then $T_A = \Sigma^{\downarrow}(A) = \{t \in T : \exists_{a \in A}\, t \leq a\}$ is a (fully embedded) sequential cascade. Indeed, for each non-empty subset B of T_A, the set $\mathrm{Max}_{T_A} B \neq \varnothing$ (any branch P passing through an element of B intersects A , thus contains a maximal element of B), and if $t \notin \mathrm{Max}\, T_A$ (which is equal to A) then $T_A^{+}(t) \subset T_A$ because $A \cap \Sigma^{\uparrow}(t)$ is a section of $\Sigma^{\uparrow}(t)$. ∎

If A is a section of a sequential cascade T, then $T_A = \{t \in T : \exists_{a \in A}\, t \leq a\}$ is a sequential cascade, and $T^{\uparrow}(a)$ is a sequential cascade for every $a \in A$. Of course,

$$T = T_A \hookleftarrow_{a \in A} T^{\uparrow}(a).$$

4. Monotone cascades as indecomposable ordinals

This section is based on [16] by S. Watson and the present author. Recall that a cascade is called assigned, if for each non maximal $t \in T$, an order $\blacktriangleleft_t$ of the type ω is defined on $T^+(t)$.

Theorem 4.1. *For every assignment $\blacktriangleleft$ of a sequential cascade $(T, \leq)$, there is a coarsest well-order $\lhd$ on T, which is finer than the inverse of $\leq$ (that is, $t_0 \geq t_1$ implies $t_0 \lhd t_1$) and which agrees on $T^+(t)$ with $\blacktriangleleft_t$ for every non maximal $t \in T$.*

Proof. If t_0, t_1 are comparable for $\leq$ then let $t_0 \lhd t_1$ if and only if $t_0 \geq t_1$. If t_0 and t_1 are not comparable for $\leq$, let $t = t_0 \wedge t_1$ where $\wedge$ stands for the infimum with respect to $\leq$; then there are unique elements s_0, s_1 of $T^+(t)$ such that $s_0 \leq t_0$ and $s_1 \leq t_1$; set $t_0 \lhd t_1$ if $s_0 \blacktriangleleft_t s_1$. The relation $\lhd$ is an order. The only property, which is maybe not completely evident, is that $\lhd$ is antisymmetric, so let $t_0 \lhd t_1$ and $t_1 \lhd t_0$. If t_0 and t_1 are comparable in $\leq$ then $t_0 = t_1$, because $\leq$ is antisymmetric. Otherwise if $t = t_0 \wedge t_1$ where $\wedge$ is the infimum with respect to $\leq$, then there exist unique $s_0, s_1 \in T^+(t)$ such that $s_0 \leq t_0$ and $s_1 \leq t_1$, thus $s_0 = s_1$ because $\blacktriangleleft_t$ is antisymmetric, which contradicts $t = t_0 \wedge t_1$.

The constructed order is obviously linear. To show that it is a well-order, we proceed by induction on the rank. If $r(T) = 0$ then $\lhd$ is the order 1, and if $r(T) = 1$ then the order is that of $\omega_0 + 1$. Suppose now that $\beta > 1$ and that the claim is true for all cascades of rank less than β. If $r(T) = \beta$ then $\beta = \sup_{n<\omega}(r(T^\uparrow(n)) + 1)$, and by the inductive assumption, $T^\uparrow(n)$ is ordered by $\lhd$ as an ordinal γ_n. If $n_0 \blacktriangleleft_o n_1$, $n_0 \neq n_1$, and $t_0 \in T^\uparrow(n_0), t_1 \in T^\uparrow(n_1)$, then $t_0 \lhd t_1$. Therefore T is well-ordered by $\lhd$ and the order type is $\sum_{n<\omega} \gamma_n + 1$ (where ω is ordered by $\blacktriangleleft_o$). ∎

Denote by $\eta(T) = \eta(T, \leq, \blacktriangleleft)$ the order constructed above. Notice that if $r(S) < r(T)$ then $\eta(S) < \eta(T)$. In fact, on inducing on the rank, we infer that there is $n_0 < \omega$ such that $r(S) \leq r(T^\uparrow(n_0))$, so that $\eta(S) < \sum_{n<\omega} \eta(T^\uparrow(n)) + 1 = \eta(T)$.

An ordinal π is *decomposable* if there exist $\alpha, \beta < \pi$ such that $\alpha + \beta = \pi$; otherwise it is *indecomposable*. It can be easily seen [12, Lemma 11.1] that π is indecomposable if and only if $\alpha + \pi = \pi$ for each $\alpha < \pi$. Remember that for every ordinal β there exist $0 \leq k < \omega$, a unique finite sequence $\beta_0 > \beta_1 > \ldots > \beta_k$ of indecomposable ordinals, and unique $n_0, n_1, \ldots, n_k < \omega$ such that

$$\beta = \beta_0 n_0 + \beta_1 n_1 + \ldots + \beta_k n_k.$$

On the other hand

Lemma 4.2. *If $(\beta_n)_n$ is a non-decreasing sequence of indecomposable ordinals, then for each $p : \omega \to \omega$ such that $\lim_n p(n) = \omega$,*

$$\sum\nolimits_{n<\omega} \beta_n = \sum\nolimits_{n<\omega} \beta_{p(n)}.$$

is an indecomposable ordinal. Every countable indecomposable ordinal (greater than 1*) is a sum of a non decreasing sequence of indecomposable ordinals.*

Proof. Indeed, if $(\beta_n)_n$ is a non-decreasing sequence of indecomposable ordinals and $p : \omega \to \omega$ tends to infinity, then either there is a strictly increasing subsequence $(n_k)_k$ of natural numbers such that $p(\beta_n) < p(\beta_{n_k})$ for each $n < n_k$, or $(\beta_n)_n$ is stationary. In the first case, $\sum_{n \leq n_k} p(\beta_n) = p(\beta_{n_k})$ for each $k < \omega$, because of indecomposability; therefore

$$\sum_{n<\omega} p(\beta_n) = \sup_{k<\omega} p(\beta_{n_k}) = \sup_{n<\omega} \beta_n = \sum_{n<\omega} \beta_n. \tag{4.1}$$

In the second case, there is a least $n_0 \geq 0$ such that $\beta_n = \beta_{n_0}$ for each $n \geq n_0$ hence $\sum_{n<\omega} p(\beta_n) = \beta_{n_0}\omega$. In both cases $\beta = \sum_{n<\omega} \beta_n$ is indecomposable: in the first, because each supremum of indecomposable ordinals is indecomposable;[10] in the second case, for if $\alpha < \beta_{n_0}\omega$, then there is $k < \omega$ such that $\alpha < \beta_{n_0}k$ hence $\alpha + \beta \leq \beta_{n_0}k + \beta_{n_0}\omega = \beta_{n_0}(k + \omega) = \beta_{n_0}\omega = \beta$.

Conversely, let β be a countable indecomposable ordinal, and let A be the set of indecomposable ordinals less than β. If $\operatorname{Max} A = \varnothing$ then there is an increasing sequence $(\alpha_n)_n$ of elements of A such that $\sup_{n<\omega} \alpha_n = \sup A = \beta$, because the limit of an increasing sequence of indecomposable ordinals is indecomposable. If α is the greatest element of A then $\beta = \alpha\omega = \sup_{n<\omega} \alpha n$, because $\alpha\omega$ is indecomposable. ∎

Theorem 4.3. *A countable ordinal β is indecomposable if and only if there exists a monotone sequential cascade T such that $\eta(T) = \beta + 1$.*

Proof. If a monotone cascade T is of rank 1 then $\eta(T) = \omega + 1$. Suppose that each monotone cascade of rank less than $\beta > 1$, the claim holds, and let $r(T) = \beta$. Then the sequence $(T^{\uparrow}(n))$ of monotone cascades such that the sequence $(r(T^{\uparrow}(n)) + 1)_n$ is non decreasing and converging to β. Then $\eta(T) = \sum_{n<\omega} \eta(T^{\uparrow}(n)) + 1$ and since by the inductive assumption $\eta(T_n)$ are indecomposable and non decreasing, $\sum_{n<\omega} \eta(T^{\uparrow}(n))$ is indecomposable.

One proves the converse by induction starting from the fact that, by Lemma 4.2, each indecomposable ordinal is a non decreasing sum of smaller indecomposable ordinals. ∎

Corollary 4.4. *If S, T are assigned monotone cascades, then $r(S) = r(T)$ if and only if $\eta(S) = \eta(T)$. In particular, η does not depend on the choice of an assignment.*

Proof. If $0 = r(S) \leq r(T)$, then $1 = \eta(S) \leq \eta(T)$. If $0 < r(S) \leq r(T)$, then by inductive assumption $\eta(S) = \sup_{n<\omega} \eta(S^{\uparrow}(n)) + 1 \leq \sup_{n<\omega} \eta(T^{\uparrow}(n)) + 1 = \eta(T)$. The converse follows from our previous observation that $\eta(S) \leq \eta(T)$ implies $\eta(S) \leq \eta(T)$ for all sequential cascades S and T. ∎

[10] If $\alpha < \beta$ then there is $k < \omega$ such that $\alpha < \sum_{n \leq k} \beta_n = \beta_k$ hence by (4.1) $\alpha + \beta \leq \sum_{k \leq n < \omega} \beta_n = \beta$.

5. Natural convergence and natural topology

A *convergence* on a set X is a relation[11] between filters on X and elements of X, denoted by

$$x \in \lim \mathcal{F}$$

such that $\mathcal{F} \subset \mathcal{G}$ implies that $\lim \mathcal{F} \subset \lim \mathcal{G}$, and the principal ultrafilter determined by x converges to x for each $x \in X$.[12]

A filter $\mathcal{F}$ converges to t in the *natural convergence* ζ on the tree Σ if $\mathcal{F}$ is finer than the infimum of two filters: the *principal ultrafilter* of t and the *cofinite filter* of the set of immediate successors of t.[13] Therefore for every $t \in \Sigma$, the coarsest filter that converges to t is generated by a sequence. Hence the natural convergence is a *pretopology*, that is, for every $t \in \Sigma$ there exists the coarsest filter $\mathcal{V}(t)$ on Σ that converges to t. The elements of $\mathcal{V}(t)$ are called *vicinities*.

A subset O of a convergence space is *open* if $O \cap \lim \mathcal{F} \neq \varnothing$ implies that $O \in \mathcal{F}$. The set of open sets of a convergence ξ is a topology called the *topologization* of ξ and is denoted by $T\xi$. Each topology defines a convergence by $x \in \lim \mathcal{F}$ whenever $\mathcal{F}$ contains each open set O such that $x \in O$. Note that $\lim_\xi \mathcal{F} \subset \lim_{T\xi} \mathcal{F}$ for every convergence ξ (and for each filter $\mathcal{F}$).

The topologization $T\zeta$ of the natural convergence ζ of Σ is called the *natural topology* of Σ. It is easy to observe that a subset O of Σ is open in ζ whenever $t \in O$ implies that $\Sigma^+(t)\backslash O$ is finite for every $t \in \Sigma$. Obviously $T\zeta$ is *sequential*, that is, each sequentially open set is open. $T\zeta$ is regular.[14]

Recall that $t \frown s$ stands for the concatenation of $t \in \Sigma$ with $s \in \Sigma$.

Proposition 5.1. *For each $t \in S$ the map $f_t(s) = t \frown s$ is a (convergence and topological) homeomorphism of Σ onto $\Sigma^{\uparrow}(t)$, and each homeomorphism f of Σ onto $\Sigma^{\uparrow}(t)$ fulfills $f(o) = t$, and restricted to $\Sigma^+(s)$ is a bijection onto $\Sigma^+(f(s))$ for every $s \in \Sigma$.*

It follows that Σ is homogeneous. Notice that the set $\Sigma^{\uparrow}(t)$ is clopen (that is, closed and open) subset of Σ for each $t \in \Sigma$.

The *sequential order* $\sigma(X)$ of a convergence space X is the least ordinal β such that $x \in \operatorname{adh}^{\beta}_{\mathrm{Seq}} A$ for every $A \subset X$ and every $x \in \operatorname{cl}_{\mathrm{Seq}} A$. The *sequential order* $\sigma(x; A)$ of x with respect to A is the least ordinal β such that $x \in \operatorname{adh}^{\beta}_{\mathrm{Seq}} A$. In other words, $\sigma(x; A) = \beta$ if and only if (1.1) holds.

[11]If $\mathcal{F}$ and x are related, then we say that $\mathcal{F}$ *converges* to x (x is a *limit* point of $\mathcal{F}$).

[12]A map f from a convergence space to another is *continuous* if $f(\lim \mathcal{F}) \subset \lim f(\mathcal{F})$ for every filter $\mathcal{F}$.

[13]The *principal filter* of a subset A of X is $\{B \subset X : A \subset B\}$; the *cofinite filter* of a subset A of X is $\{B \subset X : \operatorname{card}(A\backslash B) < \infty\}$. If A is a countably infinite subset of X, then the cofinite filter of A is the filter generated by an arbitrary sequence that enumerates A. The principal filter of a singleton is generated by a constant sequence.

[14]Because Σ is countable and the natural pretopology is regular [25, Theorem 2.4], that is, $\mathcal{V}(t) \subset \operatorname{adh}^{\natural} \mathcal{V}(t)$ (where $\operatorname{adh}^{\natural} \mathcal{H}$ is the filter generated by $\{\operatorname{adh} H : H \in \mathcal{H}\}$) for every $t \in \Sigma$. Indeed, $\mathcal{V}(t)$ admits a filter base consisting of closed sets, namely of the type: the union of $\{t\}$ and of a subset of $\Sigma^+(t)$.

Notice that $\sigma(x;A)$ is well defined whenever $x \in \mathrm{cl}_{\mathrm{Seq}} A$.[15] An easy proof that $\sigma(x;A)$ is countable (for every $x \in \mathrm{cl}_{\mathrm{Seq}} A$) is based on the fact that the limit of a sequence of countable ordinals is countable. We set

$$\sigma(x) = \sup\{\sigma(x;A) : A \subset X, x \in \mathrm{cl}_{\mathrm{Seq}} A\}. \tag{5.1}$$

and observe that $\sigma(X) = \sup_{x\in X} \sigma(x)$. If $\sigma(x)$ is an isolated ordinal, then there exists a set A such that $\sigma(x) = \sigma(x;A)$ because then the supremum in (5.1) is attained, but this is in general not the case if $\sigma(x)$ is a limit ordinal. It follows that the sequential order of a convergence space is not greater than ω_1.

To begin with, let us observe that

Proposition 5.2. *If $x \in \lim(x_n)_n$, then*

$$\sigma(x;A) \leq \liminf{}_n(\sigma(x_n;A)+1)_n. \tag{5.2}$$

Proof. Let $(x_{n_k})_k$ be a subsequence such that

$$\beta = \sup{}_{k<\omega}(\sigma(x_{n_k};A)+1) = \liminf{}_n(\sigma(x_n;A)+1)_n.$$

Then $\{x_{n_k} : k < \omega\} \subset \mathrm{adh}^{<\beta}_{\mathrm{Seq}} A$, hence by definition $x \in \mathrm{adh}_{\mathrm{Seq}}\{x_{n_k} : k < \omega\} \subset \mathrm{adh}^{\beta}_{\mathrm{Seq}} A$, which shows that $\sigma(x;A) \leq \beta$. ∎

Theorem 5.3. *The natural topology of Σ is homogeneous of sequential order ω_1.*

Proof. Indeed, if $\beta = \sigma(o)$ were an isolated ordinal, then there would exist $A \subset \Sigma$ such that $\beta = \sigma(o;A)$. If $f_n : \Sigma \to \Sigma^{\uparrow}(n)$ is a homeomorphism for each $n < \omega$, then by Proposition 5.2,

$$\begin{aligned} \beta &= \sigma(o) \geq \sigma(o; \bigcup_{n<\omega} f_n(A)) \\ &\geq \liminf{}_{n<\omega}(\sigma(n; \bigcup_{k<\omega} f_k(A)) + 1) \\ &= \liminf{}_{n<\omega} \sigma(n; f_n(A)) = \beta + 1, \end{aligned}$$

which is a contradiction. If $\sigma(o) = \beta$ were a limit ordinal less than ω_1, then let (α_k) be an increasing sequence such that $\lim_{k<\omega} \alpha_k = \lim_{k<\omega}(\alpha_k+1)_n = \beta$. Let A_k be such that $\sigma(o;A_k) \geq \alpha_k$ and $f_{n,k} : \Sigma \to \Sigma^{\uparrow}(n,k)$ be a homeomorphism for each $n,k < \omega$. Then $\sigma(n; \bigcup_{k<\omega} f_{n,k}(A_k)) = \beta$, thus $\sigma(o; \bigcup_{n<\omega} \bigcup_{k<\omega} f_{n,k}(A_k)) = \beta + 1$, which is a contradiction. ∎

A filter $\mathcal{F}$ converges to $t \in T \setminus \mathrm{Max}\, T$ in the *natural convergence* ζ_T of a sequential cascade T if $\mathcal{F}$ is finer than the infimum of the principal ultrafilter

[15]It is immediate that if $A \subset D$, then $\sigma(x;D) \leq \sigma(x;A)$. More precisely,

$$\sigma(x; A \cup B) = \sigma(x;A) \wedge \sigma(x;B).$$

Indeed, $\mathrm{adh}^{\alpha}_{\mathrm{Seq}}(A \cup B) = \mathrm{adh}^{\alpha}_{\mathrm{Seq}} A \cup \mathrm{adh}^{\alpha}_{\mathrm{Seq}} B$; thus if β is the least ordinal such that $x \in \mathrm{adh}^{\beta}_{\mathrm{Seq}}(A \cup B)$, then either $x \in \mathrm{adh}^{\beta}_{\mathrm{Seq}} A$ or $x \in \mathrm{adh}^{\beta}_{\mathrm{Seq}} B$.

of t and the cofinite filter of the set of immediate successors of t.[16] The *natural topology* of T is the topologization $T\zeta_T$ of ζ_T. Therefore, the natural convergence of a sequential cascade T is induced by an order-isomorphism of T with a subset of Σ. This convergence is independent of a particular order-isomorphism. Also the natural topology $T\zeta_T$ is induced by $T\zeta$, that is, $(T\zeta)_T = T\zeta_T$.[17]

Proposition 5.4. *If a sequential cascade T is a full subset of Σ, then*

$$\sigma(t) = \limsup_n (\sigma(t,n) + 1)_n$$

for every non-maximal $t \in T$.

Proof. We proceed by induction. The claim is obvious if $\sigma(t) = 1$.

If $\alpha < \sigma(t)$ then there is $A \subset T$ such that $\alpha < \sigma(x; A)$ thus, by Proposition 5.2, $\alpha < \limsup_n(\sigma(t,n)+1)_n$. Conversely, assume that $\alpha < \limsup_n(\sigma(t,n)+1)_n$, that is, if there is a sequence (n_k) such that $\alpha < \sigma(t, n_k) + 1$. Then either $\alpha < \sigma(t, n_k)$ for almost all k, and thus there exist $A_k \subset T^{\uparrow}(t, n_k)$ such that $\alpha = \alpha_k < \sigma((t, n_k); A_k)$, or $\alpha = \alpha_k = \sigma(t, n_k)$ for infinitely many k. Then either α is isolated, thus there is $A_k \subset T^{\uparrow}(t, n_k)$ such that $\sigma(t, n_k) = \sigma((t, n_k); A_k)$, or α is limit, and thus there exists an increasing sequence (α_k) and $A_k \subset T^{\uparrow}(t, n_k)$ such that $\alpha_k < \sigma((t, n_k); A_k)$ and $\sup_{k<\omega} \alpha_k = \alpha$. In every case if $A = \bigcup_{k<\omega} A_k$ then $\sigma(t) \geq \sigma(t; A) \geq \alpha$. ∎

Though no element of Σ is isolated, each sequential cascade is *scattered*, that is, each non-empty subset admits an isolated element.[18]

A scattered topological space can be partitioned similarly to a well-founded set. If X is a scattered topological space, then let X_0 be the set of isolated points of X, and let X_β be the set of isolated points of $X \setminus \bigcup_{\alpha<\beta} X_\alpha$. Hence there is the least ordinal γ such that $X = \bigcup_{\alpha<\gamma} X_\alpha$. The *scattering order* or *Cantor-Bendixson rank* ρ_X is an ordinal function on X defined: $\rho_X(x) = \beta$ if and only if $x \in X_\beta$. If a topological space X is sequential and scattered, then $\sigma_X(x) \leq \rho_X(x)$ for every $x \in X$.

Proposition 5.5. *If T is a sequential cascade, then the sequential order and the Cantor-Bendixson rank are equal for every $t \in T$.*

6. Upper and lower ranks

We define by (well-founded) induction the *upper rank* r_T^+ and the *lower rank* r_T^- of T. For maximal elements of T, both the ranks are 0. If r_T^+ and r_T^- have been defined for $\{s \in T : t < s\}$, then

$$r_T^+(t) = \limsup_{s \in T^+(t)} (r_T^+(s) + 1), \; r_T^-(t) = \liminf_{s \in T^+(t)} (r_T^-(s) + 1),$$

[16] The only filter converging to a maximal element is its principal ultrafilter.

[17] In general, the topologization of the restriction of a convergence is finer than the restriction of the topologization; the converse is not true in general.

[18] Maximal elements of a subset A of a cascade are isolated in A.

where the upper and lower limits are considered with respect to the *cofinite filter* on $T^+(t)$. Of course, $r_T^-(t) \leq r_T^+(t) \leq r_T(t)$ for every $t \in T$. A sequential cascade T is *normal* if

$$r_T^+(t) = r_T(t)$$

for every $t \in T$; *asymptotically monotone* if

$$r_T^-(t) = r_T^+(t)$$

for every $t \in T$; *essentially monotone* if

$$r_T^-(t) = r_T(t)$$

for every $t \in T$.

Proposition 6.1. *Every cascade has an eventual normal subcascade.*

Proof. Indeed, if $\{A_n : n < \omega\}$ is a base of the cofinite filter of $T^+(t)$ such that $A_n \supset A_{n+1}$, then $r_T^+(t) = \inf_{n<\omega} \sup_{w \in A_n}(r_T^+(w) + 1)$ and thus there is $n_0 < \omega$ such that $r_T^+(t) = \sup_{w \in A_{n_0}}(r_T^+(w) + 1)$, because each decreasing sequence of ordinals is stationary. For a given cascade T define by induction an eventual subcascade S by setting $S^+(t)$ to be a cofinite subset of $T^+(t)$ for which $r_T^+(t) = \sup_{w \in S^+(t)}(r_T^+(w) + 1)$. ∎

On the other hand,

Proposition 6.2. *If S is an eventual subcascade of T, then $r_T^-(t) = r_S^-(t)$ and $r_T^+(t) = r_S^+(t)$ for every $t \in S$.*

Proof. Obvious for maximal elements. If the claim holds for all the successors in S of $t \in S$ then the lower and the upper limits over $T^+(t)$ and its cofinite subset $S^+(t)$ are the same. ∎

Proposition 6.3. *A subcascade S of an essentially monotone cascade T is essentially monotone, and $r_T(t) = r_S(t)$ for every $t \in S$. Hence $r(S) = r(T)$.*

Proof. In fact, because $S \subset T$, if $t \in S$ then $r_S^-(t) \geq r_T^-(t)$, and the latter is not smaller than $r_T(t) \geq r_S(t)$, hence $r_S^-(t) = r_S(t) = r_T(t)$. ∎

In particular, the rank of a subcascade of an essentially monotone cascade is equal to the rank of the cascade.

Proposition 6.4. *An asymptotically monotone cascade has an eventual essentially monotone subcascade.*

Proof. If T is asymptotically monotone, then we take its eventual subcascade S which is normal, and since $r_T^- = r_S^-$ and $r_T^+ = r_S^+$, the normal subcascade S is essentially monotone. ∎

Observe that if R, S are subcascades of T and R is eventual in T, then $R \cap S$ is a subcascade of T. We say that S is an *almost subcascade* of T if there is R which is a subcascade of T and an eventual subcascade of S.

Proposition 6.5. *If T_{n+1} is an almost subcascade of a cascade T_n for each $n < \omega$, then there exists a sequential cascade T_ω, which is an almost subcascade of T_n for every $n < \omega$.*

Proof. Let R_n be an eventual subcascade of T_n and a subcascade of T_{n-1} for $n > 0$. Then $S_n = R_1 \cap R_2 \cap \ldots \cap R_n$ is such that S_{n+1} is a subcascade of S_n and of $T_{n+1}, T_n, \ldots, T_0$. Pick $t_n \in S_n(o) \backslash \{t_k : k < n\}$ and let $T_\omega^+(o) = \{t_n : n < \omega\}$, and for each n let $T_\omega^\uparrow(t_n) = S_n^\uparrow(t_n)$. This defines a sought subcascade. Indeed, $\bigcup_{k \geq n} T_\omega^\uparrow(t_k)$ is an eventual subcascade of T_ω and a subcascade of T_n. ∎

If all the cascades in Proposition 6.5 are asymptotically monotone, then we can pick T_ω to be essentially monotone.

We have seen in the proof of Theorem 3.1 that the assignment to every non-maximal element t of a sequential cascade of an order $\blacktriangleleft_t$ (of type ω) on $T^+(t)$ defines an order-isomorphic embedding of T into Σ. Let us call such a cascade an *assigned cascade*. In other words, an *assignment* $\blacktriangleleft$ of a cascade is a map associating to each $t \in T \backslash \operatorname{Max} T$ an order of the type ω on $T^+(t)$. If $\blacktriangleleft$ is an arbitrary assignment of a sequential cascade, then

$$\begin{aligned} r_T^+(t) &= \inf_{s \in T^+(t)} \sup_{w \blacktriangleright_t s} (r_T^+(w) + 1), \\ r_T^-(t) &= \sup_{s \in T^+(t)} \inf_{w \blacktriangleright_t s} (r_T^-(w) + 1). \end{aligned}$$

An assigned cascade is *monotone* if for every $t \in T \backslash \operatorname{Max} T$, the map $r_T : T^+(t) \to \mathrm{Ord}$ is non-decreasing with respect to $\blacktriangleleft_t$ and to the (natural) order of Ord. Eventual and essential monotonicity as well as normality are independent of particular assignment. On the other hand,

Proposition 6.6. *Each essentially monotone cascade admits an assignment for which it is monotone.*

Proof. This follows from the fact that if a sequence $(\alpha_n)_n$ of ordinals is such that $\sup_{n<\omega} \alpha_n = \liminf_n \alpha_n$, then the sequence $(n_k)_k$ defined by

$$n_k = \min\{n \notin \{n_0, \ldots, n_{k-1}\} : \alpha_{n_k} = \min\{\alpha_n : n \notin \{n_0, \ldots, n_{k-1}\}\}$$

constitutes a permutation of ω. ∎

Example 6.7. *The subset $\{o\} \cup \{(n) : n < \omega\} \cup \{(0,k) : k < \omega\}$ of Σ is asymptotically monotone cascade, which is not essentially monotone. The subset $\{o\} \cup \{(n) : n < \omega\} \cup \{(n,k) : \frac{n}{2} \in \omega, k \in \omega\}$ of Σ is a cascade, which is not asymptotically monotone.*

Proposition 2.6 can be now refined.

Proposition 6.8. *For every $\beta < \omega_1$ there exists a monotone sequential cascade of rank β.*

Proof. We know that there exist monotone sequential cascades of rank 0. Suppose that for $\beta > 0$ and for each $\alpha < \beta$, there exists a monotone sequential cascade of rank α. Let $(\alpha_n)_n$ be a non-decreasing sequence of ordinals such that $\beta = \sup_{n<\omega}(\alpha_n + 1)$ and $(T_n)_n$ be a sequence of monotone sequential cascades of the respective ranks $(\alpha_n)_n$. If we fully embed T_n in $\Sigma^{\uparrow}(n)$, then $\{o\} \cup \bigcup_{n<\omega} T_n$ is a monotone sequential cascade such that $r_T(o) = \sup_{n<\omega}(\alpha_n + 1) = \beta$. ∎

The notion of monotonicity relates the natural order of a cascade with the orders introduced by an assignment on the sets of immediate successors of all non-maximal elements. Normality, eventual and essential monotonicity relate the natural order of a cascade with its natural convergence.

Proposition 6.9. *If T is a sequential cascade, then $r_T^+(t) = \sigma(t)$ for every $t \in T$.*

Proof. By Theorem 3.1 one can assume without loss of generality that a sequential cascade is a full subset of Σ. If $t \in \operatorname{Max} T$, then $r_T^+(t) = 0 = \sigma(t)$. Suppose that the claim holds for every $s \in T^+(t)$. Then by Proposition 5.4,

$$r_T^+(t) = \limsup_n (r_T^+(t,n) + 1) = \limsup_n (\sigma(t,n) + 1) = \sigma(t).$$

∎

Because there exist sequential cascades of each rank (Proposition 6.8) hence of every sequential order in virtue of the proposition above, and Σ includes such sequential cascades as closed subsets, we get another proof of $\sigma(\Sigma) = \omega_1$.

7. Multisequences

A *multisequence* on a set A is a map from the set of maximal elements of a sequential cascade to A. If a multisequence $f : \operatorname{Max} T \to X$ is defined and T is assigned (that is, embedded in the sequential tree as a full, downwards closed subset), then we shall talk about an *assigned multisequence.* This is irrelevant from the point of view of convergence, but can be handy in some situations.

The *rank* $r(f)$ of a multisequence $f : \operatorname{Max} T \to X$ is by definition the rank of the underlying cascade T. Every sequential cascade of rank 1 is order-isomorphic to that considered in Example 2.2. Accordingly, a multisequence of rank 1 on X is a map f from an infinite countable set to X. If this cascade is assigned, then f becomes an *assigned sequence*, that is, a sequence in the classical sense, because the set of maximal elements of the underlying cascade is now ordered by the type ω. But of course, in topology and convergence theory the only relevant aspect of a sequence is the image of the cofinite filter of a countably infinite set.

We shall talk about *bisequences, trisequences*, and so on, referring to multisequences of rank $2, 3$ and so on. A multisequence $f : \operatorname{Max} T \to X$ is, respectively, *asymptotically monotone, essentially monotone,* and so on, whenever its T is asymptotically monotone, essentially monotone, and so on.

If X is a convergence space, then a multisequence $f : \operatorname{Max} T \to X$ *converges* to $x \in X$

$$x \in \lim f \tag{7.1}$$

if there exists an extension $\hat{f}$ of f to the whole of T such that $\hat{f}(o_T) = x$ and that the image by $\hat{f}$ of the cofinite filter of $T^+(t)$ converges to $\hat{f}(t)$ for each non-maximal $t \in T$.

In other words, if (X, ξ) is a convergence space, then $f : \operatorname{Max} T \to X$ converges to $x \in X$ if there exists an extension $\hat{f}$ of f such that $\hat{f}(o_T) = x$, which is continuous from (T, ζ_T) to (X, ξ). In this situation, by a handy abuse of terminology, we say that the extension $\hat{f}$ *converges* to x and write $x \in \lim \hat{f}$.

If X is a Hausdorff[19] convergence space and $f : T \to X$ is a convergent multisequence, then the extension $\hat{f}$ of f postulated by the definition of convergence is unique, thus the limit of f is a singleton.

If T is embedded as a full downwards closed subset of Σ, then the continuity of $\hat{f}$ is equivalent to

$$\hat{f}(t) \in \lim(\hat{f}(t, n))_n$$

for every $t \in T \setminus \operatorname{Max} T$. Such a cascade is assigned; if it is monotone, then each multisequence defined on its maximal elements is called *monotone.*

A map from a sequential cascade to X is called an *extended multisequence.* If $f : T \to X$ is an extended multisequence and if $g_t : S_t \to X$ is an extended multisequence such that $g_t(o) = f(t)$ for every $t \in \operatorname{Max} T$, then we define a confluence of multisequences $f \looparrowleft_{t \in \operatorname{Max} T} g_t$. To this end, we suppose that $S_{t_0} \cap S_{t_1} = \varnothing$ if $t_0 \neq t_1$, and that $T \cap S_t = \operatorname{Min} S_t$, and consider the confluence of cascades $T \looparrowleft_{t \in \operatorname{Max} T} S_t$. Then the extended multisequences f and g_t can be glued to a map from $T \looparrowleft_{t \in \operatorname{Max} T} S_t$ to X (that is, to an extended multisequence) denoted by

$$f \looparrowleft_{t \in \operatorname{Max} T} g_t$$

and called the *confluence* of (extended) *multisequences* $\{g_t : t \in \operatorname{Max} T\}$ to an extended multisequence f.

Proposition 7.1. *Let $f : T \to X$ be an extended multisequence and let $g_t : S_t \to X$ be extended multisequences for every $t \in \operatorname{Max} T$. If $x \in \lim f$ and $f(t) \in \lim g_t$ for every $t \in \operatorname{Max} T$, then $x \in \lim(f \looparrowleft_{t \in \operatorname{Max} T} g_t)$.*

An extended multisequence $g : T \to X$ is *free* (resp., *stationary*) at t if the sequence $g : T^+(t) \to X$ is *free* (resp., *stationary*). Both these terms admit two interpretations, broad and narrow. A sequence is *free* if it generates a cofinite filter of a countably infinite set (*free in the narrow sense* if it is injective). A sequence is *stationary* if it generates a principal ultrafilter, that is, if it is constant on a cofinal subset of its domain (*stationary in the*

[19] A convergence is *Hausdorff* if $\lim \mathcal{F}$ is at most a singleton for each filter $\mathcal{F}$.

narrow sense if it is constant). From the point of view of convergence only the broad sense is meaningful.

If a map between two sequential cascades is continuous (with respect to the natural convergences) and maps maximal elements into maximal elements, then it does not increase the rank.

Proposition 7.2. *If T, W are sequential cascades, and if $f : (T, \zeta_T) \to (W, \zeta_W)$ is a continuous map such that $f(\operatorname{Max} T) \subset \operatorname{Max} W$, then $r_T^+(t) \geq r_W^-(f(t))$ for each $t \in T$.*

Proof. We proceed by well-founded induction on W. If $w \in \operatorname{Max} W$, then $0 = r_W(w) \leq r_T(t)$ for every $t \in T$, hence for each $t \in f^-(w)$. Let $w \in f(T) \subset W \setminus \operatorname{Max} W$ and suppose that the claim holds for all the predecessors of w. Let t be an element of $f^-(w) \subset T$; by the assumption $t \notin \operatorname{Max} T$, and by continuity, $f(T^+(t)) \setminus W^+(w)$ is finite. Hence by the inductive assumption, $r_W^-(f(s)) \leq r_T^+(s)$ for almost all the elements s of $T^+(t)$, and

$$r_W^-(w) = \liminf_{W^+(w)}(r_W^- + 1) \leq \limsup_{T^+(t)}(r_T^+ + 1) = r_T^+(t).$$

∎

The converse is not true in general. A map from one sequential cascade to another can be order-preserving, but not continuous.[20]

If moreover T and W are essentially monotone, then $r_W(f(t)) \leq r_T(t)$ for each $t \in T$. The inequality becomes equality if f is injective:

Proposition 7.3. *If T and W are essentially monotone and $f : (T, \zeta_T) \to (W, \zeta_W)$ is an injective continuous map such that $f(\operatorname{Max} T) \subset \operatorname{Max} W$, then $r_T(t) = r_W(f(t))$ for each $t \in T$.*

Proof. If $r_T(t) = 0$ then $r_W(f(t)) = 0$ by assumption. If $r_T(t) > 0$ then $f(T^+(t)) \setminus W^+(w)$ is finite and $f(t) \cap f(T^+(t)) = \varnothing$, because f is continuous and injective. Therefore, by the inductive assumption,

$$r_W(f(t)) = \lim_{W^+(f(t))}(r_W + 1) = \lim_{T^+(t)}(r_T + 1) = r_T(t).$$

∎

Corollary 7.4. *If T and W are essentially monotone, $f : (T, \zeta_T) \to (W, \zeta_W)$ is a finite-to-one continuous map such that $f(\operatorname{Max} T) \subset \operatorname{Max} W$, then there exists a subcascade R of T such that $r_T(t) = r_W(f(t))$ for each $t \in R$.*

Proof. If $f : T \to W$ is a finite-to-one mapping, level by level, pick one element of $\{s \in T^+(t) : f(s) = w\}$ for every $w \in W^+(v)$ where $f(t) = v$. If $t = o$ then the obtained set is an infinite subset of $T^+(o)$. By induction, we

[20]For instance take the following subsets T, W of the sequential tree Σ:

$$T = \{o\} \cup \{(n) : n < \omega\}, \; W = \{o\} \cup \{(n) : n < \omega\} \cup \{(n, k) : n, k < \omega\},$$

and a map $f : T \to W$ defined by $f(n) = (n, n)$ and $f(o) = o$.

obtain a subcascade R of T such that the restriction of f to R is injective, so that we can use Proposition 7.3 to conclude. ∎

According to the classical definition, a *sequence* on a set X is a map f from the set ω (of natural numbers naturally ordered) to X. As was said before, from the point of view of convergence, the only pertinent aspect of a sequence f is the image by f of the cofinite filter of ω.[21] As f is a partial map from $\omega + 1 = \omega \cup \{\omega\}$, the image by f of the cofinite filter of ω, is equal to $f(\mathcal{N}(\omega))$, where $\mathcal{N}(\omega)$ is the neighborhood filter of ω in the usual topology of ordinal numbers.

It is instructive to compare two sequences (in the classical sense) from the point of view of convergence. We say that a sequence $g : \omega \to X$ is *finer* than a sequence $f : \omega \to X$ (in symbols, $f \prec g$) if $g(\mathcal{N}(\omega))$ is finer than $f(\mathcal{N}(\omega))$.

Proposition 7.5. *A sequence g is finer than a sequence f if and only if there exists a cofinite subset A of ω and a continuous map j from $A \cup \{\omega\}$ to $\omega \cup \{\omega\}$ (with respect to the topology of ordinals) such that $g|_A = f \circ j|_A$.*

Proof. By definition, g is finer than f if and only if for every $n < \omega$ there exists $k_n < \omega$ such that

$$g(\{k_n, \ldots\}) \subset f(\{n, \ldots\}). \tag{7.2}$$

Hence there exists a strictly increasing sequence $(k_n)_n$ for which (7.2) holds (if the sequence is not strictly increasing, then we can replace k_n by $\sup\{k_0, \ldots, k_n\} + n$ for example). Let $A_n = \{k : k_n \leq k < k_{n+1}\}$. Therefore for each $n < \omega$ there exists a map $j_n : A_n \to \{n, \ldots\}$ such that $g(k) = f(j_n(k))$ for $k \in A_n$. The map $j : A = \bigcup_{n<\omega} A_n \cup \{\omega\} \to \omega \cup \{\omega\}$ defined by $j(\omega) = \omega$ and $j(k) = j_n(k)$ for $k \in A_n$ is continuous, because $j(\bigcup_{m\leq n<\omega} A_n) \subset \{n, \ldots\}$ for every $m < \omega$.

Conversely, if j is a required map, then by continuity for every n there is k_n such that $j(k) \geq n$ if $k \geq k_n$, that is, $j(\{k_n, \ldots\}) \subset \{n, \ldots\}$ hence

$$g(\{k_n, \ldots\}) = f \circ j(\{k_n, \ldots\}) \subset f(\{n, \ldots\}).$$

∎

In other words, g is finer than f if and only if g is a subsequence of f in the broad sense. This approach was developed in [7] where the equivalence of sequences in broad sense was characterized. Proposition 7.5 can serve as a model for a concept of submultisequence.

A multisequence $g : \operatorname{Max} T \to A$ is a *submultisequence* of $f : \operatorname{Max} W \to A$ if there exist an eventual subcascade R of T and a continuous map j :

---

[21] Hence a *sequence* on X, seen as a multisequence of rank 1, is just a map from an infinitely countable set (the set of maximal elements of a sequential cascade of rank 1) to X. If however we embed the cascade of rank 1 in the sequential tree as a full closed downwards subset, then the sequence becomes a sequence in the classical sense.

$(R, \zeta_R) \to (W, \zeta_W)$ such that

$$j(o_T) = o_W, \tag{7.3}$$

$$\forall_{t \in R \setminus \operatorname{Max} R}\, j(R^+(t)) \subset W^+(j(t)), \tag{7.4}$$

$$j(\operatorname{Max} R) \subset \operatorname{Max} W, \tag{7.5}$$

$$g|_{\operatorname{Max} R} = f \circ j|_{\operatorname{Max} R}. \tag{7.6}$$

It follows that the image by j of the cofinite filter of $R^+(t)$ is a free filter containing $W^+(j(t))$ for every non-maximal $t \in R$, that is, j is free at t for every $t \in R \setminus \operatorname{Max} R$. In an analogous way[22], we define an *extended submultisequence* $g : T \to A$ of an extended multisequence $f : W \to A$.

Proposition 7.6. *If $j : R \to W$ fulfills the conditions (7.3)(7.4)(7.5), then $j(R)$ is a subcascade of W and j is finite-to-one.*

Proof. The element o_W belongs to $j(R)$, because $j(o_T) = o_W$. If $w \in j(R) \setminus \operatorname{Max} W$ then there exists $t \in R$ such that $w = j(t)$, thus by (7.5) $t \notin \operatorname{Max} R$, and because j is continuous and (7.4), the set $j(R) \cap W^+(j(t))$ is infinite. Finally, if $o_W \neq w \in j(R)$ then let w_0 be such that $w \in W^+(w_0)$. Then there exist s and t in R such that $j(s) = w$ and $s \in R^+(t)$, hence $j(t) = w_0$, which proves that $j(R)$ is closed downwards. Observe that if $j(t) = o_W$ then $t = o_T$ because otherwise, there is a predecessor s of t, thus $o_W \notin W^+(j(s)) \ni j(t)$. If the claim holds for all w up to level n, and $l_W(v) = n+1$ then for the predecessor w of v the preimage $j^-(w)$ is finite by the inductive assumption. For each $t \in j^-(w)$ the set $j(R^+(t))$ is included in $W^+(w)$ and by the continuity of j the intersection $j^-(v) \cap R^+(t)$ is finite, hence $j^-(v)$ is finite. ∎

If $j : R \to W$ is not injective, then on inducing on levels of R, we can find a (not necessarily eventual) subcascade S of R such that the restriction of j to S is injective. If j in the definition of submultisequence above is injective, then we call g an injective submultisequence of f. Therefore,

Proposition 7.7. *If $g : T \to X$ is an extended submultisequence of $f : W \to X$, then there is a subcascade S of T such that $g|_S$ is an injective extended submultisequence of f.*

It follows from Proposition 7.3 that if g is a monotone submultisequence of a monotone multisequence f, then $r(g) = r(f)$. Moreover if f is free (resp. stationary) at $j(t)$ and $g = f \circ j$, then g is free (resp. stationary) at t.

Proposition 7.8. *Each multisequence in a sequentially compact topological space admits a convergent submultisequence.*

Proof. The claim amounts to the definition if the rank of a multisequence is 1. Suppose that the claim holds for the multisequences of rank less than

[22] The modification concerns merely Condition 7.6, which is replaced by $g|_R = f \circ j$.

$\beta > 1$ and let $f : T \to X$ be an extended multisequence of rank β. Then for every $t \in T^+(o)$ let $f_t(s) = f(t, s)$ defines a multisequence $f_t : \operatorname{Max} T^{\uparrow}(t) \to X$ of rank less than β. Hence by the inductive assumption there exists a submultisequence $g_t : \operatorname{Max} S_t \to X$ of f_t, which converges to an element $x(t)$ of X. Therefore $x : T^+(o) \to X$ is a sequence (multisequence of rank 1), hence there is a subsequence g_∞ of x and an element x_∞ of X such that $x_\infty \in \lim g_\infty$. Call $\hat{g}_t$ a convergent extension of g_t, and $\hat{g}_\infty$ a convergent extension of g_∞. Then the restriction to $\bigcup_{t \in T^+(o)} \operatorname{Max} S_t$ of the confluence $\hat{g}_\infty \looparrowleft_{t \in T^+(o)} \hat{g}_t$ is a submultisequence of f, which converges to x_∞. ∎

Proposition 7.9. *Let $(T_n)_n$ be a sequence of sequential cascades and let $f_n : T_n \to X$ be an extended multisequence such that f_{n+1} is an injective extended submultisequence of f_n for every $n < \omega$. Then there exists a sequential cascade T_∞ and $f_\infty : T_\infty \to X$, which is an extended submultisequence of f_n for every $n < \omega$.*

Proof. By definition, for every $n < \omega$ there is an eventual subcascade R_{n+1} and a continuous map $j_n : R_{n+1} \to T_n$, which fulfills the conditions corresponding to (7.3)(7.4)(7.5) and such that $f_{n+1}|_{\operatorname{Max} R_{n+1}} = f_n \circ j_n|_{\operatorname{Max} R_{n+1}}$. As $R_n \cap j_n(R_{n+1})$ is a subcascade of T_n, because R_n is an eventual subcascade of T_n, it is enough to carry out a proof in the case where $T_n = R_n$ and j_n are injective for every $n < \omega$, hence a homeomorphism from T_{n+1} to $j_n(T_{n+1})$. On defining $A_n = j_0 \circ j_1 \circ \ldots \circ j_{n-1}(T_n)$, we get a sequence of cascades such that A_{n+1} is a subcascade of A_n. By Proposition 6.5 there exists a cascade T_∞ such that for every $n < \omega$ there is an eventual subcascade B_n of T_∞, which is also a subcascade of A_n. We set $f_\infty = f_0|_{\operatorname{Max} T_\infty}$. Fix $n < \omega$. To see that $f_\infty : \operatorname{Max} T_\infty \to X$ is a submultisequence of $f_n : \operatorname{Max} T_n \to X$, take $g_n : B_n \to T_n$ to be $(j_0 \circ \ldots \circ j_{n-1})^-$ restricted to B_n. By construction, g_n is continuous. Because $f_n = f_{n-1} \circ j_{n-1} = \ldots = f_0 \circ j_0 \circ \ldots \circ j_{n-1}$, hence $f_0 = f_n \circ (j_0 \circ \ldots \circ j_{n-1})^-$, and thus $f_\infty|_{B_n} = f_0|_{B_n} = f_n \circ g_n|_{B_n}$, the proof is complete. ∎

8. Extensions

A convergence space X is said to be *prime* if it there exists $\infty \in X$, called an *origin*, such that $X \backslash \{\infty\}$ is discrete. Discrete topologies are prime and the only prime convergences for which every element is an origin (equivalently, an origin of a prime convergence is not unique if and only if the convergence is a discrete topology with the underlying set of cardinality greater than 1).

The restriction of the natural topology $T\zeta_T$ of a sequential cascade T (of non-zero rank) to $\operatorname{Ext} T$ is prime and o_T is the origin. The trace of the neighborhood filter of o_T on $\operatorname{Ext} T$ is called the *contour* of T.

The following result (established in [14, Theorem 3.3] for monotone sequential cascades) implies that the natural topology of a sequential cascade can be recovered from the contour on a subcascade.

Theorem 8.1. *Let $(S, T\zeta_S)$ and $(W, T\zeta_W)$ be sequential cascades endowed with the natural topologies, and let $f : \operatorname{Ext} S \to \operatorname{Ext} W$ be a continuous map such that $f^-(o_W) = \{o_S\}$. Then there exists a subcascade R of S and a continuous map $\hat{f} : (R, T\zeta_R) \to (W, T\zeta_W)$ such that $\hat{f}|_{\operatorname{Ext} R} = f|_{\operatorname{Ext} R}$. Moreover if f is injective, then $\hat{f}$ can be taken injective.*

Proof. We proceed by well-founded induction. If $t \in \operatorname{Ext} S$ then f restricted to $\{t\}$ is continuous and equal to the extension. Let now S be a cascade of rank β. For every $t \in S$ consider the set

$$F_S(t) = \{w \in W : \exists_{Q \in \mathcal{N}_S(t)} \forall_{s \in Q \cap \operatorname{Max} S}\, w \le f(s)\}. \tag{8.1}$$

The set $\operatorname{Max} F_S(t)$ is a singleton and we denote its element by $\hat{f}(t)$. Indeed, $o_W \in F_S(t)$ and if $w_0, w_1 \in \operatorname{Max} F_S(t)$, then, by definition, there exist $Q_0, Q_1 \in \mathcal{N}_S(t)$ such that $w_0 \le f(s)$ and $w_1 \le f(s)$ for every $s \in Q_0 \cap Q_1 \cap \operatorname{Max} S$, and of course $Q_0 \cap Q_1 \in \mathcal{N}_S(t)$. As W is a tree and $w_0, w_1 \le f(s)$ for some s , the elements w_0, w_1 are comparable, hence $w_0 = w_1$ as maximal elements of (8.1).

Moreover, for every $t_0, t \in S$ such that $t_0 \le t$, the sets $F_S(t)$ and $F_{S^{\uparrow}(t_0)}(t)$ coincide, because $S^{\uparrow}(t)$ is a neighborhood of t in S and in $S^{\uparrow}(t_0)$. Therefore $\hat{f}(t)$ depends only on the restriction of f to $\operatorname{Max} S^{\uparrow}(t)$.

It turns out that $\hat{f}(o_S) = f(o_S)$. Otherwise there would exist $w \in W$ different from $o_W = f(o_S)$ and a neighborhood Q of o_S such that $f(Q \cap \operatorname{Max} S$) would be a subset of $W^{\uparrow}(w)$, which is closed and $o_W \notin W^{\uparrow}(w)$, contrary to the continuity of f.

Now we observe that the image by $\hat{f}$ of the cofinite of $S^{+}(o_S)$ converges to o_W. Otherwise there would exist a (closed downwards) neighborhood S of o_W and an infinite subset P of $S^{+}(o_S)$ such that $\hat{f}(P) \cap S = \varnothing$. This implies the existence of neighborhoods Q_p of p for each $p \in P$ such that $f(\bigcup_{p \in P} Q_p \cap \operatorname{Max} S\ ) \cap S = \varnothing$. But $o_S \in \operatorname{cl}_S(\bigcup_{p \in P} Q_p \cap \operatorname{Max} S\ )$, hence $o_W \in \operatorname{cl}_W f(\bigcup_{p \in P} Q_p \cap \operatorname{Max} S\ )$, which is in contradiction with the continuity of f.

There is an infinite subset A_o of $S^{+}(o)$ such that $\hat{f}$ is continuous at every element of A_o. Otherwise there is a cofinite subset B of $S^{+}(o)$ such that for every $n \in B$ there exist a neighborhood X_n of $\mathcal{N}_W(\hat{f}(n))$ and a subset of D_n of $\operatorname{Max} S^{\uparrow}(n)$ such that $n \in \operatorname{cl}_S D_n$ and $X_n \cap f(D_n) = \varnothing$.

If there were an infinite subset B_0 of B such that $\hat{f}(B_0) \subset W^{+}(o)$ then o_S is in the closure of $\bigcup_{n \in B_0} D_n$ but o_W is not in the closure of $\hat{f}(\bigcup_{n \in B_0} D_n)$, contrary to the continuity assumption.

If there were an infinite subset B_0 of B such that $\hat{f}(B_0) = \{o_W\}$ then $X_n \in \mathcal{N}(o_W)$ and thus can be written

$$X_n = \{o_W\} \cup \bigcup_{k < \omega} X_{n,k},$$

where $X_{n,k} \in \mathcal{N}(k)$ for each $k \in W^+(o)$. Clearly $X_{n,n} \cap f(D_n) = \varnothing$ for every $n \in B_0$, hence the neighborhood $\{o_W\} \cup \bigcup_{k<\omega} X_{n,n}$ of o_W is disjoint from $f(\bigcup_{n\in B_0} D_n)$, but o_S is in the closure of $\bigcup_{n\in B_0} D_n$. This contradicts the continuity.

By upward induction we construct $A_s \in S^{\,+}(s)$ for every $s \in A_t \in S^{\,+}(t)$ (when these exist) so that $\hat{f}$ is continuous at every element of A_s and the union of all these is a subcascade of S on which $\hat{f}$ is continuous. ∎

9. Hypersequences

The topology of the Baire space ω^ω (of sequences of natural numbers) is that of pointwise convergence. If $f \in \omega^\omega$, then $f \restriction n$ denotes the restriction of f to the set $\{0, 1, \dots, n-1\} \subset \omega$, that is, $f \restriction n$ is the finite sequence $(f(0), \dots, f(n-1))$ for $n > 0$ and $f \restriction 0 = o$ (the empty sequence). Therefore the subset $\{f \restriction n : n < \omega\}$ of the sequential tree Σ is a maximal chain, that is, a branch of Σ . The map which associates with $f \in \omega^\omega$ the branch $\{f \restriction n : n < \omega\}$ of Σ, is a bijection of ω^ω onto Σ. Therefore we can identify the set of all branches of Σ and ω^ω.

The *extended sequential tree* is the disjoint union $\Sigma_\infty = \Sigma \cup \omega^\omega$, to which the order of Σ is extended by $t < f$, whenever $f \restriction h_\Sigma(t) = t$ (where $h_\Sigma(t)$ is the level of t in Σ). We know that ω^ω with its pointwise topology is homeomorphic (via continued fractions) with the set $\mathbb{P}$ of irrational numbers with the natural topology; therefore we can view Σ_∞ as tree completed from above by $\mathbb{P}$.

If S is a section in Σ, then for every branch f of Σ there is a unique element of $S \cap f$ denoted by $\pi_S(f)$. In this way we define a map $\pi_S : \omega^\omega \to S$, called the *projection* on S.

The *universal filter* on ω^ω is defined with the aid of the following base:

$$V_g = \{f \in \omega^\omega : g(f \restriction n) \le f(n)\},$$

where $g : \Sigma \to \omega$. In other words, V_g consists of all those branches P for which $t \in P$ implies $(t, k) \in P$ provided that $g(t) \le k$.

Proposition 9.1. *The image by $\pi_{\operatorname{Max} T}$ of the universal filter on ω^ω is equal to the contour of T.*

Proof. A neighborhood base of $o \in \Sigma$ in the natural topology $T\zeta$ is composed of sets of the form

$$Q_g = \{(n_0, \dots, n_k) \in \Sigma : g(n_{j-1}) \le n_j, 0 \le j \le k\},$$

where $g : \Sigma \to \omega$ and n_{-1} is identified with o. Indeed, all open neighborhoods of a given point t (of any topologization of a pretopology) can be constructed by induction as $V = \bigcup_{n<\omega} V_n$, where V_1 is a pretopological vicinity of t, and for every $n > 1$ one defines $V_{n+1} = \bigcup_{s\in V_n} W(s)$, where $W(s)$ is a pretopological vicinity of s for every $s \in V_n$. In our case, each pretopological vicinity of $s \in \Sigma$ can be chosen as a subset of $\{s\} \cup \Sigma^+(s)$ and, more precisely, of the form $\{s\} \cup \{(s, k) : g(s) \le k\}$ with $g(s) \in \omega$. By the way, notice that each neighborhood Q_g is a cofinal subtree of Σ, clopen in Σ and homeomorphic to Σ. Consequently, V_g is the set of branches of Σ that lie in Q_g.

If T is a cascade fully embedded in Σ, then open neighborhoods can be constructed from pretopological vicinities in the analogous way (as for every topology obtained as the topologization of a pretopology). The countable procedure is the same as in the case of Σ, but now vicinities $W(s)$ are replaced by their restrictions to T, that is, $W_T(s) = W(s) \cap T$. Hence when $s \in \operatorname{Max} T$ then $W_T(s) = \{s\}$. As the natural topology $T\zeta_T$ of T is the restriction of the natural topology $T\zeta_\Sigma$ of Σ, a neighborhood base of T consists of the restrictions of the elements of a neighborhood base of Σ,

$$Q_g^T = \{(n_0, \ldots, n_k) \in T : g(n_{j-1}) \leq n_j, 0 \leq j \leq k\}.$$

If now we consider the restriction of $T\zeta_T$ to $\operatorname{Ext} T$ (thus of $T\zeta$ to $\operatorname{Ext} T$), then a neighborhood base of $\operatorname{Ext} T$ consists of the restrictions of the elements of a neighborhood base of Σ, that is,

$$Q_g^{\operatorname{Max} T} = \{(n_0, \ldots, n_k) \in \operatorname{Max} T : g(n_{j-1}) \leq n_j, 0 \leq j \leq k\}.$$

∎

By definition,

$$\pi_{\operatorname{Max} T} V_g = \{t \in \operatorname{Max} T : \exists_{f \in V_g} \{t\} = f \cap \operatorname{Max} T\} = Q_g^{\operatorname{Max} T}.$$

A *hypersequence* on X is a map from ω^ω to X, which is continuous from the Baire topology to the discrete topology. This notion was introduced in [21] under the name of multisequence. Let (X, τ) be a topological space. A hypersequence $F : \omega^\omega \to X$ converges to x if $x \in \lim_\tau F(\mathcal{F})$, where $\mathcal{F}$ is the universal filter.

Proposition 9.2. *A map $F : \omega^\omega \to X$ is a hypersequence if and only if there exists a sequential cascade T (fully embedded in Σ) and a multisequence $f : \operatorname{Max} T \to X$ such that $f \circ \pi_{\operatorname{Max} T} = F$.*

Proof. If $F : \omega^\omega \to X$ is a hypersequence, then for every $g \in \omega^\omega$ there exists the least $t(g) \in \Sigma$ such that $F(j) = F(g)$ for ever $j > t(g)$. The set $A = \{t(g) :\in \omega^\omega\}$ is a section of Σ. We need only to show that it is an antichain: if $t(g_0) < t(g_1)$, then $F(j) = F(g_0)$ for every $j > t(g_0)$ hence $F(g_1) = F(g_0)$ and thus $t(g_1) \leq t(g_0)$. Then $\pi_A(g) = t(g)$. Let $T = \Sigma^{\downarrow}(A)$. It is a sequential cascade and $\operatorname{Max} T = A$. For every $t \in \operatorname{Max} T$ the hypersequence F is constant on $\pi^-_{\operatorname{Max} T}(t)$, hence $f(t) = F(g)$ if $\pi_{\operatorname{Max} T}(g) = t$ is well-defined. Clearly $F = f \circ \pi_{\operatorname{Max} T}$. Conversely, if $f : \operatorname{Max} T \to T$, then $f \circ \pi_{\operatorname{Max} T}$ is a hypersequence. ∎

We have seen that each hypersequence uniquely determines a multisequence for which $F = f \circ \pi_{\operatorname{Max} T}$, and vice versa. Kratochvíl expressed this fact in a more complicated way in terms of his notion of generator [22].

Proposition 9.3. *Let $f : \operatorname{Max} T \to X$ be a fully embedded multisequence. Then the hypersequence $F = f \circ \pi_{\operatorname{Max} T}$ converges to x, if and only if the extension $\tilde{f} : \operatorname{Ext} T \to X$ defined by $\tilde{f}(o) = x$, is continuous.*

Proof. If $\mathcal{F}$ stands for the universal filter on ω^ω, then by Proposition 9.1, $\pi_{\operatorname{Max} T}(\mathcal{F})$ is the contour of T. In other words, $F = f \circ \pi_{\operatorname{Max} T}$ converges to x whenever $x \in \lim f(\pi_{\operatorname{Max} T}(\mathcal{F}))$, and the last formula is equivalent to the continuity of $\tilde{f}$. ∎

10. Classifications of multisequences

Recall that for each filter $\mathcal{F}$, there exists a unique pair of (possibly degenerate) filters $\mathcal{F}^\circ, \mathcal{F}^\bullet$ such that $\mathcal{F}^\circ$ is free, $\mathcal{F}^\bullet$ is principal, and

$$\mathcal{F} = \mathcal{F}^\circ \wedge \mathcal{F}^\bullet \text{ and } \mathcal{F}^\circ \vee \mathcal{F}^\bullet = 2^X. \tag{10.1}$$

By analogy with stationary sequences, I say that a filter is *stationary* if it is a principal ultrafilter.[23]

If T is a sequential cascade, and $\mathcal{V}(t)$ stands for the vicinity filter of $t \in T$ with respect to ζ_T, then $\mathcal{V}(t)^\circ$ is the cofinite filter of $T^+(t)$ if t is not maximal, and is degenerate if t is maximal. As the set $T^+(t)$, of immediate successors of t, is countably infinite, its cofinite filter is a free sequential filter. Therefore for every $f : T \to X$ and $t \in T \setminus \operatorname{Max} T$, the filter $f(\mathcal{V}(t)^\circ)$ is sequential and non-degenerate. Of course, $f(\mathcal{V}(t)^\circ)$ admits a decomposition into the free and the principal parts.

When T is a sequential cascade, then a map $f : T \to X$ is called *freedom classifiable* if $f(\mathcal{V}(t)^\circ)$ is either free or stationary for every $t \in T \setminus \operatorname{Max} T$. We associate with $f : T \to X$ a *freedom function* $\varphi = \varphi_f : T \to \{0, 1, *\}$ as follows

$$\varphi(t) = \varphi_f(t) = \begin{cases} 0 & \text{if } f|_{T^+(t)} \text{ is stationary;} \\ 1 & \text{if } f|_{T^+(t)} \text{ is free;} \\ * & \text{otherwise.} \end{cases}$$

In practice, the symbol $*$ will be also used if either the value of φ at t is unknown or irrelevant for an argument. I shall say that a multisequence is *stationary* (respectively, *free*) *at* t if $\varphi(t) = 0$ (respectively, $\varphi(t) = 1$); *free* if it is free at every non-maximal point.

Proposition 10.1. *Every extended multisequence admits a freedom classifiable (extended) submultisequence.*

Proof. Let $f : T \to X$ be an extended multisequence. Let $R_0 = \{o\}$. There is an infinite subset R of $T^+(o)$ such that for every $t \in R$ the sequence $f|_{R_1}$ is either free or principal. Let $R_1 = \{o\} \cup R$. If we have already found a subcascade R_n of $\{t \in T : l_T(t) \leq n\}$ such that f restricted to R_n is freedom classifiable, then for every t of level n there is a subset $R_{n+1}(t)$

[23] Let $\mathcal{F}^\bullet$ be the principal filter of $F_\bullet = \bigcap_{F \in \mathcal{F}} F$. Then $\mathcal{F}^\circ = \mathcal{F} \vee F_\bullet^c$ is free, and obviously (10.1) holds. If now $\mathcal{N}_\iota(A)$ is a principal filter finer than $\mathcal{F}$, then $A \subset F_\bullet$, hence $\mathcal{F} \vee A^c$ is free if and only if $A = F_\bullet$, which shows the uniqueness of the decomposition.

such that $f|_{R_{n+1}(t)}$ is either free or principal. Then f restricted to $R_{n+1} = R_n \cup \bigcup\{R_{n+1}(t) : l_T(t) = n\}$ is a freedom classifiable multisequence. Finally, f restricted to $\bigcup_{n<\omega} R_n$ is a freedom classifiable submultisequence of f. ∎

A convergent (extended) multisequence $f : T \to X$ is called *transverse* (resp., *antitransverse)* at $t \in T$ if $t_n > (t, n)$ for each $n < \omega$ implies that $f(t) \in \lim f(t_n)_n$ (resp., if $f(t) \notin \mathrm{adh_{Seq}}\, f(t_n)_n$). An extended multisequence $f : T \to X$ is called *transversely closed* at t if $\mathrm{adh_{Seq}}\, f(t_n)_n \subset \{f(t)\}$ for every sequence $t_n \geq (t, n)$, where

$$\mathrm{adh_{Seq}}(x_n)_n = \bigcup\nolimits_{(y_k)_k \succ (x_n)_n} \lim\, (y_k)_k.$$

The notions above are meaningful for t of rank greater than 1; those of rank 0 and 1 are trivially transverse and antitransverse.

With a convergent (extended) multisequence $f : T \to X$ we associate a *transversality function* $\zeta = \zeta_f$ from T to $\{-, +, *\}$ defined by

$$\zeta(t) = \zeta_f(t) = \begin{cases} - & \text{if } f \text{ is antitransverse at } t; \\ + & \text{if } f \text{ is transverse at } t; \\ * & \text{otherwise.} \end{cases}$$

We shall also write $\zeta_f(t) = *$ if the value is either unknown or irrelevant. An extended multisequence is said to be *transversality classifiable* if it is either transverse or antitransverse at every element (of rank greater than 1). An extended multisequence is called *classifiable* if it is both freedom and transversality classifiable at every point. In this case, we use the symbol

$$\varphi(t)_{\zeta(t)} \tag{10.2}$$

to denote the *freedom* and the *transversality* of t.

As we shall see the sequential order can be estimated from above by the rank of convergent multisequences (Proposition 11.1), and from below by the rank of convergent, antitransverse multisequences with closed range (Proposition 11.10). Example 11.5 shows that in general multisequences used for such estimates are not free.

Certain multisequences, alternating free and stationary points, turned out to be a useful tool for estimation of sequential order of products of topologies in terms of some ordinal functions on the component topologies [13],[12],[11]. A *multifan* is a multisequence which is stationary at all the non-maximal points of even level, and free at all the non-maximal points of odd level. An *arrow* is a multisequence which is free at all the non-maximal points of even level, and stationary at all the non-maximal points of odd level. A multifan (arrow) is *correct* if all the stationary points are antitransverse and all the free points are transverse. The supremum of the ranks of correct multifans, which converge to x, is called the *fascicularity* of x and is denoted by $\lambda(x)$, and the supremum of the ranks of correct arrows, which converge to x, is called the *sagittality* of x and is denoted $\mu(x)$.

11. Sequentiality and sequential order

Proposition 11.1. *If $\sigma(x;A)=\beta$ then there exists on A a monotone multisequence of rank β that converges to x.*

Proof. If $\sigma(x;A)=0$, then $x\in A$, thus the constant multisequence $f:\{o\}\to A$ with $f(o)=x$, converges to x and is of sequential order 0. Suppose that the claim holds for the sequential order less than $\beta>0$ and let $\sigma(x;A)=\beta$. This means that $x\in\mathrm{adh}^{\beta}_{\mathrm{Seq}}A$ but not in $\mathrm{adh}^{<\beta}_{\mathrm{Seq}}A$, in particular, there is a (assigned) sequence f on $\mathrm{adh}^{<\beta}_{\mathrm{Seq}}A$ such that $x\in\lim f$. Therefore there exists a non-decreasing sequence $(\alpha_n)_n$ of ordinals less than β such that $f(n)\in\mathrm{adh}^{\alpha_n}_{\mathrm{Seq}}A$, and $\lim_n(\alpha_n+1)_n=\beta$. By the inductive assumption, there exists a monotone multisequence f_n on A of rank α_n such that $f(n)\in\lim f_n$. Hence the confluence $\hat{f}\looparrowleft_{n<\omega}\hat{f}_n$ (where $\hat{f},\hat{f}_n$ are convergent multisequences which extend f and f_n respectively) is monotone of rank not greater than β, and converges to x. In fact, it is of rank β , for if $g:T\to A$ is an extended multisequence that converges to x and is of rank $\alpha<\beta$, then $g(t)\in\mathrm{adh}^{\alpha}_{\mathrm{Seq}}A$ contrary to the assumption. ∎

As a corollary of Proposition 11.1, we get the following instrumental characterization of sequentiality and sequential order of topologies in terms of multisequences.

Theorem 11.2. *A topology is sequential (of order $\le\beta$) if and only if $x\in\mathrm{cl}\,A$ implies the existence of a (monotone) multisequence (of rank $\le\beta$) on A that converges to x.*

Proof. A topological space X is sequential (of order $\le\beta$) if and only if is the least ordinal α such that $\mathrm{cl}\,A=\mathrm{adh}^{\alpha}_{\mathrm{Seq}}A$ for every $A\subset X$, is less than or equal to β. Then, by Proposition 11.1, $x\in\mathrm{cl}\,A$ implies the existence on A of a multisequence of rank less than or equal to β that converges to x. As every multisequence admits a monotone submultisequence, the necessity follows. Conversely, if X is not sequential (of order $\le\beta$), then there exist $A\subset X$ and $x\in X$ such that $\sigma_X(x,A)>\beta$. By Proposition 11.1, there is no multisequence on A of rank less than or equal to β. ∎

Kratochvíl proved [22, Theorem 1] that if A is a subset of a sequential topological space, then $x\in\mathrm{cl}\,A$ if and only there is a hypersequence on A that converges to x, in other words there is $F:\omega^\omega\to A$ and $x\in\lim F(\mathcal{F})$ where $\mathcal{F}$ is the universal filter on ω^ω. Of course, the existence of a filter on A converging to x implies $x\in\mathrm{cl}\,A$ for any topology. The converse implication is an immediate consequence of Theorem 11.2 and Propositions 9.2 and 9.3. Indeed, if a multisequence on a subset A of a topological space converges to x, then the image by that multisequence of its contour is a filter on A that converges to x. Therefore, as we shall see in Section 13, [22, Theorem 1] holds also in subsequential spaces.

Proposition 11.3. *A topology τ is sequential if and only if* $\mathrm{cl}_\tau A = \mathrm{adh}^{\sigma(\tau)}_{\mathrm{Seq}\,\tau} A$ *for every* $A \subset |\tau|$.

Proof. By definition, a topology τ is sequential if and only if $\mathrm{cl}_\tau A \subset \mathrm{cl}_{\mathrm{Seq}\,\tau} A$ for every $A \subset |\tau|$. A subset A of $|\tau|$ is sequentially closed if and if $\mathrm{adh}_{\mathrm{Seq}\,\tau} A \subset A$. This implies that $\mathrm{adh}^{\gamma}_{\mathrm{Seq}\,\tau} A \subset A$ for every ordinal γ, hence in particular $\mathrm{adh}^{\sigma(\tau)}_{\mathrm{Seq}\,\tau} A \subset A$. Because $\mathrm{adh}^{\sigma(\tau)}_{\mathrm{Seq}\,\tau}$ is a topological closure operation, every sequentially closed set is closed if and only if $\mathrm{cl}_\tau A \subset \mathrm{adh}^{\sigma(\tau)}_{\mathrm{Seq}\,\tau} A$ for every $A \subset |\tau|$. The inclusion $\mathrm{cl}_{\mathrm{Seq}\,\tau} A \subset \mathrm{cl}_\tau A$ holds for every topology τ. ∎

Proposition 11.4. *If X is either a closed or an open subset of a sequential topological space Y, then X is sequential of order not greater than $\sigma(Y)$.*

Proof. Let Y be sequential, X a subset of Y and let $x \in \mathrm{cl}_X A$, where $A \subset X$. Then there exists a multisequence $f : \mathrm{Max}\, T \to A$ such that $x \in \lim_Y f$. If X is closed, then $\mathrm{cl}_Y A \subset \mathrm{cl}_Y X = X$, hence $x \in \lim_X f$, which shows that X is sequential and $\sigma(X) \le \sigma(Y)$. Suppose now that X is open. As $x \in \lim_Y f$, there is an extension $\hat{f} : T \to Y$ such that $\hat{f}(o_T) = x$ and $\hat{f}(t) = \lim_Y(\hat{f}(t,n))_n$ for every $t \in T \setminus \mathrm{Max}\, T$. As $x \in X$ and X is open in Y, by induction, for every $t \in T \setminus \mathrm{Max}\, T$, there exists there is $n_t < \omega$ such that $\hat{f}(t,n) \in X$ for $n \ge n_t$. It follows that there exists a subcascade S of T such that $\hat{f}(S) \subset X$ hence the restriction of f to $\mathrm{Max}\, S$ is a multisequence on A that converges to x in X. Moreover $\sigma(X) \le \sigma(Y)$, because $r(S) = r(T)$. ∎

The fact that $\sigma_\tau(x) = \beta$ does not imply that there exists a free multisequence of rank β that converges to x in τ. In fact,

Example 11.5. *Consider a disjoint union $X = \bigcup_{n<\omega} X_n$, where X_n is a monotone sequential cascade of rank n, and let Y be the quotient of X obtained after the identification of $\{o_{T_n} : n < \omega\}$ to o. Then $\sigma_Y(o) = \omega_0$, but all the free convergent multisequences are of finite rank. If $f_n : \mathrm{Max}\, X_n \to \mathrm{Max}\, X_n$ stands for the identity multisequence and $g : \omega \to Y$ is the constant sequence $g(n) = o$, then $g \looparrowleft_n \{f_n : n < \omega\}$ is a multisequence on $\mathrm{Max}\, Y$ of rank ω_0 that converges to o, but it is not free.*

However

Proposition 11.6. *If $\sigma(x)$ is finite greater than 0, then there exists a set A and a free multisequence $f : T \to A$ of rank $\sigma(x)$ which converges to x.*

Proof. Obvious for $\sigma(x) = 1$ and almost obvious for each finite $\sigma(x)$. In fact, if $n > 1$ and the claim holds for the sequential order n, then $x \in \mathrm{adh}^{n+1}_{\mathrm{Seq}} A \setminus \mathrm{adh}^{n}_{\mathrm{Seq}} A$, then there is on $\mathrm{adh}^{n}_{\mathrm{Seq}} A$ a free sequence f, which converges to x and $f(k) \notin \mathrm{adh}^{n-1}_{\mathrm{Seq}} A$ for almost all k for otherwise x would belong to $\mathrm{adh}^{n}_{\mathrm{Seq}} A$. By the inductive assumption, for almost every k there exists a free multisequence f_k on A, of rank n, which converges to $f(k)$. The confluence $f \looparrowleft_{k<\omega} f_k$ is a sought free multisequence. ∎

In case of infinite rank, there are two types of sequential orders: one that can be characterized with the aid of free multisequences (like in the construction used to prove Proposition 2.6); the other, where this is not possible (Example 11.5).

If in a Hausdorff sequential topological space x is of sequential order 1 (that is, Fréchet), then there is a convergent injective sequence (that is, a multisequence of rank 1), which is a homeomorphic embedding (because of compactness), and this statement can be easily generalized to Hausdorff sequential topological spaces of finite order. This is no longer the case for infinite order.

Example 11.7. [13, Example 1.2] *Let $T \subset \Sigma$ be a sequential cascade of rank ω_0 in which the convergence of sequences is defined as follows: $o \in \lim(x_k)_k$ whenever there exists $m < \omega$ and a sequence $(n_k)_k$ of natural numbers tending to ∞ such that $n_k \leq x_k$ and $h_T(x_k) \leq m$ for all k. If $o \neq t \in \lim(x_k)_k$, then $(x_k)_k$ is finer than the cofinite filter of $T^+(t)$. The associated topology is sequential and coarser than the natural topology of the cascade. Hence its sequential order is not greater than ω_0. Moreover the sequential order is not finite. Indeed, if a multisequence $f : \operatorname{Max} S \to T$ of finite rank converges to o, then there exist $m < \omega$ and an eventual subcascade S_0 of S such that the level of $f(\operatorname{Max} S)$ is bounded by m. As the level is unbounded on $\operatorname{Max} T$ and $o \in \mathrm{cl}_{\mathrm{Seq}}(\operatorname{Max} T)$, the sequential order of o is infinite. Take now any free bisequence $f : S \to T$, which converges to o. By definition, there is $m < \omega$ such that $h_T(f(k)) \leq m$ for each $k < \omega$ and a sequence $(n_k)_k$ tending to ∞ and such that $n_k \leq f(k)$ for almost all k. Hence, for almost each $k < \omega$ there is p_k such that $h_T(f(k, p_k)) \leq m + 1$, and thus $(f(k, p_k))_k$ converges to o_T, but, of course, $(k, p_k))_k$ does not converge to o_S.*

Lemma 11.8. *Let T be a sequential cascade, X a Hausdorff topological space, and $f : T \to X$ be a free, antitransverse continuous map. If a sequence of distinct elements $(f(t_k))$ converges to $f(t)$, then $t_k \in T^+(t)$ for almost all k.*

Proof. Suppose that on the contrary the set $A = \{k < \omega : t_k \notin T^+(t)\}$ is infinite, and let t_∞ be a maximal element of T with the property that $t_\infty \leq t_k$ for infinitely many k from A. Since the terms of (t_k) are distinct, there is a sequence (n_k) tending to ∞ such that $(t_\infty, n_k) \leq t_k$ for infinitely many $k \in A$. Because f is antitransverse and $\lim(f(t_k))_k \neq f(o)$, one has $(t_\infty, n_k) = t_k$ for infinitely many $k \in A$ and $t_\infty = t$. Hence $t_k \in T^+(t)$ for (infinitely many) $k \in A$, which is a contradiction. ∎

Proposition 11.9. *Let T be a sequential cascade, X a Hausdorff topological space, and $f : T \to X$ a free, antitransverse continuous map with closed image. If a free multisequence $g : \operatorname{Max} S \to f(\operatorname{Max} T)$ converges to $f(o_T)$ then g is a submultisequence of $f|_{\operatorname{Max} T}$.*

Proof. As g restricted to $S^+(s)$ is a free sequence which converges to $g(s)$ for every $s \in S \setminus \operatorname{Max} S$, there is an injective map $j_s : S^+(s) \cup \{s\} \to T$ such that

$g(s,k) = f(j_s(s,k))$ and $g(s) = f(j_s(s))$. By Lemma 11.8, $j_s(k) \in T^+(j_s(s))$ for almost all k. By the well-founded induction, if an injective map j_w is defined so that $j_w(S^+(w)) \subset T^+(j_w(w))$ and $g(w,k) = f(j_w(w,k))$ and $g(w) = f(j_w(w))$ for every successor w of s, then $j_s(w) = j_w(w)$ for almost all $w \in S^+(s)$. In this way, proceeding by induction on levels of S we construct an eventual subcascade R of S and a map $j : R \to T$ which is continuous, maps o_R onto o_T and $\operatorname{Max} R$ into $\operatorname{Max} T$, and such that $j(R^+(s)) \subset T^+(j(s))$ for each non-maximal element s of R, and $g(s) = f(j(s))$ for each $s \in \operatorname{Max} R$. ∎

The following proposition is partially converse of Proposition 11.1.

Proposition 11.10. *If f is a free antitransverse transversely closed multisequence which converges to x in a Hausdorff topological space, then $\sigma(x) \geq r(f)$.*

Proof. Inducing on the rank with the aid of Lemma 11.8. ∎

The existence of free, antitransverse continuous maps with closed image restricts the applicability of the criterion above. Indeed, the only such maps in locally sequentially compact topological spaces are trivial (more precisely, convergent sequences).

12. Sequentiality and sequential order of products

Typical applications of multisequences consist of some estimates of sequential order, which often results in establishing the sequentiality or Fréchetness of some classes of spaces.

A most eloquent example of the method of multisequences is a proof [27] of the following classical theorem of T. K. Boehme [4].[24]

Theorem 12.1. *The product of a (sequentially regular) locally countably compact, sequential topology and of a sequential topology is sequential.*

Recall that if Ω is a subset of a Cartesian product $X \times Y$, then $\Omega x = \{y : (x,y) \in \Omega\}$ and $\Omega^- y = \{x : (x,y) \in \Omega\}$ are respectively, the image of x, and the inverse image of y by the relation Ω. More generally, for $A \subseteq X$ and for $B \subseteq Y$,

$$\Omega A = \bigcup_{x \in A} \Omega x, \quad \Omega^- B = \bigcup_{y \in B} \Omega^- y.$$

Let T be a sequential cascade and $f_k : T \to X_k$ an extended multisequence for $1 \leq k \leq m$. Then the *diagonal product* is defined by

$$\Big(\bigotimes_{1 \leq k \leq m} f_k\Big)(t) = (f_1(t), f_2(t), \ldots, f_m(t)).$$

[24] I have given already this example in [6]

Proof. Let X be a sequential, countably compact (hence sequentially compact) topological space and let Y be a sequential topological space. Consider $A \subset X \times Y$ and $(x, y) \in \operatorname{cl} A$. By definition, $(V \times W) \cap A \neq \varnothing$ for every neighborhood V of x and each neighborhood W of y; equivalently, $AV \cap W \neq \varnothing$, that is $y \in \operatorname{cl}_Y(AV)$ for every (closed) neighborhood V of x. Since Y is sequential, by Theorem 11.2 there exists a multisequence on AV such that its extension $g_V : T_V \to Y$ converges to y. It follows that for every $t \in \operatorname{Max} T_V$, there is $f_V(t) \in V$ so that $(f_V(t), g_V(t)) \in A$. Because V is sequentially compact, by Proposition 7.8, f_V admits a submultisequence converging to some $x_V \in X$. Let us denote by $f_V^0 : T_V^0 \to X$ a converging extension of a submultisequence of f_V and by $g_V^0 : T_V^0 \to Y$ the corresponding extension of the submultisequence of g_V. Of course, g_V^0 converges to y.

Consider now the collection $\mathcal{N}(x)$ of the sequentially closed neighborhoods of x. Given $V \in \mathcal{N}(x)$, the limit x_V of f_V^0 belongs to V, and since X is sequentially regular, $x \in \operatorname{cl}_X\{x_V : V \in \mathcal{N}(x)\}$. Because X is sequential, there exists a multisequence on $\{x_V : V \in \mathcal{N}(x)\}$ whose extension $j : S \to X$ converges to x.

The confluence $S \looparrowleft \{T_V^0 : V \in \mathcal{N}(x)\}$ is defined so that $(s, t) \in S \looparrowleft \{T_V^0 : V \in \mathcal{N}(x)\}$ whenever $s \in \operatorname{Max} S$, $j(s) = x_V$ and $t \in T_V^0$. The corresponding confluence of extensions

$$j \looparrowleft_{\mathcal{N}(x)} f_V^0 = j \looparrowleft \{f_V^0 : V \in \mathcal{N}(x)\}$$

converges to x. On the other hand, for the constant multisequence $k : S \to Y$ such that $k(s) = y$ for every $s \in S$, the confluence

$$k \looparrowleft_{\mathcal{N}(x)} g_V^0 = k \looparrowleft \{g_V^0 : V \in \mathcal{N}(x)\}$$

converges to y. Therefore their diagonal product

$$(j \looparrowleft_{\mathcal{N}(x)} f_V^0) \otimes (k \looparrowleft_{\mathcal{N}(x)} g_V^0)$$

converges to (x, y) and its restriction to $\operatorname{Max}(S \looparrowleft_{\mathcal{N}(x)} T_V^0)$ is a multisequence on A. By Theorem 11.2, $X \times Y$ is sequential. ∎

The proof above enables us to estimate the sequential order of $X \times Y$, namely

$$\sigma(X \times Y) \leq \sigma(Y) + \sigma(X).$$

Indeed, the converging multisequences in Y can be taken of rank less than or equal to $\sigma(Y)$, and the converging multisequences in X of rank less than or equal to $\sigma(X)$. The rank of the resulting confluence is less than or equal to $\sigma(Y) + \sigma(X)$ because of (2.10). Similar results were obtained in [27] with the aid of multisequences.

A similar technique was used in [10, Two-fold theorem] to establish a sufficient condition for the Fréchetness of the product of two Fréchet topologies. This result was a common generalization of all the (known to us) theorems concerning the question ([23],[1],[26]). Since then, a characterization in [20] of topologies whose product with every strongly Fréchet topology is (strongly) Fréchet (as *productively Fréchet* topologies), generalized [10, Two-fold theorem].

Now we can use Propositions 11.1 and 11.10 together with the following two facts

Proposition 12.2. *If there is k such that f_k is free (resp., antitransverse) at t, then $\bigotimes_{1\leq k\leq m} f_k$ is free (resp., antitransverse) at t.*

Proposition 12.3. *If f_k is stationary (resp., transverse, transversely closed) at t for every k, then $\bigotimes_{1\leq k\leq m} f_k$ is stationary (resp., transverse, transversely closed) at t.*

In terms of freedom and transversality functions of extended multisequences, we seek to construct in product spaces, convergent transversely closed multisequences which are free and antitransverse, that is, such that $\varphi(t)_{\zeta(t)} = 1_-$ (see (10.2)) for every t of rank greater than 1. By virtue of Proposition 11.10, the sequential order is not less than the rank of such multisequences.

Suppose that $f : T \to X$ is a correct multifan and $g : T \to Y$ a correct arrow. In other words, the elements of T on the levels $0, 1, 2, 3, \ldots$ are of the type $0_-, 1_+, 0_-, 1_+, \ldots$ for f and $1_+, 0_-, 1_+, 0_-, \ldots$ for g. Therefore the corresponding elements for $f \otimes g$ are of the type $1_-, 1_-, 1_-, 1_-, \ldots$, which yields a lower bound for the sequential order of the product, provided that $f \otimes g$ is transversely closed. The last property holds if the component multisequences are transversely closed at t of the type 0_-.

On generalizing a result of Nogura and Tanaka [24, Theorem 3.8], it was shown in [13, Theorem 3.1] that each correct multifan in a regular Fréchet topology admits an (extended) submultisequence which is transversely closed (at every t of the type 0_-). Therefore, on recalling the definitions of sequential order σ, of fascicularity λ and of sagittality μ,

Theorem 12.4. [13, Theorem 3.3] *If X, Y are regular Fréchet topological spaces, then the sequential order σ fulfills*

$$\sigma(x, y) \geq 1 + \max[\min(\lambda(x), \mu(y)), \min(\mu(x), \lambda(y))]$$

for each $x \in X, y \in Y$.

On the other hand, in *sequentially subtransverse* spaces (such that each convergent extended multisequence, which is free at o, admits a submultisequence transverse at o), every convergent extended multisequence in $X \times Y$, which is antitransverse and free at every point, admits a submultisequence of the form $f \otimes g$ where one of the components is a correct multifan and the other a correct arrow. An important classical example of a sequentially subtransverse, Fréchet (topological) spaces are *Lašnev spaces* (closed images of metrizable spaces). Hence

Theorem 12.5. [13, Theorem 5.4] *If X, Y are Lašnev spaces, then the sequential order σ fulfills*

$$\sigma(x, y) = 1 + \max[\min(\lambda(x), \mu(y)), \min(\mu(x), \lambda(y))]$$

for each $x \in X, y \in Y$.

Recall that Fréchet topologies are precisely the sequential topologies with the sequential order not greater than 1. It is possible to construct regular Lašnev topologies of arbitrary fascicularity and sagittality.

If we consider products of more than 2 topologies, then estimates of sequential order are much more complex. In this situation we can compose multifans and arrows on splitting stationary indices to several successive levels of stationary indices. The optimization of this procedure in order to construct free, antitransverse (transversely closed) multisequences of maximal rank leads to some problems of transfinite combinatorics [12].

For example, consider 4 spaces with a correct arrow of rank 3 and correct multifans of ranks $2, 2$ and 4, respectively. Then we build up in the product a free antitransverse multisequence of rank 6 with the aid of the following pattern (here the columns correspond to the levels of the resulting multisequence, while the rows correspond to respective component multisequences, and the last row to the resulting one).

1_+			0_-	1_+		
0_-	1_+			0_-	1	
	0_-	1_+				
		0_-	1_+			
1_-	1_-	1_-	1_-	1_-	1	0

In the table I put only essential indices needed to produce 1_- in the product multisequences. The entries after 1_+ and 0_- are just stationary points split into several levels. At the last column there are maximal (hence stationary) elements; the elements in the one before last are free and being of rank 1, transversality is not defined for them. If the component spaces are Fréchet, then 1_+ automatically follows 0_- in each row.

A *node* of a multisequence f is its stationary, antitransverse argument. The *nodality* $\nu(f)$ is the rank of the tree of nodes of f. The nodality is *active* if the multisequence is free at the root. The *nodality* of x is the supremum of the nodalities of multisequences that converge to x. If f is a correct multifan (resp., arrow) and $\lambda(f) = \lambda_0(f) + \lambda_1(f)$ (resp., $\mu(f) = \mu_0(f) + \mu_1(f)$) is the decomposition of the fascicularity (resp., sagittality) into its transfinite and its finite parts, then

$$\nu(f) = \lambda_0(f) + \frac{\lambda_1(f)}{2},\ \nu(f) = \mu_0(f) + \frac{\mu_1(f) - 1}{2}.$$

A *listing* g of the set of ordinals $\{\nu_1, \nu_2, \ldots, \nu_m\}$ is an injection from an ordinal γ to $\bigcup_{k=1}^{m} (\nu_k \times \{k\})$ such that no consecutive values of g belong to the same $\nu_k \times \{k\}$, and such that its restriction to $g^-(\nu_k \times \{k\})$ is increasing for every $1 \leq k \leq m$. The supremum of ordinals γ for which there exists a listing of $\{\nu_1, \nu_2, \ldots, \nu_m\}$ is called the *listing number* of $\{\nu_1, \nu_2, \ldots, \nu_m\}$ and is denoted by $\kappa(\nu_1, \nu_2, \ldots, \nu_m)$. The listing number was calculated in [12] in terms of combinatorial formulas involving the decomposition of ordinals into indecomposable ordinals.

Theorem 12.6. [12, Theorem 9.1] *If X, Y are regular Fréchet spaces,*

$$\sigma(x_1, x_2, \ldots, x_m) \geq \kappa((\nu(x_1), \nu(x_2), \ldots, \nu(x_m)),$$

for each $x \in X, y \in Y$, provided that one of the nodalities is active.

Theorem 12.7. [12, Corollary 10.4] *If X, Y are Lašnev spaces of finite nodalities,*

$$\sigma(x_1, x_2, \ldots, x_m) = \kappa((\nu(x_1), \nu(x_2), \ldots, \nu(x_m)),$$

for each $x \in X, y \in Y$, provided that one of the nodalities is active.

I do not know if the assumption of finiteness above can be removed.

The technique of multisequences has been also used to obtain precise upper bounds for the sequential order of infinite products of sequential sequentially compact topologies. Recall that a topological space X is *cofinal-cofinal* (traditionally called α_3 [1]) if for every fan $f : \operatorname{Max} T \to X$ (that is, a multisequence of rank 2, which is stationary at the root) that converges to x there is a submultisequence $g : \operatorname{Max} S \to X$ such that $x \in \lim f(\mathcal{N})$, where $\mathcal{N}$ is the cofinite filter of $\operatorname{Max} S$.

Theorem 12.8. [11, Corollary 3.2] *If X_n is sequential, sequentially compact for $n < \omega$, then $\prod_{n<\omega} X_n$ is sequential, and*

$$\sigma(\prod_{n<\omega} X_n) \leq \sup\{\sigma(\prod_{k\leq n} X_k) + 1 : n < \omega\}.$$

If moreover X_n is cofinal-cofinal for each $n < \omega$, then

$$\sigma(\prod_{n<\omega} X_n) = \sup\{\sigma(\prod_{k\leq n} X_k) : n < \omega\}.$$

This theorem improves

Corollary 12.9. [3, Theorem 3] *Each countable product of sequential, sequentially compact topologies is sequential.*

A *compact MAD topology* is obtained as the Alexandrov compactification of the topology on $\mathcal{A} \cup \omega$, where $\mathcal{A}$ is a maximal almost disjoint family of infinite subsets of ω, and $A = \lim A^{\natural}$ for each $A \in \mathcal{A}$ (where $A^{\natural}$ stands for the cofinite filter of A). Each compact MAD topology is sequential, sequentially compact and cofinal-cofinal.

Theorem 12.10. [11, Corollary 3.8] *The sequential order of each finite (hence of each countable) power of compact MAD topologies is* 2.

13. Subsequential topologies

A subspace of a sequential (topological) space is called *subsequential* . A subspace of a T_α sequential space,[25] is called T_α *-subsequential.* The *subsequential rank* of a subsequential topology X is the least ordinal α such that there exists a sequential topology of rank α in which X can be homeomorphically embedded. Therefore the restriction of every sequential cascade T to $\operatorname{Ext} T$ is a subsequential space, and

Proposition 13.1. *If T is an essentially monotone cascade of rank greater than 1, then $\operatorname{Ext} T$ (with its natural topology) is not sequential.*

Proof. Let T be an essentially monotone cascade with $r(T) > 1$. It is enough to show that the only convergent sequences on $\operatorname{Ext} T$ are stationary. Indeed, if f is a free convergent sequence on $\operatorname{Ext} T$, then $o_T \in \lim f$ because o_T is the origin of the prime topology of $\operatorname{Ext} T$.[26] Hence $\hat{f} : \omega + 1 \to T$ which extends f so that $\hat{f}(\omega) = o_T$ is continuous, thus by Proposition 7.2, $r_T^-(o_T) = 1$ and since T is essentially monotone, $r_T(o_T) = 1$. ∎

We shall see that the collection of natural topologies on the sets of extremal elements of sequential cascades of all countable ranks characterizes all the subsequential topologies. If T is a sequential cascade of rank α then the coarsest free filter that converges to o_T in $\operatorname{Ext} T$ is called a *sequential contour* of rank α.

If $\mathcal{A}$ is a family of subsets of X, then the *grill* $\mathcal{A}^\#$ of $\mathcal{A}$ is defined by

$$\mathcal{A}^\# = \{H \subset X : \forall_{A \in \mathcal{A}}\, H \cap A \neq \varnothing\}.$$

Theorem 13.2. [15] *A topological space X is subsequential of rank α if and only if $x \in \operatorname{cl}_X A$ implies the existence of a sequential contour $\mathcal{F}$ on ω of rank α and of a map $f : \omega \to A$ such that $x \in \lim_X f(\mathcal{F})$.*

Proof. Let X be a subspace of a sequential topological space Y of rank α. If $A \subset X$ and $x \in \operatorname{cl}_X A \subset \operatorname{cl}_Y A$, then by Theorem 11.2, there exists a free multisequence $f : \operatorname{Max} T \to A$ such that $x \in \lim_Y f$. The sequential contour $\int T$ of rank α is the trace of the neighborhood of o_T on $\operatorname{Max} T$. By topologicity, $f(\int T)$ converges to x in Y and since it contains X, also for the topology of X.

Conversely, if the condition is fulfilled, then let $\mathbb{F}(x)$ be a base of sequential contours of rank not greater than α, which converge to x in X. By assumption, for every x and A such that $x \in \operatorname{cl}_X A$, there exists $\mathcal{F} \in \mathbb{F}(x)$ such that $A \in \mathcal{F}^\#$. For each $x \in X$ and every $\mathcal{F} \in \mathbb{F}(x)$, let $T_\mathcal{F}$ be a sequential cascade such that $\int T_\mathcal{F}$ is homeomorphic to $\mathcal{F}$. Consider the simple sum of X and of all such $T_\mathcal{F}$, and define Y as the quotient by identifying $\operatorname{Max} T_\mathcal{F}$ with a subset of X through a homeomorphism with $\mathcal{F}$ and $o_{T_\mathcal{F}}$ with

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[25] Here T_α stands for the separation property, like T_0, T_1, T_2 or *Hausdorff*, T_3 or *regular*, T_4 or *completely regular*, T_5 or *normal.*

[26] Observe that every sequential prime topology is Fréchet.

x. Then X is a subspace of Y because $\operatorname{Max} T_{\mathcal{F}}$ is discrete and the trace of $\mathcal{N}_{T_{\mathcal{F}}}(o)$ on $\operatorname{Max} T_{\mathcal{F}}$ coincides $\mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}(x)$ and each $x \in X$.

I claim that Y is sequential. Indeed, let $y \in \operatorname{cl}_Y A$. If $y \in X$ and $A \subset X$, then by assumption, there exists $\mathcal{F} \in \mathbb{F}(y)$ such that $A \in \mathcal{F}^{\#}$, hence there is a multisequence $f : \operatorname{Max} T_{\mathcal{F}} \to A$ that converges to y in Y. If $y \in X$ and $A \subset T_{\mathcal{F}} \backslash X$ with $\mathcal{F} \in \mathbb{F}(y)$, then use the sequentiality of $T_{\mathcal{F}}$. If $y \in T_{\mathcal{F}} \backslash X$ and $A \subset T_{\mathcal{F}} \backslash X$, then use the sequentiality of $T_{\mathcal{F}}$. Finally if $y \in T_{\mathcal{F}} \backslash X$ and $A \subset X$, then there is a subset B of $\operatorname{Max} T_{\mathcal{F}}$ such that $y \in \operatorname{cl}_Y B$ and $B \subset \operatorname{cl}_Y A$. Then for each $x \in B$ there is $\mathcal{F}_x \in \mathbb{F}(x)$ with $A \in \mathcal{F}_x$, and thus a sequential cascade $T_{\mathcal{F}_x}$ converges to x. The confluence of $T_{\mathcal{F}}$ with $\{T_{\mathcal{F}_x} : x \in B\}$ is a multisequence on A that converges to x. ∎

It also follows from the proof above that a space T_1-subsequential if and only if it is subsequential and T_1.[27] It can be seen that if $\mathcal{F}$ is a sequential contour (on ω) and $f : \omega \to \omega$, then there is a sequential contour $\mathcal{G}$ such that $r(\mathcal{G}) \leq r(\mathcal{F})$ and $\mathcal{G} \geq f(\mathcal{F})$. Hence

Corollary 13.3. *A topological space X is subsequential if and only if $x \in \operatorname{cl}_X A$ implies the existence of a sequential contour $\mathcal{G}$ on A such that $x \in \lim_X \mathcal{G}$.*

A filter is called *subsequential* if it is an intersection of sequential contours. Denote by $\mathbb{S}(X)$ the set of sequential contours on X. Sequential contours are free filters with the exception of that of rank 0 (which is the principal ultrafilter of a singleton).

Corollary 13.4. *A topology is subsequential if and only if each neighborhood filter is subsequential.*

Proof. By Theorem 13.2, X is a subsequential space if and only if $A \in \mathcal{N}(x)^{\#}$ implies the existence of a sequential contour $\mathcal{F}$ such that $\mathcal{N}(x) \vee A \leq \mathcal{F}$ for every $x \in X$. Hence, $\mathcal{N}(x)^{\#} = \bigcup_{\mathcal{N}(x) \leq \mathcal{F} \in \mathbb{S}(X)} \mathcal{F}$, and by duality, $\mathcal{N}(x) = \bigcap_{\mathcal{N}(x) \leq \mathcal{F} \in \mathbb{S}(X)} \mathcal{F}^{\#} \subset \bigcap_{\mathcal{N}(x) \leq \mathcal{F} \in \mathbb{S}(X)} \mathcal{F}$, because for a sequential contour $\mathcal{F}$ and $A \in \mathcal{F}^{\#}$, the filter $\mathcal{F} \vee A$ is also a sequential contour. Therefore

$$\mathcal{N}(x) = \bigcap_{\mathcal{N}(x) \leq \mathcal{F} \in \mathbb{S}(X)} \mathcal{F},$$

which completes the proof. ∎

A class of topologies is $\mathbb{F}$-*radial* if for every topology from the class $x \in \operatorname{cl} A$ implies the existence of $\mathcal{F} \in \mathbb{F}$ and a map f such that $A \in f(\mathcal{F})$ and $x \in \lim f(\mathcal{F})$. Theorem 13.2 states that subsequential topologies are precisely those *radial* with respect to sequential contours. This is analogous to Fréchet spaces, which are precisely the radial spaces with respect to sequential filters. Subsequential topologies can be also characterized in the way analogous to

[27] In general, a subsequential topology which is T_α (for $\alpha > 2$) need not be T_α-subsequential.

that used in the definition of sequential spaces (each sequentially closed set is closed). Let

$$\operatorname{adh}_{\mathbb{F}} A = \bigcup\nolimits_{\mathcal{F}\in\mathbb{F}} \lim \mathcal{F}.$$

Proposition 13.5. *A topology is subsequential if and only if each set closed for sequential contours is closed.*

Proof. If a topology is subsequential and A is closed for sequential contours, that is, $\operatorname{adh}_{\mathbb{F}} A \subset A$ then it is closed, because $\operatorname{adh}_{\mathbb{F}} = \operatorname{cl}$ by Corollary 13.3. To prove the converse, we notice that $\operatorname{adh}_{\mathbb{F}}^2 = \operatorname{adh}_{\mathbb{F}}$. In fact, if $x_0 \in \operatorname{adh}_{\mathbb{F}}(\operatorname{adh}_{\mathbb{F}} A)$, then there is a sequential cascade T such that $\operatorname{adh}_{\mathbb{F}} A \in \int T$ and $x_0 \in \lim \int T$. Now for every $x \in \operatorname{adh}_{\mathbb{F}} A$ there is a sequential cascade S_x such that $A \in \int S_x$ and $x \in \lim \int S_x$. Therefore $x_0 \in \lim \int (T \hookleftarrow_{x\in \operatorname{adh}_{\mathbb{F}} A} S_x)$ and $A \in \int (T \hookleftarrow_{x\in \operatorname{adh}_{\mathbb{F}} A} S_x)$. ∎

We infer that subsequentiality is a pointwise property, that is, to establish that a topology is subsequential, it is enough to prove, separately, that each neighborhood filter is subsequential. Recall that if X is a topological space and $p \in X$, then the *prime factor* X_p is a topological space which consists of the same underlying set for which all the elements are isolated with the (possible) exception of p, the neighborhood filter of which is $\mathcal{N}_X(p)$. Thus we have recovered [17, Corollary 5.2] that a topology is subsequential if and only if its every prime factor is subsequential.

The Rudin-Keisler order can be extended to filters by setting $\mathcal{F} \geq_{RK} \mathcal{G}$ if there is a map f such that $f(\mathcal{F}) \geq \mathcal{G}$.

Corollary 13.6. *If $\alpha < \beta$ then each asymptotically monotone contour of rank α is Rudin-Keisler strictly less than every asymptotically monotone contour of rank β.*

Proof. Suppose that $\mathcal{F}, \mathcal{G}$ are asymptotically monotone sequential contours such that $\alpha = r(\mathcal{F}) < r(\mathcal{G}) = \beta$. If $\mathcal{F} \geq_{RK} \mathcal{G}$, then there exist asymptotically monotone sequential cascades T and W of respective ranks α and β such that $\mathcal{F} = \int T$ and $\mathcal{G} = \int W$. Therefore there is a continuous non-trivial map $f : \operatorname{Ext} T \to \operatorname{Ext} W$, hence by Theorem 8.1, there exist an asymptotically monotone subcascade R of T and a continuous map $\hat{f} : R \to W$ such $\hat{f}|_{\operatorname{Ext} R} = f|_{\operatorname{Ext} R}$. By Proposition 7.3, $r(R) \geq r(W)$ and since $\alpha = r(\mathcal{F}) = r(T) = r(R) < r(\mathcal{F}) = r(W) = \beta$ we have a contradiction. ∎

The sequential contours are traces of $\mathcal{N}_{T\zeta}(o_{\Sigma})$ on the sections of Σ.

S. Franklin and M. Rajagopalan asked in [17] if a sequential topological space can have a sequential subspace of rank strictly greater than that of whole space. That this is the case under CH was proved by S. Watson [30].

Boldjiev and Malyhin proved in [5] that the class of sequential topologies is radial with respect to a single filter $\mathcal{F}$ on ω. In [14] it was observed that if every sequential topology is radial with respect to a filter $\mathcal{H}$ (on ω), then the class of $\mathcal{H}$-radial topologies includes that of subsequential topologies.

If $\{\mathcal{F}_\alpha : \alpha < \omega_1\}$ are sequential contours such that $r(\mathcal{F}_\alpha) = \alpha$ and $\mathcal{F}_\alpha \leq \mathcal{F}_\beta$ if $\alpha \leq \beta$, then $\sup_{\alpha<\omega_1} \mathcal{F}_\alpha$ is called a *supercontour*. It was shown in [14, Proposition 6.3] that the class of subsequential topologies is radial with respect to every supercontour. On the other hand, by Corollaries 13.3 and 13.6, if for a fixed filter $\mathcal{H}$, every subsequential topology is $\mathcal{H}$-radial, then the prime topology generated by $\mathcal{H}$ is not subsequential [14, Proposition 6.4].

Question 13.7. *Let $\mathcal{H}$ be a filter on ω such that each sequential (hence, each subsequential) topology is $\mathcal{H}$-radial. Is there any non-subsequential $\mathcal{H}$-radial topology?*

Question 13.7 leads to the notion of fractal filter: a filter $\mathcal{H}$ is *fractal* if for every $A \in \mathcal{H}^\#$ the filters $\mathcal{H}$ and $\mathcal{H}|_A$ are equivalent in the sense of Rudin-Keisler. It is straightforward that the prime topology determined by $\mathcal{H}$ is $\mathcal{H}$-radial if and only if $\mathcal{H}$ is fractal. It is clear that ultrafilters and cofinite filters are fractal.[28] Moreover, each (essentially monotone) sequential contour is fractal. If a supercontour $\mathcal{F}$ were fractal, then its prime space would be non-subsequential and $\mathcal{F}$-radial. This is possible under the Continuum Hypothesis.

Theorem 13.8 (CH)**.** [14, Theorem 4.8] *There exists a supercontour which is an ultrafilter.*

In [28] Starosolski studied supercontours and fractal filters. Question 13.7 however remains open.

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