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PROPERTIES TRANSFER BETWEEN TOPOLOGIES ON FUNCTION SPACES, HYPERSPACES AND UNDERLYING SPACES

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Abstract: Each collection α of families of subsets of X determines a topology $\alpha(X,Z)$ on the space of continuous maps $C(X,Z)$. Interrelations between local properties of $\alpha(X,\mathbb{R})$ and of $\alpha(X,\$)$ (on the hyperspace $C(X,\$)$), and with properties of a topological space X are studied in a general framework, which allows to treat simultaneously several classical constructions, like pointwise convergence, compact-open topology and the Isbell topology.

1. Introduction

The interrelation of properties of $C_\alpha(X,Z)$ with those of X and Z, is a fascinating theme. Here α is a collection of (openly isotone[1]) families of subsets of X, that defines a topology $\alpha(X,Z)$ on $C(X,Z)$ by a subbase

$$\{[\mathcal{A},O] : \mathcal{A} \in \alpha, O \in \mathcal{O}_Z\}, \tag{1.1}$$

where $[\mathcal{A},O] := \{f : f^-(O) \in \mathcal{A}\}$, $f^-(O) := \{x : f(x) \in O\}$, and $\mathcal{O}_Z$ is the set of open subsets of Z.

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[1]A family $\mathcal{A}$ of open sets is *openly isotone* if $B \in \mathcal{A}$ provided that B is open and there is an element $A \in \mathcal{A}$ such that $A \subset B$.

Its very special case, that of $C_p(X,\mathbb{R})$ (the space of real-valued functions with pointwise convergence) has attracted a lot of researchers, among whom A. V. Arhangel'skii (e.g., [2]). Its intermediate case of

$$\alpha = \alpha_{\mathcal{D}} := \{\mathcal{O}_X(D) : D \in \mathcal{D}\}\,, \tag{1.2}$$

where $\mathcal{D}$ is a family of subsets of X, and $\mathcal{O}_X(D) := \{O \in \mathcal{O}_X : D \subset O\}$, is the object of a book of McCoy and Ntantu [17].

Actually the said interrelation corresponds to the upper side of a quadrilateral

$$\begin{array}{ccc} X & \leftrightarrow & C_\alpha(X,\mathbb{R}) \\ \updownarrow & & \updownarrow \\ C_\alpha(X,\$^*) & \leftrightarrow & C_\alpha(X,\$) \end{array}$$

in which, of course, one can consider also other sides, as well as diagonals. Here $\$, \$^*$ stand for the two homeomorphic variants of the Sierpiński topology on $\{0,1\}$, so that $C(X,\$)$ can be identified with the hyperspace of X, and $C(X,\$^*)$ with the set O_X of open subsets of X.

It turns out that it is fruitful to study the three other sides in order to better grasp the interrelation of the upper side $X \leftrightarrow C_\alpha(X,\mathbb{R})$. Indeed,

(1) $C_\alpha(X,\$)$ is homeomorphic to $C_\alpha(X,\$^*)$;

(2) One can establish a dictionary of easy translations of elementary properties of $C_\alpha(X,\$^*)$ and α-properties of X;

(3) Under a separation condition (by real functions) one can evidence an intimate relationship between $C_\alpha(X,\mathbb{R})$ and $C_\alpha(X,\$)$.

More precisely, if X is completely regular and α is a compact web, then the neighborhood filter for $\alpha(X,\mathbb{R})$ of the zero function $\tilde{0}$ (that is, $\tilde{0}(x) = 0$ for each $x \in X$) belongs to the same *transferable* class as the neighborhood filter of $\varnothing$ for $\alpha(X,\$)$. Roughly speaking a *web* α on X is a collection of families of open subsets of X such that for each open subset Y there is $\mathcal{A} \in \alpha$ that can be reconstructed from its trace on Y. A web is *compact* if its every element $\mathcal{A}$ is a *compact family*.[2]

Compact (openly isotone) families on a topological space X coincide with the open sets of the *Scott topology* of $C(X,\$^*)$ (see, e.g. [11]). It was shown in [6] that each such a family is of the form $\bigcup_{K\in\mathcal{D}} \mathcal{O}_X(K)$, where $\mathcal{D}$ is a subfamily of compact subsets of X, if and only if X is *consonant*.

A collection $\alpha_{\mathcal{D}}$ of the type (1.2), where $\mathcal{D}$ is a network consisting of compact subsets of X, is a compact web. Moreover, if $\mathcal{D}$ is hereditarily

[2]Precise definitions are given before Lemma 4.7.

closed in a completely regular space X, then $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ is, in particular, a topological group (e.g., [17, Th. 1.1.7]), hence homogeneous. Therefore in order to prove a local (transferable) property of $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$, it is enough to establish it for the neighborhood filter of the constant function $\tilde{0}$.

Of course, in general, a hyperspace topology $\alpha(X,\$)$ is not homogeneous. As $\alpha(X,\$)$ and $\alpha(X,\$^*)$ are homeomorphic (by complementation), a property of $\mathcal{N}_{\alpha(X,\$)}(A)$ for $A \in C(X,\$)$ is also a property of $\mathcal{N}_{\alpha(X,\$^*)}(X \setminus A)$ and, as a rule, can be characterized in terms of the space $X \setminus A$ with the induced topology. Therefore a local property of $C_\alpha(X,\$)$ can be characterized by a hereditary (with respect to open subsets) property of X.

For general compact webs α on completely regular spaces, $C_\alpha(X,\mathbb{R})$ need not be even translation invariant. Therefore, *that* $C_\alpha(X,\$)$ has a local transferable property does not necessarily imply that $C_\alpha(X,\mathbb{R})$ has the same property. The implication holds for completely regular consonant spaces, because then α is of the form (1.2).

Nevertheless, some local properties of hyperspaces pass onto the corresponding function spaces thanks to a characterization of convergence of functions valued in topological spaces in terms of the corresponding hyperspace convergence of the preimages of closed sets. Consequently, each α-topology on $C(X,\mathbb{R})$ can be, in principle, characterized in terms of the corresponding α-topology on the hyperspace $C(X,\$)$, actually on its subset consisting of functionally closed subsets of X. By the way, it is why Georgiou, Iliadis and Papadopoulos studied properties of real-valued function spaces in terms of topologies on functionally open sets [9].

The present paper restricts its scope to topologies on function spaces (almost always real-valued) and to the corresponding hyperspace topologies. This is just one aspect of a general theory of convergence function spaces and hyperspace convergences that will be discussed in [7].

2. Open-set topologies

We denote the set of open subsets of X by either $C(X,\$^*)$ or $\mathcal{O}_X$. We use the latter convention to define $\mathcal{O}_X(x) := \{O \in \mathcal{O}_X : x \in O\}$, and by $\mathcal{O}_X(A) := \{O \in \mathcal{O}_X : A \subset O\}$. If now $\mathcal{A}$ is a family of subsets of X, then $\mathcal{O}_X(\mathcal{A}) := \bigcup_{A\in\mathcal{A}} \mathcal{O}_X(A)$. A family $\mathcal{A}$ of subsets of X is *openly isotone* if $\mathcal{O}_X(\mathcal{A}) = \mathcal{A}$.

If α is a non-empty collection of openly isotone families of subsets of

X, then (1.1) is a subbase of a topology on $C(X,Z)$, denoted by $\alpha(X,Z)$. The corresponding topological space is denoted by $C_\alpha(X,Z)$.

In particular, for a non-empty family $\mathcal{D}$ of subsets of X, the collection $\alpha := \alpha_{\mathcal{D}}$ is defined by

$$\alpha_{\mathcal{D}} := \{\mathcal{O}_X(D) : D \in \mathcal{D}\}, \tag{2.1}$$

and the symbol $C_{\alpha_{\mathcal{D}}}(X,Z)$ is abridged to $C_{\mathcal{D}}(X,Z)$. It is often required (e.g., [17]) that $\mathcal{D}$ be a (closed) *network* on X, that is, a family of closed sets such that for each $x \in X$ and $O \in \mathcal{O}_X(x)$ there is $D \in \mathcal{D}$ for which $x \in D \subset O$. However (1.1) is a topology subbase for each $\alpha = \alpha_{\mathcal{D}}$ provided that $\mathcal{D} \neq \varnothing$.

If $A \subset X$ and $B \subset Z$ then $[A,B] := \{f \in C(X,Z) : f(A) \subset B\}$. Therefore, $[\mathcal{O}_X(D), O] = [D,O]$ and thus

$$\big\{[\mathcal{A},O] : \mathcal{A} \in \alpha_{\mathcal{D}}, O \in \mathcal{O}_Z\big\} = \big\{[D,O] : D \in \mathcal{D}, O \in \mathcal{O}_Z\big\}.$$

Example 2.1. If $\mathcal{D} = [X]^{<\aleph_0}$, then

$$\big\{[F,O] : F \in [X]^{<\aleph_0}, O \in \mathcal{O}_Z\big\}$$

is a base of the topological space $C_p(X,Z)$ of pointwise convergence (here p abridges $[X]^{<\aleph_0}$).

Example 2.2. If $\mathcal{D} = \mathcal{K}_X$ (the family of compact subsets of X), then

$$\big\{[K,O] : K \in \mathcal{K}_X, O \in \mathcal{O}_Z\big\}$$

is a base of the topological space $C_k(X,Z)$ of compact-open topology (here k abridges $\mathcal{K}_X$).

We consider two complementary topologies on, respectively, the hyperspace $C(X,\$)$ and the set $C(X,\$^*)$ of open subsets of X. Here $\$$ and $\$^*$ are two homeomorphic avatars of the *Sierpiński topology* on $\{0,1\}$:

$$\$:= \{\varnothing, \{1\}, \{0,1\}\} \text{ and } \$^* := \{\varnothing, \{0\}, \{0,1\}\}.$$

The *indicator function* ψ_A of a subset A of X is defined by to be 0 on A and 1 out of A. If X is a topological space, then $\psi_A \in C(X,\$)$ if and only if A is closed, and $\psi_A \in C(X,\$^*) := \mathcal{O}_X$ if and only if A is open.

The *complementation* $^c : 2^X \to 2^X$ associates $A^c := X \setminus A$ with $A \subset X$. In order to avoid ambiguity, we denote the image of $\mathcal{A} \subset 2^X$ by the complementation by

$$\mathcal{A}_c := \{A^c : A \in \mathcal{A}\}.$$

The topology $\alpha(X,\$^*)$ on the set $C(X,\$^*)$ (of all open subsets of X) has α for a subbase, because, due to our convention, the subbase consists of $\{[\mathcal{A},\{0\}]:\mathcal{A}\in\alpha\}$, and $[\mathcal{A},\{0\}]=\left\{\psi_B\in C(X,\$^*):\psi_B^-(0)\in\mathcal{A}\right\}$ (by definition, $\psi_B^-(0)=B$).

If α is stable for finite intersections, then α is a base of $\alpha(X,\$^*)$. Hence the *neighborhood filter* $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ of $Y\in C(X,\$^*)$ is generated by

$$\{\mathcal{A}\in\alpha : Y\in\mathcal{A}\}\,.$$

In particular, for $\alpha=\alpha_{\mathcal{D}}$ a subbase for open sets is of the form

$$\{\mathcal{O}_X(D):D\in\mathcal{D}\}\,,$$

and $\alpha_{\mathcal{D}}$ is stable for finite intersections provided that $\mathcal{D}$ is stable for finite unions, so that

$$\mathcal{N}_{\alpha_{\mathcal{D}}(X,\$^*)}(Y)\approx\{\mathcal{O}_X(D):Y\supset D\in\mathcal{D}\}\,.$$

The homeomorphic image of $\alpha(X,\$^*)$ by the complementation is a topology on the hyperspace $C(X,\$)$ denoted by $\alpha(X,\$)$. Accordingly, $\{\mathcal{A}_c:\mathcal{A}\in\alpha\}$ is a subbase of $\alpha(X,\$)$-open sets on the hyperspace $C(X,\$)$; the neighborhood of $H\in C(X,\$)$ with respect to $\alpha(X,\$)$ is

$$\mathcal{N}_{\alpha(X,\$)}(H)\approx\{\mathcal{A}_c:H^c\in\mathcal{A}\in\alpha\}\,.$$

In particular, a base of $\mathcal{N}_{\alpha_{\mathcal{D}}(X,\$)}(A_0)$ consists of

$$\{\{A\in C(X,\$):A\cap D=\varnothing\}:D\in\mathcal{D},\ A_0\cap D=\varnothing\}$$

This form of basic neighborhoods is at the origin of the term $\mathcal{D}$-*miss topology*.

Remark 2.3. Gruenhage introduced the so-called γ-*connection* [12]. In particular, a filter $\Gamma(Y,X)$, where Y is an open subset of X, is generated by

$$\left\{\mathcal{O}_X(F):Y\supset F\in[X]^{<\aleph_0}\right\},$$

hence $\Gamma(Y,X)$ is a neighborhood filter of Y with respect to

$$\alpha_{[X]^{<\aleph_0}}:=\left\{\mathcal{O}_X(F):F\in[X]^{<\aleph_0}\right\}.$$

3. Preimage-wise characterization

Denote by $f^-(A):=\{x:f(x)\in A\}$ and by $\mathcal{F}^-(A)$ a filter generated by

$$\left\{\left\{f^-(A):f\in F\right\}:F\in\mathcal{F}\right\}.$$

What follows is a special case of a theorem (see [7]) about $C_\alpha(X,T)$ and $C_\alpha(X,\$)$, where X is a convergence space and T is a topological space.

Theorem 3.1. *Let α be a collection of openly isotone families on a topological space X. Let $\mathcal{C}$ be a base of closed subsets of $\mathbb{R}$. If $\mathcal{F}$ is a filter on $C(X,\mathbb{R})$, then*

$$f \in \lim_{\alpha(X,\mathbb{R})} \mathcal{F} \Longleftrightarrow f^-(C) \in \lim_{\alpha(X,\$)} \mathcal{F}^-(C)$$

for each C.

Proof. By definition, $f_0 \in \lim_{\alpha(X,\mathbb{R})} \mathcal{F}$ if and only if for each open subset O of $\mathbb{R}$ and every $\mathcal{A} \in \alpha$ such that $f_0 \in [\mathcal{A}, O]$, there exists $F \in \mathcal{F}$ such that $f \in [\mathcal{A}, O]$ for each $f \in F$. In other words, if $f_0^-(O) \in \mathcal{A}$, then there exists $F \in \mathcal{F}$ such that $f^-(O) \in \mathcal{A}$ for each $f \in F$, that is, $\mathcal{F}^-(O)$ converges to $f_0^-(O)$ in $\alpha(X, \$^*)$, equivalently, $f_0^-(O^c)$ converges to $f_0^-(O^c)$ in $\alpha(X, \$)$.

Suppose that $f_0^-(C) \in \lim_{\alpha(X,\$)} \mathcal{F}^-(C)$ for each element C of a base of closed subsets of $\mathbb{R}$. Let A be a closed subset of $\mathbb{R}$ and $\mathcal{C}_A \subset \mathcal{C}$ be such that $A = \bigcap_{C \in \mathcal{C}_A} C$. If $x \notin f_0^-(A)$ then there is $C \in \mathcal{C}_A$ such that $x \notin f_0^-(C)$, hence, by assumption, there exists $F \in \mathcal{F}$ such that $x \notin f^-(C)$, and thus $x \notin f^-(A)$ for every $f \in F$, that is, $f_0^-(A) \in \in \lim_{\alpha(X,\$)} \mathcal{F}^-(A)$. $\Diamond$

Corollary 3.2. *The (infinite) tightness of* $\alpha(X, \mathbb{R})$ *is not greater than that of* $\alpha(X, \$)$.

Proof. Suppose that the tightness of $\alpha(X, \$)$ be λ and let $\mathcal{C}$ be a countable base of closed subsets of $\mathbb{R}$. If $f_0 \in \mathrm{cl}_{\alpha(X,\mathbb{R})}\mathcal{B}$, then by Th. 3.1, $f_0^-(C) \in \mathrm{cl}_{\alpha(X,\$)} \{f^-(C) : f \in \mathcal{B}\}$ for each $C \in \mathcal{C}$. Hence for each $C \in \mathcal{C}$ there is $\mathcal{B}_C \subset \mathcal{B}$ with $\mathrm{card}(\mathcal{B}_C) \leq \lambda$ such that
$f_0^-(C) \in \mathrm{cl}_{\alpha(X,\$)} \{f^-(C) : f \in \mathcal{B}_C\}$, thus $f_0^-(C) \in \mathrm{cl}_{\alpha(X,\$)} \{f^-(C) : f \in \mathcal{B}_0\}$, where $\mathcal{B}_0 := \bigcup_{C \in \mathcal{C}} \mathcal{B}_C$. Th. 3.1 implies that $f_0 \in \mathrm{cl}_{\alpha(X,\mathbb{R})}\mathcal{B}_0$ and $\mathrm{card}(\mathcal{B}_0) \leq \lambda$. $\Diamond$

Corollary 3.3. *The (infinite) character of* $\alpha(X, \mathbb{R})$ *is not greater than that of* $\alpha(X, \$)$.

Proof. Suppose that the character of $\alpha(X, \$)$ be λ and let $\mathcal{C}$ be a countable base of closed subsets of $\mathbb{R}$. Then $f \in \lim_{\alpha(X,\mathbb{R})} \mathcal{F}$ if and only if $f^-(C) \in \lim_{\alpha(X,\$)} \mathcal{F}^-(C)$ for each element $C \in \mathcal{C}$. By the assumption, for each $C \in \mathcal{C}$ there is a filter $\mathcal{E}_C \leq \mathcal{F}^-(C)$ of character not greater than λ and such that $f^-(C) \in \lim_{\alpha(X,\$)} \mathcal{E}_C$. Let $\mathcal{F}_C \subset \mathcal{F}$ be a filter on $C(X, \mathbb{R})$ such that $F \in \mathcal{F}_C$ whenever there is $E \in \mathcal{E}_C$ for which $E \subset F^-(C)$. Let $\mathcal{C}$ be ranged in a sequence $\{C_n : n < \omega\}$. Then there is a sequence $(\mathcal{F}_{C_n})_n$ such that $\mathcal{F}_{C_n} \subset \mathcal{F}_{C_{n+1}} \subset \mathcal{F}$ and $f^-(C_n) \in \lim_{\alpha(X,\$)} \mathcal{F}_{C_k}^-(C_n)$ for each $k \leq n$. Consequently $\left(\bigcup_{k<\omega} \mathcal{F}_{C_k}\right)^-(C_n)$ converges to $f^-(C_n)$ in $\alpha(X, \$)$ for each $n < \omega$, and the character of $\bigcup_{k<\omega} \mathcal{F}_{C_k}$ is not greater than λ. By Th. 3.1, $f \in \lim_{\alpha(X,\mathbb{R})} \bigcup_{k<\omega} \mathcal{F}_{C_k}$. $\Diamond$

As we have seen, no assumptions on X or α were needed to get the corollaries above. The converse inequality will be established in the case of compact webs in completely regular spaces.

4. Compact families

An openly isotone family $\mathcal{A}$ is *compact* if each family $\mathcal{P}$ of open sets such that $\bigcup\mathcal{P} \in \mathcal{A}$ has a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup\mathcal{P}_0 \in \mathcal{A}$. We denote by $\kappa(X)$ the collection of all compact families on X. Here are fundamental examples:

$$K \text{ compact } \Rightarrow \mathcal{O}_X(K) \in \kappa(X);$$

$$x \in \lim_X \mathcal{F} \Rightarrow \mathcal{O}_X(\mathcal{F} \wedge \{x\}) \in \kappa(X),$$

where $\mathcal{F} \wedge \{x\} := \{\{F \cup \{x\}\} : F \in \mathcal{F}\}$.

The collection of (openly isotone) compact families fulfill the following properties:

$$\varnothing, \mathcal{O}_X \in \kappa(X);$$

$$\alpha \subset \kappa(X) \Rightarrow \bigcup_{\mathcal{A}\in\alpha} \mathcal{A} \in \kappa(X);$$

$$\mathcal{A}_0, \mathcal{A}_1 \in \kappa(X) \Rightarrow \mathcal{A}_0 \cap \mathcal{A}_1 \in \kappa(X).$$

Therefore

Corollary 4.1. *$\kappa(X)$ is the collection of open sets of a topology on $\mathcal{O}_X = C(X, \$^*)$.*

The topology of Cor. 4.1 is called the *Scott topology* (see [11], [3]).

Example 4.2. If $\kappa = \kappa(X)$ is the collection of (openly isotone) compact families on X, then

$$\{[\mathcal{A}, O] : \mathcal{A} \in \kappa(X), O \in \mathcal{O}_Z\}$$

is a subbase of the *Isbell topology* on $C(X, Z)$. In particular, $\kappa(X)$ is the collection of open sets of $C_\kappa(X, \$^*)$.

Lemma 4.3. *If $\mathcal{A} = \mathcal{O}(\mathcal{A})$ is a compact family of subsets of a completely regular topological space X, and F is a closed subset of X with $F^c \in \mathcal{A}$, then there is $A \in \mathcal{A}$ and $h \in C(X, [0,1])$ such that $h(A) = \{0\}$ and $h(F) = \{1\}$.*

Proof. By complete regularity, for every $x \notin F$, there is an open neighborhood O_x of x and $f_x \in C(X, [0,1])$ such that $f_x(O_x) = \{0\}$ and $f_x(F) = \{1\}$. Therefore $F^c = \bigcup_{x \notin F} O_x \in \mathcal{A}$, so that by the compactness of $\mathcal{A}$ there is $n < \omega$ and $x_1, \ldots, x_n \notin F$ such that $A = \bigcup_{1 \le i \le n} O_{x_i} \in \mathcal{A}$. The continuous function $\min_{1 \le i \le n} f_{x_i}$ is 0 on A and 1 on F. $\Diamond$

If $\mathcal{A}$ is an openly isotone family on X and C is a subset of X, then
$$\mathcal{A} \vee C := \mathcal{O}_X \left(\{ A \cap C : A \in \mathcal{A} \} \right).$$

Lemma 4.4. *If $\mathcal{A}$ is a compact openly isotone family on X and C is a closed subset of X, then $\mathcal{A} \vee C$ is compact.*

Proof. Indeed, if $\mathcal{P}$ is a family of open sets such that $\bigcup \mathcal{P} \in \mathcal{O}(\{A \cap C : A \in \mathcal{A}\})$, then $\bigcup \mathcal{P} \cup (X \setminus C) \in \mathcal{A}$, hence there exists a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup \mathcal{P}_0 \cup (X \setminus C) \in \mathcal{A}$, thus $\bigcup \mathcal{P}_0 \in \mathcal{O}(\{A \cap C : A \in \mathcal{A}\})$. $\Diamond$

The concept of *network* is well-known. Here we introduce a notion of web that extends and weakens that of network. A collection α of openly isotone families is a *web* in X if for every $x \in X$ and each $O \in \mathcal{O}_X(x)$ there is $\mathcal{A} \in \alpha$ such that $\mathcal{A}$ is generated by a filter on O. In particular, $\alpha_{\mathcal{D}}$ (2.1) is a web if for each $x \in X$ and every $O \in \mathcal{O}_X(x)$ there is $D \in \mathcal{D}$ such that $D \subset O$. This is a weaker property than that of $\mathcal{D}$ being a network. A collection of openly isotone families is called a *compact web* if it is a web consisting of compact families.

Proposition 4.5. *If $\mathcal{D}$ is a compact network, then $\alpha_{\mathcal{D}}$ is a compact web.*

Indeed, in this case, $\alpha_{\mathcal{D}}$ is a collection of compact families. It is a web, because it includes $\left\{ \mathcal{O}_X(\{x\}) : x \in X \right\}$. For instance, $\left\{ \mathcal{O}_X(F) : F \in [X]^{<\aleph_0} \right\}$ and $\left\{ \mathcal{O}_X(K) : K \in \mathcal{K}(X) \right\}$ are compact webs. Therefore,

Corollary 4.6. *$\kappa(X)$ is a compact web on X.*

In fact, $\kappa(X)$ is a web, because it includes a web, for example, $\{ \mathcal{O}_X(K) : K \in \mathcal{K}(X) \}$. The following result extends [17, Th. 1.1.5].

Lemma 4.7. *If Z is Hausdorff and α is a web, then $C_\alpha(X, Z)$ is Hausdorff.*

Proof. If $f_0 \neq f_1$ then there is $x \in X$ such that $f_0(x) \neq f_1(x)$, and because Z is Hausdorff, there exist disjoint open sets O_0 and O_1 such that $f_0(x) \in O_0$ and $f_1(x) \in O_1$. Therefore $W := f_0^-(O_0) \cap f_1^-(O_1) \in \mathcal{O}_X(x)$, and since α is a web, there exists $\mathcal{A} \in \alpha$ such that $\mathcal{A}$ is generated by a filter on W. Therefore $f_0 \in [\mathcal{A}, O_0], f_1 \in [\mathcal{A}, O_1]$ and $[\mathcal{A}, O_1] \cap [\mathcal{A}, O_0]$ is empty, for if $f \in [\mathcal{A}, O_1] \cap [\mathcal{A}, O_0]$ then there exist $W \supset A_0, A_1 \in \mathcal{A}$ such that $A_0 \subset f^-(O_0), A_1 \subset f^-(O_1)$ and $A := A_0 \cap A_1 \in \mathcal{A}$, hence $f(A) \subset O_0 \cap O_1 = \varnothing$. $\Diamond$

A family $\mathcal{D}$ of closed subsets of X is called *hereditarily closed* provided that $F \subset D \in \mathcal{D}$ and F is closed implies that $F \in \mathcal{D}$.[3] It is proved in [17, Th. 1.1.7] that

Theorem 4.8. *If $\mathcal{D}$ is a hereditarily closed compact network and Z is a topological group, then $C_{\mathcal{D}}(X, Z)$ is a topological group.*

In particular, the topology $\alpha_{\mathcal{D}}$ of Th. 4.8 is homogeneous. Of course, families of all closed compact subsets and of all finite subsets of T_1 topologies are hereditarily closed compact networks, so that, in particular, $C_p(X, \mathbb{R})$ and $C_k(X, \mathbb{R})$ are topological groups, in fact, topological vector spaces.

Nevertheless, there exists a topological space X (satisfying high separation axioms) and a collection α of compact families including all families generated by compact sets, for which $C_\alpha(X, \mathbb{R})$ is not a translation invariant. Of course, such a space X must be *dissonant*.

Example 4.9. Consider the *Arens topology* on $X := \{x_\infty\} \cup \bigcup_{n<\omega} X_n$ where $X_n := \{x_{n,k} : k < \omega\}$: each $x \neq x_\infty$ is isolated, and $O \in \mathcal{O}_X(x_\infty)$ whenever there is n_O and a map $h : \omega \to \omega$ such that

$$\{x_\infty\} \cup \{x_{n,k} : n \geq n_O, k \geq h(n)\} \subset O.$$

The Arens topology is a *prime topology*, that is, all the elements but possibly one are isolated. Each prime topology has strong separation properties, in particular, is zero-dimensional and paracompact. Every compact subset of the Arens space is finite. A compact family $\mathcal{S}$ is *simple* if either $\mathcal{S} = \mathcal{O}_X(F)$ where F is a compact (hence, finite) subset of X, or $\mathcal{S} \subset \mathcal{O}_X(x_\infty)$. Every compact family on the Arens space is a union of simple families. It is known [6] that the Arens topology is dissonant, in other words, there exists a compact family $\mathcal{S}$ that is not of the form $\mathcal{O}_X(F)$ with compact set F, hence $\mathcal{S} \subsetneq \mathcal{O}_X(x_\infty)$.

Let $\mathcal{D} \subset \mathcal{O}_X(x_\infty)$ be the compact family such that $D \cap X_n \neq \varnothing$ for each $n < \omega$ and every $D \in \mathcal{D}$, and let $\alpha := \{\mathcal{D}\} \cup \{\mathcal{O}_X(F) : F \in [X]^{<\aleph_0}\}$. Then $C_\alpha(X, \mathbb{R})$ is not translation invariant.

Indeed, let $D_0 \in \mathcal{D}$ be such that $X_n \setminus D_0 \neq \varnothing$ for each $n < \omega$. Define $f(D_0) = \{0\}$ and $f(X \setminus D_0) = \{1\}$. Then the translation $g \mapsto f + g$ is not continuous at $\tilde{0}$. Indeed, $f + \tilde{0} \in [\mathcal{D}, B(0, \varepsilon)]$ where $\varepsilon = \frac{1}{2}$. Take any finite set F and $0 < \delta < \varepsilon$, and consider a neighborhood

$$W_\delta := [\mathcal{D}, B(0, \delta)] \cap [\mathcal{O}_X(F), B(0, \delta)]$$

of the zero function $\tilde{0}$. Then there is $n_F < \omega$ such that $X_{n_F} \cap F = \varnothing$. Let $D_1 \in \mathcal{D}$ be such that $X_{n_F} \cap D_1 \cap D_0 = \varnothing$. On the other hand, $X_{n_F} \cap D_0 \neq \varnothing$ and $X_{n_F} \cap D_1 \neq \varnothing$ by the definition of $\mathcal{D}$. Set $g(D_1 \cup F) = \{0\}$

[3] By analogy to *openly isotone* one could call this property *closedly antitone*.

and $g(x) = 1$ elsewhere, so that $g \in W_\delta$ for each $\delta > 0$. Notice that $f(x) + g(x) \in \{1, 2\}$ for each $x \in X_{n_F}$, and since $X_{n_F} \cap D \neq \varnothing$ for every $D \in \mathcal{D}$, $(f + g)(D) \cap \{1, 2\} \neq \varnothing$ and thus $(f + g) \notin [\mathcal{D}, B(0, \varepsilon)]$.

5. Polar topologies

Recall that if $\Omega \subset V \times W$, then the Ω-*polar* $\Omega^* A$ of a subset A of V is the greatest subset B of W such that $A \times B \subset \Omega$. Dual topologies can be represented in terms of polarity.

For every open subset O of $\mathbb{R}$ we define a relation $\Omega_O := \{(x, f) : : f(x) \in O\}$. Accordingly, for each $A \in C(X, \$^*)$, the set $[A, O]$ is the Ω_O-polar of A. Indeed,

$$[A, O] = \left\{f : A \subset f^-(O)\right\} = \Omega_O^* A. \tag{5.1}$$

On the other hand, Ω_O^* is a relation on $C(X, \$^*) \times C(X, \mathbb{R})$, namely

$$\Omega_O^* = \left\{(A, f) : A \subset f^-(O)\right\},$$

so that if $\mathcal{A}$ is a subset of $C(X, \$^*)$, then $\Omega_O^* \mathcal{A} = \bigcup_{A \in \mathcal{A}} [A, O] = [\mathcal{A}, O]$. Hence for a filter (base) α on $C(X, \$^*)$, our convention yields

$$\Omega_O^* \alpha \approx \{[\mathcal{A}, O] : \mathcal{A} \in \alpha\}.$$

Finally

$$\mathcal{N}_{\alpha(X,\mathbb{R})}(\tilde{0}) \approx \bigvee_{O \in \mathcal{N}_{\mathbb{R}}(0)} \Omega_O^* \alpha \approx \{[\mathcal{A}, O] : \mathcal{A} \in \alpha, O \in \mathcal{N}_{\mathbb{R}}(0)\}.$$

In case of homogeneity, it is enough to establish a property of $\mathcal{N}_{\alpha(X,\mathbb{R})}(\tilde{0})$ in order to prove that property for every neighborhood filter of $C_\alpha(X, \mathbb{R})$ (for $\alpha = \alpha_{\mathcal{D}}$ with a compact network $\mathcal{D}$ on a completely regular space X).

On the other hand, it follows from Th. 3.1 that the function $\tilde{0} \in \in \lim_{\alpha(X,\mathbb{R})} \mathcal{F}$ implies, in particular, $\tilde{0}^-(C) \in \lim_{\alpha(X,\$)} \mathcal{F}^-(C)$ for each closed subset C of $\mathbb{R}$. If $0 \in C$ then $\tilde{0}^-(C) = X$, hence $\tilde{0}^-(C) \in \in \lim_{\alpha(X,\$)} \mathcal{F}^-(C)$ for every $\mathcal{F}$. Hence the only case to consider is that of $0 \notin C$ that is equivalent to $\tilde{0}^-(C) = \varnothing$.

This observation implies that properties of $\mathcal{N}_{\alpha(X,\$)}(\varnothing)$ are intimately related to properties of $\mathcal{N}_{\alpha(X,\mathbb{R})}(\tilde{0})$, hence to local properties of $C_\alpha(X, \mathbb{R})$, thanks to homogeneity (for $\alpha = \alpha_{\mathcal{D}}$ with a compact network $\mathcal{D}$ on a completely regular space X). As $\alpha(X, \$)$ and $\alpha(X, \$^*)$ are homeomorphic by complementation, the properties of $\mathcal{N}_{\alpha(X,\$)}(\varnothing)$ and $\mathcal{N}_{\alpha(X,\$^*)}(X)$ are the same. On the other hand, $\mathcal{N}_{\alpha(X,\$^*)}(X)$ has a filter subbase α.

If $\Gamma \subset X_1 \times \ldots \times X_m$ is a relation, then for $1 \leq k \leq m$, let $\Gamma_k : : \Gamma \to X_k$ be the restriction to Γ of the k-th projection of $X_1 \times \ldots \times X_m$.

Consider the *fundamental relation* $\Gamma \subset C(X,\mathbb{R}) \times C(X,\$^*) \times C(\mathbb{R},\$^*)$ defined by

$$\Gamma := \{(f, A, O) : f \in [A, O]\}\,.$$

The last component of Γ is valued in (open) subsets of $\mathbb{R}$, and not in $\mathbb{R}$, because Γ results from a polarity. Therefore, we need to define a filter on $\mathcal{O}_{\mathbb{R}}(0)$ such that its projection on $\mathbb{R}$ coincides with $\mathcal{N}_{\mathbb{R}}(0)$. A base for such a filter (denoted by $\bar{\mathcal{N}}_{\mathbb{R}}(0)$) is given by $\{P \in \mathcal{O}_{\mathbb{R}}(0) : P \subset O\}$ with $O \in \mathcal{O}_{\mathbb{R}}(0)$.

Theorem 5.1. $\mathcal{N}_{\alpha(X,\mathbb{R})}(\tilde{0}) \approx \Gamma_1(\Gamma_2^- \alpha \vee \Gamma_3^- \bar{\mathcal{N}}_{\mathbb{R}}(0))$.

Proof. By definition, $\Gamma_2^- \mathcal{A} = \{(f, A, O) : f \in [A, O], A \in \mathcal{A}\}$, and $\Gamma_3^- O = = \{(f, A, O) : f \in [A, O]\}$, hence $\Gamma_1(\Gamma_2^- \mathcal{A} \vee \Gamma_3^- O) = [\mathcal{A}, O]$, so that $\mathcal{N}_{\alpha(X,\mathbb{R})}(\tilde{0}) = \Gamma_1(\Gamma_2^- \alpha \vee \Gamma_3^- \bar{\mathcal{N}}_{\mathbb{R}}(0))$. $\Diamond$

Let Δ be the following subset of $C(X,\$^*) \times C(X,\$^*)$:

$$\Delta := \left\{(A, G) : \exists_{\theta \in C(X,[0,1])}\ \theta(A) = \{0\}\,, \theta(X \setminus G) = \{1\}\right\}.$$

Let $\Theta : \Delta \to C(X,[0,1])$ be such that

$$\Theta(A, G)(A) = \{0\} \text{ and } \Theta(A, G)(X \setminus G) = \{1\}\,.$$

Denote by Δ_2 the projection of Δ on the second component.

Theorem 5.2. *If α is a compact web, and X is completely regular, then*

$$\alpha \approx \Delta_2(\Theta^- \mathcal{N}_{\alpha(X,\mathbb{R})}(\tilde{0})).$$

Proof. If $G \in \Delta_2(\Theta^-[\mathcal{A}, (-\frac{1}{n}, \frac{1}{n})])$ then there is an open subset D of X such that $\Theta(D, G) \in [\mathcal{A}, (-\frac{1}{n}, \frac{1}{n})]$, that is, there $A \in \mathcal{A}$ such that $\Theta(D, G)(A) \subset (-\frac{1}{n}, \frac{1}{n})$ hence $A \subset G$, and thus $G \in \mathcal{A}$. It follows that $\Delta_2(\Theta^-[\mathcal{A}, (-\frac{1}{n}, \frac{1}{n})]) \subset \mathcal{A}$ for each $n < \omega$.

Conversely, if $G \in \mathcal{A}$ then, by Lemma 4.3, there is $A \in \mathcal{A}$ such that $(A, G) \in \Delta$, thus $\Theta(A, G)(A) = \{0\}\,, \Theta(A, G)(X \setminus G) = \{1\}$. Hence $\Theta(A, G) \in [\mathcal{A}, (-\frac{1}{n}, \frac{1}{n})]$ for every $n < \omega$. In other words, $(A, G) \in \in \Theta^-[\mathcal{A}, (-\frac{1}{n}, \frac{1}{n})]$ and so $G \in \Delta_2(\Theta^-[\mathcal{A}, (-\frac{1}{n}, \frac{1}{n})])$, showing that $\mathcal{A} \subset \subset \Delta_2(\Theta^-[\mathcal{A}, (-\frac{1}{n}, \frac{1}{n})])$ for every $n < \omega$. $\Diamond$

6. Transfer of properties

Let $\mathbb{B}$ be a class of filters. A topology is $\mathbb{B}$-*based* if and only if each neighborhood filter is in $\mathbb{B}$. For each class $\mathbb{B}$, the $\mathbb{B}$-based topologies form a concretely coreflective subcategory of topologies. For example, classes of topologies of a given character, or of a given tightness, can be represented as those of $\mathbb{B}$-based topologies for appropriate classes $\mathbb{B}$. Other instnaces of classes of $\mathbb{B}$-based topologies for appropriate classes of filters $\mathbb{B}$ are sequentiality, Fréchetness, strong Fréchetness, productive Fréchetness, bisequentiality, and others (see, e.g., [4]).

Thms. 5.1 and 5.2 enable us to transfer some such coreflective properties from $C_\alpha(X,\mathbb{R})$ to $C_\alpha(X,\$)$ and vice versa.

If $H \subset X \times Y$, then $Hx := \{y \in Y : (x,y) \in H\}$, and if $A \subset X$ then $HA := \bigcup_{x\in A} Hx$. If now $\mathcal{F}$ and $\mathcal{H}$ are families of subsets of X and $X \times Y$ respectively, then

$$\mathcal{HF} := \{HF : F \in \mathcal{F}, H \in \mathcal{H}\}$$

is a family of subsets of Y. If $\mathcal{F}$ and $\mathcal{H}$ are filters, then, by a handy abuse of notation, $\mathcal{HF}$ stands also for the filter it generates.

Let $\mathbb{F}_\lambda$ denote the class of filters admitting a filter base of cardinality less than $\aleph_\lambda$. In particular, $\mathbb{F}_0$ is the class of *principal* filters, and $\mathbb{F}_1$ is the class of *countably based* filters. The class of all filters is denoted by $\mathbb{F}$.

A class $\mathbb{B}$ of filters is $\mathbb{H}$-*composable* if $\mathcal{HF} \in \mathbb{B}$ for each $\mathcal{F} \in \mathbb{B}$ and every $\mathcal{H} \in \mathbb{H}$ (see [8], [13], [16]). A class $\mathbb{B}$ of filters is $\mathbb{H}$-*steady* if $\mathcal{H} \vee \mathcal{F} \in \mathbb{B}$ for each $\mathcal{F} \in \mathbb{B}$ and each $\mathcal{H} \in \mathbb{H}$ (see [13], [16]).

If $\mathbb{H}$ is a class of filters and γ is a filter subbase, then $\gamma \in \mathbb{H}$ means that the filter generated by γ belongs to $\mathbb{H}$.

By Th. 5.1,

Proposition 6.1. *Let $\mathbb{B}$ be $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady. If X is completely regular, α is a compact web, and $\alpha \in \mathbb{B}$, then $C_\alpha(X,\mathbb{R})$ is $\mathbb{B}$-based at $\tilde{0}$. If moreover $\mathcal{D}$ is a hereditarily closed compact network, then $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ is $\mathbb{B}$-based.*

Proof. If $\alpha \in \mathbb{B}$ then $\Gamma_2^- \alpha \in \mathbb{B}$, because $\mathbb{B}$ is $\mathbb{F}_0$-composable. On the other hand, $\Gamma_3^- \bar{\mathcal{N}}_{\mathbb{R}}(0)$ is a countably based filter, because $\mathcal{N}_{\mathbb{R}}(0)$ is countably based. Therefore, $\Gamma_2^- \alpha \vee \Gamma_3^- \bar{\mathcal{N}}_{\mathbb{R}}(0) \in \mathbb{B}$, because $\mathbb{B}$ is $\mathbb{F}_1$-steady. Finally, $\mathcal{N}_{\alpha(X,\mathbb{R})}(\tilde{0}) \in \mathbb{B}$ as the image by a map of a filter from $\mathbb{B}$. Therefore $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ is $\mathbb{B}$-based because $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ is homogeneous by Th. 4.8. $\Diamond$

Proposition 6.2. *Let $\mathbb{B}$ be $\mathbb{F}_0$-composable. If α is a compact web, X is completely regular, and $C_\alpha(X,\mathbb{R})$ is $\mathbb{B}$-based, then $\alpha \in \mathbb{B}$.*

Proof. If $C_\alpha(X,\mathbb{R})$ is $\mathbb{B}$-based, $\mathcal{N}_{\alpha(X,\mathbb{R})}(\tilde{0}) \in \mathbb{B}$, hence by Th. 5.2, $\alpha \in \mathbb{B}$, because $\mathbb{B}$ is $\mathbb{F}_0$-composable. $\Diamond$

Theorem 6.3. *Let $\mathbb{B}$ be $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady, and let $\mathcal{D}$ be a hereditarily closed compact network on a completely regular space X. Then $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ is $\mathbb{B}$-based if and only if $\alpha_{\mathcal{D}} \in \mathbb{B}$.*

F. Jordan established in [13, Th. 3] a special case of Th. 6.3 for $\alpha = \left\{\mathcal{O}(D) : D \in [X]^{<\aleph_0}\right\}$, hence concerning $C_p(X,\mathbb{R})$, in terms of γ-connection (see Rem. 2.3). It is enough to replace in his proofs $[X]^{<\aleph_0}$ by any (additively stable) family $\mathcal{D}$ of compact sets, in order that the proofs remain valid for $\alpha = \{\mathcal{O}(D) : D \in \mathcal{D}\}$ and $C_{\mathcal{D}}(X,\mathbb{R})$.

Since α is a filter subbase of $\mathcal{N}_{\alpha(X,\$^*)}(X)$, and $\alpha(X,\$^*)$ is homeomorphic to $\alpha(X,\$)$ by complementation, we have

Corollary 6.4. *Let $\mathbb{B}$ be $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady, and let $\mathcal{D}$ be a hereditarily closed compact network on a completely regular space X. Then $C_{\alpha_D}(X,\mathbb{R})$ is $\mathbb{B}$-based if and only if $\mathcal{N}_{\alpha_D(X,\$)}(\varnothing) \in \mathbb{B}$.*

7. Transferable properties

We shall review several $\mathbb{F}_0$-composable $\mathbb{F}_1$-steady classes of filters, in other words, transferable local properties. Several results on composability and steadiness can be found in [13], [16].

We say that a property of topological spaces is *local* if there is a class $\mathbb{P}$ of filters[4] such that a topology has the property whenever each neighborhood filter belongs to $\mathbb{P}$. Character and tightness are examples of local properties.

Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of a given set *mesh* (in symbols, $\mathcal{A}\#\mathcal{B}$) if $A \cap B \neq \varnothing$ for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The *grill* $\mathcal{A}^{\#}$ of a family $\mathcal{A}$ of subsets of X is defined as $\{H \subset X : H\#\mathcal{A}\}$, where $H\#\mathcal{A}$ is an abbreviation for $\{H\}\#\mathcal{A}$. The *character* $\chi(\mathcal{F})$ of a filter $\mathcal{F}$ is the least cardinal τ such that $\mathcal{F}$ has a filter base of cardinality τ. The *tightness* $t(\mathcal{F})$ of a filter $\mathcal{F}$ is the least cardinal τ for which if $A \in \mathcal{F}^{\#}$ then there is $B \subset A$ of cardinality τ such that $B \in \mathcal{F}^{\#}$. It was proved in [15] that

Proposition 7.1. *(Infinite) character and tightness are $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady.*

A filter $\mathcal{F}$ is $\mathbb{G}$ *to* $\mathbb{E}$ *refinable* [14] ($\mathcal{F} \in (\mathbb{G}/\mathbb{E})_{\geq}$) if for each filter $\mathcal{G} \in \mathbb{G}$ with $\mathcal{G}\#\mathcal{F}$ there exists a filter $\mathcal{E} \in \mathbb{E}$ such that $\mathcal{E} \geq \mathcal{F} \vee \mathcal{G}$; a filter $\mathcal{F}$ is $\mathbb{G}$ *to* $\mathbb{E}$ *me-refinable* [14] ($\mathcal{F} \in (\mathbb{G}/\mathbb{E})\#_{\geq}$) if for each filter $\mathcal{G} \in \mathbb{G}$ with $\mathcal{G}\#\mathcal{F}$ there exists a filter $\mathcal{E} \in \mathbb{E}$ such that $\mathcal{E} \geq \mathcal{F}$ and $\mathcal{E}\#\mathcal{G}$. The following two facts were observed in [14] in special cases of countably based filters.

Lemma 7.2. *The property $(\mathbb{F}_\kappa/\mathbb{F}_\lambda)_{\geq}$ is $\mathbb{F}_\mu$-steady if $\mu \leq \kappa$.*

[4] possibly depending on the topology.

Proof. Let $\mathcal{F} \in (\mathbb{F}_\kappa/\mathbb{F}_\lambda)_\geq, \mathcal{E} \in \mathbb{F}_\kappa$ and $\mathcal{D} \in \mathbb{F}_\mu$ be such that $\mathcal{D}\#(\mathcal{E} \vee \mathcal{F})$. Then $(\mathcal{D} \vee \mathcal{E})\#\mathcal{F}$ and $\mathcal{D} \vee \mathcal{E} \in \mathbb{F}_\kappa$, because $\mu \leq \kappa$; thus there is $\mathcal{G} \in \mathbb{F}_\lambda$ such that $\mathcal{G} \geq \mathcal{D} \vee \mathcal{E} \vee \mathcal{F}$. ♢

Lemma 7.3. *The property* $(\mathbb{F}_\kappa/\mathbb{F}_\lambda)_\geq$ *is* $\mathbb{F}_\mu$*-composable if* $\mu \leq \kappa \wedge \lambda$.

Proof. If $\mathcal{F} \in (\mathbb{F}_\kappa/\mathbb{F}_\lambda)_\geq, \mathcal{E} \in \mathbb{F}_\kappa$ and $\mathcal{M} \in \mathbb{F}_\mu$ be such that $\mathcal{E}\#(\mathcal{M}\mathcal{F})$. Then $\mathcal{M}^-\mathcal{E}\#\mathcal{F}$ and $\mathcal{M}^-\mathcal{E} \in \mathbb{F}_\kappa$ provided that $\mu \leq \kappa$. As $\mathcal{F} \in (\mathbb{F}_\kappa/\mathbb{F}_\lambda)_\geq$ there is $\mathcal{G} \in \mathbb{F}_\lambda$ such that $\mathcal{G} \geq \mathcal{M}^-\mathcal{E} \vee \mathcal{F}$. Thus $\mathcal{M}\mathcal{G} \geq \mathcal{M}(\mathcal{M}^-\mathcal{E} \vee \mathcal{F}) \geq$ $\geq \mathcal{E} \vee \mathcal{M}\mathcal{F}$ and $\mathcal{M}\mathcal{G} \in \mathbb{F}_\lambda$ provided that $\mu \leq \lambda$. ♢

Fréchetness, strong Fréchetness, productive Fréchetness and bisequentiality are other examples of local properties that can be expressed in terms of refinable and me-refinable filters with respect to various classes (see [15] and a pioneering paper [5]). A filter $\mathcal{F}$ is

(1) *Fréchet* $\Longleftrightarrow \mathcal{F} \in (\mathbb{F}_0/\mathbb{F}_1)_\geq$: A filter $\mathcal{F}$ is *Fréchet* if for each set A such that $A\#\mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $A \in \mathcal{E} \geq \mathcal{F}$.

(2) *strongly Fréchet* $\Longleftrightarrow \mathcal{F} \in (\mathbb{F}_1/\mathbb{F}_1)_\geq$: A filter $\mathcal{F}$ is *strongly Fréchet* if for each countably filter $\mathcal{G}$ such that $\mathcal{G}\#\mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $\mathcal{E} \geq \mathcal{F} \vee \mathcal{G}$.

(3) *productively Fréchet* $\Longleftrightarrow \mathcal{F} \in \big((\mathbb{F}_1/\mathbb{F}_1)_\geq/\mathbb{F}_1\big)_\geq$: A filter $\mathcal{F}$ is *productively Fréchet* if for each strongly Fréchet filter $\mathcal{G}$ such that $\mathcal{G}\#\mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $\mathcal{E} \geq \mathcal{F} \vee \mathcal{G}$.

(4) *bisequential* $\Longleftrightarrow \mathcal{F} \in (\mathbb{F}/\mathbb{F}_1)_{\#\geq}$: A filter $\mathcal{F}$ is bisequential if for each filter $\mathcal{G}$ such that $\mathcal{G}\#\mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $\mathcal{E} \geq \mathcal{F}$ and $\mathcal{E}\#\mathcal{G}$.

Of course, in the first three conditions (but not in the fourth) the existence of a countably based filter $\mathcal{E}$ is equivalent to the existence of a *sequential filter*[5] $\mathcal{E}$. All these properties are $\mathbb{F}_0$-composable. Not all are $\mathbb{F}_1$-steady.

Proposition 7.4. *Classes of strongly Fréchet, productively Fréchet and bisequential filters are* $\mathbb{F}_1$*-steady; the class of Fréchet filters is not* $\mathbb{F}_1$*-steady. All the listed properties are* $\mathbb{F}_0$*-composable.*

Proof. All the cases are proved in [16] except for bisequential filters. Let $\mathcal{F}$ be bisequential and $\mathcal{E} \in \mathbb{F}_1$. If $\mathcal{D}$ is any filter such that $\mathcal{D}\#(\mathcal{E} \vee \mathcal{F})$, then $(\mathcal{D} \vee \mathcal{E})\#\mathcal{F}$, hence there is $\mathcal{G} \in \mathbb{F}_1$ such that $\mathcal{G} \geq \mathcal{F}$ and $\mathcal{G}\#(\mathcal{D} \vee \mathcal{E})$. The filter $\mathcal{G} \vee \mathcal{E} \in \mathbb{F}_1$ and $\mathcal{G} \vee \mathcal{E}$ meshes with $\mathcal{D}$ and $\mathcal{G} \vee \mathcal{E} \geq \mathcal{G} \geq \mathcal{F}$. Let $\mathcal{F}$ be bisequential and A a relation. If $\mathcal{D}$ is a filter such that $\mathcal{D}\#A\mathcal{F}$,

[5] A filter is *sequential* if it is generated by the queues of a sequence.

then $A^{-}\mathcal{D}\#\mathcal{F}$, hence there is $\mathcal{H} \in \mathbb{F}_1$ such that $\mathcal{H}\#A^{-}\mathcal{D}$ and $\mathcal{H} \geq \mathcal{F}$. Thus $A\mathcal{H}\#\mathcal{D}$ and $A\mathcal{H} \geq A\mathcal{F}$.

If $\mathcal{F}$ is Fréchet but not strongly Fréchet, then there is $\mathcal{E} \in \mathbb{F}_1$ such that $\mathcal{G} \geq \mathcal{E} \vee \mathcal{F}$ for no $\mathcal{G} \in \mathbb{F}_1$. Hence $\mathcal{E} \vee \mathcal{F}$ is not Fréchet. $\Diamond$

8. Dictionary $X \longleftrightarrow \mathcal{O}_X$

Here there is a list of elementary equivalences that will be used to establish equivalences of more convoluted equivalences between properties of $C_\alpha(X, \$^*)$ and X. We consider only those collections α that are *finitely stable*, that is, $\mathcal{A}_0, \mathcal{A}_1 \in \alpha$ implies that $\mathcal{A}_0 \cap \mathcal{A}_1 \in \alpha$.

Let $Y \subset X$. A family $\mathcal{B}$ of (open) subsets of X is called an *α-cover* of Y if $\mathcal{B} \cap \mathcal{A} \neq \varnothing$ for every $\mathcal{A} \in \alpha$ such that $Y \in \mathcal{A}$. In particular, if $\alpha = \left\{\mathcal{O}(D) : D \in [X]^{<\aleph_0}\right\}$, then an α-cover is an *ω-cover*, that is, for each finite set D there is $B \in \mathcal{B}$ such that $D \subset B$.

Lemma 8.1. *A family $\mathcal{B}$ meshes with $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ if and only if $\mathcal{B}$ is an α-cover of Y.*

Proof. A family $\mathcal{B}$ meshes with $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ if and only if $\mathcal{B} \cap \mathcal{A} \neq \varnothing$ for each $\mathcal{A} \in \alpha$ such that $Y \in \mathcal{A}$. This means exactly that $\mathcal{B}$ is an *α-cover* of Y. $\Diamond$

Let $\mathcal{A}, \mathcal{B}$ be families of subsets of a given set. We say that $\mathcal{A}$ is *coarser* than $\mathcal{B}$ (equivalently, $\mathcal{B}$ is *finer* than $\mathcal{A}$)

$$\mathcal{A} \leq \mathcal{B}$$

if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $B \subset A$. A collection of families of subsets of X can be considered as a family of subsets of 2^X. In this sense, we say that a collection is *finer* (*coarser*) than another collection. The following facts are just rewording of definitions, but we formulate them as lemmas for easy reference.

Lemma 8.2. *A collection γ is finer than $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ if and only if for each $\mathcal{A} \in \alpha$ such that $Y \in \mathcal{A}$ there is $\mathcal{G} \in \gamma$ such that $\mathcal{G} \subset \mathcal{A}$.*

Lemma 8.3. *A collection γ is coarser than $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ if and only if for each $\mathcal{G} \in \gamma$ there is $\mathcal{A} \in \alpha$ such that $Y \in \mathcal{A} \subset \mathcal{G}$.*

In particular, a sequence $(G_n)_n$, that is, a family $\gamma := \left\{\{G_n : n \geq m\} : m < \omega\right\}$ is finer than $\mathcal{N}_{\alpha(X,\$^*)}(Y)$ if for every $\mathcal{A} \in \alpha$ with $Y \in \mathcal{A}$ there is $n_{\mathcal{A}} < \omega$ such that $G_n \in \mathcal{A}$ for each $n \geq n_{\mathcal{A}}$.

8.1. Tightness

Recall that (see e.g., [17]) the *α-Lindelöf number* of a topological space X is the least cardinal τ such that for each α-cover there exists an α-subcover of cardinality less than or equal to τ.[6]

By Lemma 8.1,[7]

Theorem 8.4. *The tightness of $C_\alpha(X,\$)$ is equal to the supremum of the α-Lindelöf numbers of open subsets of X.*

Hence, by Cor. 3.2 and Th. 6.3,

Theorem 8.5. *If α is a compact web on a completely regular space X, then $C_\alpha(X,\mathbb{R})$ is τ-tight if and only if the α-Lindelöf number of X is τ.*

These facts specialize, in an obvious way, to *compact-open* topologies $C_k(X,Z)$, when $\alpha = \{\mathcal{O}(K) : K \in \mathcal{K}\}$ where $\mathcal{K}$ is the family of compact subsets of X, to *Isbell* topologies $C_\kappa(X,Z)$, when $\alpha = \kappa(X)$ is the collection of compact families. If $\alpha = \left\{\mathcal{O}_X(D) : D \in [X]^{<\aleph_0}\right\}$ then Th. 8.5 specializes with $\tau = \aleph_0$ to

Proposition 8.6. *If X is completely regular, then $C_p(X,\mathbb{R})$ is countably tight if and only if each open ω-cover of X has a countable ω-subcover of X.*

Recall that a family $\mathcal{P}$ is an *ω-cover* of X if for each finite subset F of X there is $P \in \mathcal{P}$ such that $F \subset P$.

The following theorem is due to Arhangel'skii [1] and Pytkeev [19]:

Theorem 8.7. *If X is completely regular, then $C_p(X,\mathbb{R})$ is countably tight if and only if X^n is Lindelöf for every $n < \omega$.*

8.2. Character

A subset γ of a collection α (of openly isotone families) is a *base* of α if for each $\mathcal{A} \in \alpha$ there is $\mathcal{G} \in \gamma$ such that $\mathcal{G} \subset \mathcal{A}$. The least cardinality τ such α has a base of cardinality τ is called the *character* $\chi(\alpha)$ of α.

[6] More generally, if $\kappa \le \lambda$ are cardinals, then we say that X is *$\lambda/\kappa[\alpha]$-compact* if for every open α-cover of X of cardinality $< \lambda$ there is an α-subcover of cardinality $< \kappa$ of X. In particular, a topological space is *$[\alpha]$-compact* if it is $\lambda/\aleph_0[\alpha]$-compact for each cardinal λ, *countably $[\alpha]$-compact* if it is $\aleph_1/\aleph_0[\alpha]$-compact, *$[\alpha]$-Lindelöf* if it is $\lambda/\aleph_1[\alpha]$-compact for every λ.

[7] Similar characterizations can be formulated for λ/κ-tightnes with $\kappa \ge \aleph_0$. We say that a filter $\mathcal{F}$ is *λ/κ-tight* if for each $H \in \mathcal{F}^{\#}$ with $\operatorname{card} H < \lambda$ there is $B \subset H$ such that $\operatorname{card} B < \kappa$ and $B \in \mathcal{F}^{\#}$. A topological space is *λ/κ-tight* if its every neighborhood filter is λ/κ-tight.

Because the character of α is a hereditary property, Lemma 8.2 implies that

Theorem 8.8. *The character of $C_\alpha(X,\$)$ is equal to the character of α.*

It follows from Cor. 3.3 and Prop. 6.2 that

Theorem 8.9. *If α is a compact web on a completely regular space X, then the character of $C_\alpha(X,\mathbb{R})$ is equal to the character of α.*

Corollary 8.10. *If X is T_1, then $C_p(X,\$)$ is of countable character if and only if X is countable.*

Proof. By Th. 8.8, the character of $\mathrm{C}_p(X,\$)$ is countable, if and only if for every open subset Y of X there is a sequence $(x_n)_n \subset Y$ such that $\{\mathcal{O}_X(\{x_1,\ldots,x_n\}) : n<\omega\}$ is finer than $\left\{\mathcal{O}_X(F) : F \in [X]^{<\aleph_0}\right\}$, that is, for every finite subset F of Y there is $n<\omega$ such that $\{x_1,\ldots,x_n\} \subset O$ implies $F \subset O$ for each open set O. Since X is T_1, this means that $F \subset \{x_1,\ldots,x_n\}$. $\Diamond$

Corollary 8.11. *If X is T_1, then $C_k(X,\$)$ is of countable character if and only if X is hereditarily hemicompact.*

Proof. Let Y be an open subset of X. The neighborhood filter $\mathcal{N}_{\mathcal{K}(X,\$^*)}(Y)$ is countably based if and only if there exists a sequence $(K_n)_n$ of compact subsets of Y such that for every $K \in \mathcal{K}_Y$ there exists n such that $\mathcal{O}_X(K_n) \subset \mathcal{O}_X(K)$, which, for a T_1-topology, is equivalent $K \subset K_n$. $\Diamond$

It is well-known that a (Hausdorff) topological vector space is metrizable if and only if it is of countable character. Therefore, we recover [17, p. 60]

Corollary 8.12. *If X is completely regular, then $C_p(X,\mathbb{R})$ is metrizable if and only if it is of countable character if and only if X is countable.*

Corollary 8.13. *If X is completely regular, then $C_k(X,\mathbb{R})$ is metrizable if and only if it is of countable character if and only if X is hemicompact.*

8.3. Variants of Fréchetness

Here we characterize some of the properties $(\mathbb{H}/\mathbb{E})_{\geq}$ of hyperspaces in terms of their underlying spaces.

Proposition 8.14. *$C_\alpha(X,\$)$ is $(\mathbb{F}_\kappa/\mathbb{F}_\lambda)_{\geq}$-based if and only if X enjoys the following property: For each open subset Y of X, for every collection γ of α-covers of Y with* $\mathrm{card}\,(\gamma) \leq \aleph_\kappa$, *there exists a collection ζ of families of open sets with* $\mathrm{card}\,(\zeta) \leq \aleph_\lambda$ *such that for every $\mathcal{A} \in \alpha$ with $Y \in \mathcal{A}$, and each $\mathcal{G} \in \gamma$ there exists $\mathcal{Z} \in \zeta$ such that $\mathcal{Z} \subset \mathcal{A} \cap \mathcal{G}$.*

As we have observed in preliminary considerations, the property exhibited in the proposition above is necessarily hereditary for open sets. The class $(\mathbb{F}_0/\mathbb{F}_1)_{\geq}$ is that of *Fréchet filters*, and $(\mathbb{F}_1/\mathbb{F}_1)_{\geq}$ that of *strongly Fréchet filters*. In the following corollaries we will use sequence characterizations of these properties: a filter $\mathcal{F}$ is *Fréchet* if for each $H \in \mathcal{F}^{\#}$ there is a sequence $(x_n)_n \subset H$ that is finer than $\mathcal{F}$; a filter $\mathcal{F}$ is *strongly Fréchet* if for each decreasing sequence $(H_n)_n$ such that $H_n \in \mathcal{F}^{\#}$ for each n, there is a sequence $(x_n)_n$ finer than $\mathcal{F}$ and such that $x_n \in H_n$ for each n. Prop. 8.14 specializes as follows:

Corollary 8.15. *$C_\alpha(X, \$)$ is Fréchet at $X_0 \in C(X, \$)$ if and only if for each family $\mathcal{G}$ of open sets such that $\mathcal{G} \cap \mathcal{A} \neq \varnothing$ for each $X \setminus X_0 \in \mathcal{A} \in \alpha$, there exists a sequence $(G_n)_n \subset \mathcal{G}$ such that for each $X \setminus X_0 \in \mathcal{A} \in \alpha$, there is $n_\mathcal{A} < \omega$, for which $G_n \in \mathcal{A}$ for every $n \geq n_\mathcal{A}$.*

Corollary 8.16. *$C_\alpha(X, \$)$ is strongly Fréchet at $X_0 \in C(X, \$)$ if and only if for each decreasing sequence $(\mathcal{G}_n)_n$ of families of open sets such that $\mathcal{G}_n \cap \mathcal{A} \neq \varnothing$ for each $X \setminus X_0 \in \mathcal{A} \in \alpha$, there exists a sequence $(G_n)_n$ with $G_n \in \mathcal{G}_n$ such that for each $X \setminus X_0 \in \mathcal{A} \in \alpha$, there is $n_\mathcal{A} < \omega$, for which $G_n \in \mathcal{A}$ for every $n \geq n_\mathcal{A}$.*

Of course, the sequence $(G_n)_n$ fulfills the condition above if and only if it converges to $X \setminus X_0$ in $C_\alpha(X, \$^*)$. In the case of $\alpha = \alpha_\mathcal{D}$, where $\mathcal{D} = [X]^{<\aleph_0}$, it is equivalent to $X \setminus X_0 \subset \underline{\mathrm{Lim}}_n G_n := \bigcup_{n<\omega} \bigcap_{k>n} G_k$ (the set-theoretic lower limit). In particular, for $X_0 = \varnothing$ the condition above is the condition (γ) of Gerlits and Nagy [10]: if $\mathcal{G}$ is an open ω-cover of X, then there is a sequence $G_n \in \mathcal{G}$ with $\underline{\mathrm{Lim}}_n G_n = X$.

As we have seen in Prop. 7.4, Fréchetness is not $\mathbb{F}_1$-steady. Nevertheless, it is known that a Fréchet topological group is strongly Fréchet (see [18]). Therefore

Theorem 8.17. *If $\mathcal{D}$ is a compact network on a completely regular space X, then $C_{\alpha_\mathcal{D}}(X, \mathbb{R})$ is Fréchet if and only if it is strongly Fréchet if and only if for every $\mathcal{D}$-cover $\mathcal{P}$ of X there is a sequence $(P_n)_n \subset \mathcal{P}$ such that for each $D \in \mathcal{D}$ there is $n_D < \omega$ such that $P_n \in \mathcal{A}$ for each $n \geq n_D$.*

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