PROPERTIES TRANSFER BETWEEN TOPOLOGIES ON FUNCTION SPACES, HYPERSONSACES AND UNDERLYING SPACES

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Abstract: Each collection \( \alpha \) of families of subsets of \( X \) determines a topology \( \alpha(X, Z) \) on the space of continuous maps \( C(X, Z) \). Interrelations between local properties of \( \alpha(X, \mathbb{R}) \) and of \( \alpha(X, \mathbb{S}) \) (on the hyperspace \( C(X, \mathbb{S}) \)), and with properties of a topological space \( X \) are studied in a general framework, which allows to treat simultaneously several classical constructions, like pointwise convergence, compact-open topology and the Isbell topology.

1. Introduction

The interrelation of properties of \( C_\alpha(X, Z) \) with those of \( X \) and \( Z \), is a fascinating theme. Here \( \alpha \) is a collection of (openly isotone\(^1\)) families of subsets of \( X \), that defines a topology \( \alpha(X, Z) \) on \( C(X, Z) \) by a subbase

\[
\{ [\mathcal{A}, O] : \mathcal{A} \in \alpha, O \in \mathcal{O}_Z \},
\]

where \( [\mathcal{A}, O] := \{ f : f^{-1}(O) \in \mathcal{A} \}, f^{-1}(O) := \{ x : f(x) \in O \} \), and \( \mathcal{O}_Z \) is the set of open subsets of \( Z \).

\(^{1}\) A family \( \mathcal{A} \) of open sets is openly isotone if \( B \in \mathcal{A} \) provided that \( B \) is open and there is an element \( A \in \mathcal{A} \) such that \( A \subset B \).

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Its very special case, that of $C_0(X, \mathbb{R})$ (the space of real-valued functions with pointwise convergence) has attracted a lot of researchers, among whom A. V. Arhangel’skii (e.g., [2]). Its intermediate case of (1.2) $\alpha = \alpha_D := \{O_X(D) : D \in \mathcal{D}\}$, where $\mathcal{D}$ is a family of subsets of $X$, and $O_X(D) := \{O \in O_X : D \subset O\}$, is the object of a book of McCoy and Ntantu [17].

Actually the said interrelation corresponds to the upper side of a quadrilateral

$$
\begin{array}{c}
X & \leftrightarrow & C_\alpha(X, \mathbb{R}) \\
\uparrow & & \uparrow \\
C_\alpha(X, \star) & \leftrightarrow & C_\alpha(X, \$)
\end{array}
$$

in which, of course, one can consider also other sides, as well as diagonals. Here $\$, $\star$ stand for the two homeomorphic variants of the Sierpiński topology on $\{0, 1\}$, so that $C(X, \, \) can be identified with the hyperspace of $X$, and $C(X, \, \star)$ with the set $O_X$ of open subsets of $X$.

It turns out that it is fruitful to study the three other sides in order to better grasp the interrelation of the upper side $X \leftrightarrow C_\alpha(X, \mathbb{R})$. Indeed,

1. $C_\alpha(X, \, \)$ is homeomorphic to $C_\alpha(X, \, \star)$;

2. One can establish a dictionary of easy translations of elementary properties of $C_\alpha(X, \, \star)$ and $\alpha$-properties of $X$;

3. Under a separation condition (by real functions) one can evidence an intimate relationship between $C_\alpha(X, \mathbb{R})$ and $C_\alpha(X, \, )$.

More precisely, if $X$ is completely regular and $\alpha$ is a compact web, then the neighborhood filter for $\alpha(X, \mathbb{R})$ of the zero function $\hat{0}$ (that is, $\hat{0}(x) = 0$ for each $x \in X$) belongs to the same transferable class as the neighborhood filter of $\emptyset$ for $\alpha(X, \, )$. Roughly speaking a web $\alpha$ on $X$ is a collection of families of open subsets of $X$ such that for each open subset $Y$ there is $A \in \alpha$ that can be reconstructed from its trace on $Y$.

A web is \textit{compact} if its every element $A$ is a \textit{compact family}.\footnote{Precise definitions are given before Lemma 4.7.}

Compact (openly isotone) families on a topological space $X$ coincide with the open sets of the \textit{Scott topology} of $C(X, \, \star)$ (see, e.g. [11]). It was shown in [6] that each such a family is of the form $\bigcup_{K \in \mathcal{D}} O_X(K)$, where $\mathcal{D}$ is a subfamily of compact subsets of $X$, if and only if $X$ is \textit{consonant}.

A collection $\alpha_D$ of the type (1.2), where $\mathcal{D}$ is a network consisting of compact subsets of $X$, is a compact web. Moreover, if $\mathcal{D}$ is hereditarily

...
closed in a completely regular space $X$, then $C_{\alpha}(X, \mathbb{R})$ is, in particular, a topological group (e.g., [17, Th. 1.1.7]), hence homogeneous. Therefore in order to prove a local (transferable) property of $C_{\alpha}(X, \mathbb{R})$, it is enough to establish it for the neighborhood filter of the constant function $0$.

Of course, in general, a hyperspace topology $\alpha(X, \mathcal{S})$ is not homogeneous. As $\alpha(X, \mathcal{S})$ and $\alpha(X, \mathcal{S}^*)$ are homeomorphic (by complementation), a property of $\mathcal{N}_{\alpha(X, \mathcal{S})}(A)$ for $A \in C(X, \mathcal{S})$ is also a property of $\mathcal{N}_{\alpha(X, \mathcal{S}^*)}(X \setminus A)$ and, as a rule, can be characterized in terms of the space $X \setminus A$ with the induced topology. Therefore a local property of $C_{\alpha}(X, \mathcal{S})$ can be characterized by a hereditary (with respect to open subsets) property of $X$.

For general compact webs $\alpha$ on completely regular spaces, $C_{\alpha}(X, \mathbb{R})$ need not be even translation invariant. Therefore, that $C_{\alpha}(X, \mathcal{S})$ has a local transferable property does not necessarily imply that $C_{\alpha}(X, \mathbb{R})$ has the same property. The implication holds for completely regular consonant spaces, because then $\alpha$ is of the form (1.2).

Nevertheless, some local properties of hyperspaces pass on to the corresponding function spaces thanks to a characterization of convergence of functions valued in topological spaces in terms of the corresponding hyperspace convergence of the preimages of closed sets. Consequently, each $\alpha$-topology on $C(X, \mathbb{R})$ can be, in principle, characterized in terms of the corresponding $\alpha$-topology on the hyperspace $C(X, \mathcal{S})$, actually on its subset consisting of functionally closed subsets of $X$. By the way, it is why Georgiou, Iliadis and Papadopoulos studied properties of real-valued function spaces in terms of topologies on functionally open sets [9].

The present paper restricts its scope to topologies on function spaces (almost always real-valued) and to the corresponding hyperspace topologies. This is just one aspect of a general theory of convergence function spaces and hyperspace convergences that will be discussed in [7].

2. Open-set topologies

We denote the set of open subsets of $X$ by either $C(X, \mathcal{S}^*)$ or $\mathcal{O}_X$. We use the latter convention to define $\mathcal{O}_X(x) := \{O \in \mathcal{O}_X : x \in O\}$, and by $\mathcal{O}_X(A) := \{O \in \mathcal{O}_X : A \subset O\}$. If now $\mathcal{A}$ is a family of subsets of $X$, then $\mathcal{O}_X(\mathcal{A}) := \bigcup_{A \in \mathcal{A}} \mathcal{O}_X(A)$. A family $\mathcal{A}$ of subsets of $X$ is open isotone if $\mathcal{O}_X(\mathcal{A}) = \mathcal{A}$.

If $\alpha$ is a non-empty collection of open isotonous families of subsets of
X, then (1.1) is a subbase of a topology on \( C(X, Z) \), denoted by \( \alpha(X, Z) \).

The corresponding topological space is denoted by \( C_\alpha(X, Z) \).

In particular, for a non-empty family \( \mathcal{D} \) of subsets of \( X \), the collection \( \alpha := \alpha_\mathcal{D} \) is defined by
\[
\alpha_\mathcal{D} := \{ \mathcal{O}_X(D) : D \in \mathcal{D} \},
\]
and the symbol \( C_{\alpha_\mathcal{D}}(X, Z) \) is abridged to \( C_\mathcal{D}(X, Z) \). It is often required (e.g., [17]) that \( \mathcal{D} \) be a (closed) network on \( X \), that is, a family of closed sets such that for each \( x \in X \) and \( O \in \mathcal{O}_X(x) \) there is \( D \in \mathcal{D} \) for which \( x \in D \subset O \). However (1.1) is a topology subbase for each \( \alpha = \alpha_\mathcal{D} \) provided that \( \mathcal{D} \neq \emptyset \).

If \( A \subset X \) and \( B \subset Z \) then \( [A, B] := \{ f \in C(X, Z) : f(A) \subset B \} \).
Therefore, \( \mathcal{O}_X(D), O] := [D, O] \) and thus
\[
\{ [A, O] : A \in \alpha_\mathcal{D}, O \in \mathcal{O}_Z \} = \{ [D, O] : D \in \mathcal{D}, O \in \mathcal{O}_Z \}.
\]

**Example 2.1.** If \( \mathcal{D} = [X]^{<\aleph_0} \), then
\[
\{ [F, O] : F \in [X]^{<\aleph_0}, O \in \mathcal{O}_Z \}
\]
is a base of the topological space \( C_p(X, Z) \) of pointwise convergence (here \( p \) abridges \( [X]^{<\aleph_0} \)).

**Example 2.2.** If \( \mathcal{D} = \mathcal{K}_X \) (the family of compact subsets of \( X \)), then
\[
\{ [K, O] : K \in \mathcal{K}_X, O \in \mathcal{O}_Z \}
\]
is a base of the topological space \( C_k(X, Z) \) of compact-open topology (here \( k \) abridges \( \mathcal{K}_X \)).

We consider two complementary topologies on, respectively, the hyperspace \( C(X, \$) \) and the set \( C(X, \$^*) \) of open subsets of \( X \). Here \( \$ \) and \( \$^* \) are two homeomorphic avatars of the Sierpiński topology on \( \{0, 1\} \):
\[
\$ := \{ \emptyset, \{1\}, \{0, 1\} \} \quad \text{and} \quad \$^* := \{ \emptyset, \{0\}, \{0, 1\} \}.
\]
The indicator function \( \psi_A \) of a subset \( A \) of \( X \) is defined by to be 0 on \( A \) and 1 out of \( A \). If \( X \) is a topological space, then \( \psi_A \in C(X, \$) \) if and only if \( A \) is closed, and \( \psi_A \in C(X, \$^*) := \mathcal{O}_X \) if and only if \( A \) is open.

The complementation \( c : 2^X \to 2^X \) associates \( A^c := X \setminus A \) with \( A \subset X \). In order to avoid ambiguity, we denote the image of \( A \subset 2^X \) by the complementation by
\[
\mathcal{A}_c := \{ A^c : A \in \mathcal{A} \}.
\]

The topology \( \alpha(X, \$^*) \) on the set \( C(X, \$^*) \) (of all open subsets of \( X \)) has \( \alpha \) for a subbase, because, due to our convention, the subbase consists
of \(\{[A, \{0\}] : A \in \alpha\}\), and \([A, \{0\}] = \{\psi_B \in C(X, S^*) : \psi_B(0) \in A\}\) (by definition, \(\psi_B(0) = B\)).

If \(\alpha\) is stable for finite intersections, then \(\alpha\) is a base of \(\alpha(X, S^*)\).

Hence the neighborhood filter \(N_{\alpha(X, S^*)}(Y)\) of \(Y \in C(X, S^*)\) is generated by
\[
\{A \in \alpha : Y \in A\}. 
\]
In particular, for \(\alpha = \alpha_D\) a subbase for open sets is of the form
\[
\{O_X(D) : D \in D\},
\]
and \(\alpha_D\) is stable for finite intersections provided that \(D\) is stable for finite unions, so that
\[
N_{\alpha_D(X, S^*)}(Y) \approx \{O_X(D) : Y \supset D \in D\}.
\]

The homeomorphic image of \(\alpha(X, S^*)\) by the complementation is a topology on the hyperspace \(C(X, S)\) denoted by \(\alpha(X, S)\). Accordingly, \(\{A_c : A \in \alpha\}\) is a subbase of \(\alpha(X, S)\)-open sets on the hyperspace \(C(X, S)\); the neighborhood of \(H \in C(X, S)\) with respect to \(\alpha(X, S)\) is
\[
N_{\alpha(X, S)}(H) \approx \{A_c : H^c \in A \in \alpha\}. 
\]
In particular, a base of \(N_{\alpha_D(X, S)}(A_0)\) consists of
\[
\{\{A \in C(X, S) : A \cap D = \emptyset\} : D \in D, A_0 \cap D = \emptyset\}.
\]

This form of basic neighborhoods is at the origin of the term \(D\)-miss topology.

**Remark 2.3.** Gruenhage introduced the so-called \(\gamma\)-connection [12]. In particular, a filter \(\Gamma(Y, X)\), where \(Y\) is an open subset of \(X\), is generated by
\[
\{O_X(F) : Y \supset F \in [X]<\aleph_0\},
\]
hence \(\Gamma(Y, X)\) is a neighborhood filter of \(Y\) with respect to
\[
\alpha_{[X]<\aleph_0} := \{O_X(F) : F \in [X]<\aleph_0\}.
\]

### 3. Preimage-wise characterization

Denote by \(f^-(A) := \{x : f(x) \in A\}\) and by \(F^-(A)\) a filter generated by
\[
\{\{f^-(A) : f \in F\} : F \in F\}.
\]
What follows is a special case of a theorem (see [7]) about \(C_a(X, T)\) and \(C_a(X, S)\), where \(X\) is a convergence space and \(T\) is a topological space.

**Theorem 3.1.** Let \(\alpha\) be a collection of openly isotone families on a topological space \(X\). Let \(C\) be a base of closed subsets of \(\mathbb{R}\). If \(F\) is a filter on \(C(X, \mathbb{R})\), then
\[ f \in \lim_{\alpha \in (X, R)} F \iff f^-(C) \in \lim_{\alpha \in (X, S)} F^-(C) \]

for each \( C \).

**Proof.** By definition, \( f_0 \in \lim_{\alpha \in (X, R)} F \) if and only if for each open subset \( O \) of \( \mathbb{R} \) and every \( A \in \alpha \) such that \( f_0 \in [A, O] \), there exists \( F \in F \) such that \( f \in [A, O] \) for each \( f \in F \). In other words, if \( f_0^{-}(O) \in A \), then there exists \( F \in F \) such that \( f^{-}(O) \in A \) for each \( f \in F \), that is, \( F^{-}(O) \) converges to \( f_0^{-}(O) \) in \( \alpha(X, S) \), equivalently, \( f_0^{-}(O^c) \) converges to \( f_0^{-}(O^c) \) in \( \alpha(X, S) \).

Suppose that \( f_0^{-}(C) \in \lim_{\alpha \in (X, S)} F^{-}(C) \) for each element \( C \) of a base of closed subsets of \( \mathbb{R} \). Let \( A \) be a closed subset of \( \mathbb{R} \) and \( \mathcal{C}_A \subset C \) be such that \( A = \bigcap_{C \in \mathcal{C}_A} C \). If \( x \notin f_0^{-}(A) \) then there is \( C \in \mathcal{C}_A \) such that \( x \notin f_0^{-}(A) \), hence, by assumption, there exists \( F \in F \) such that \( x \notin F^{-}(C) \), and thus \( x \notin f^{-}(A) \) for every \( f \in F \), that is, \( f_0^{-}(A) \in \lim_{\alpha \in (X, S)} F^{-}(A) \).

**Corollary 3.2.** The (infinite) tightness of \( \alpha(X, R) \) is not greater than that of \( \alpha(X, S) \).

**Proof.** Suppose that the tightness of \( \alpha(X, S) \) be \( \lambda \) and let \( \mathcal{C} \) be a countable base of closed subsets of \( \mathbb{R} \). If \( f_0 \in \text{cl}_{\alpha(X, R)} \mathcal{B} \), then by Th. 3.1, \( f_0^{-}(C) \in \text{cl}_{\alpha(X, S)} \{f^{-}(C) : f \in \mathcal{B}\} \) for each \( C \in \mathcal{C} \). Hence for each \( C \in \mathcal{C} \) there is \( \mathcal{B}_C \subset \mathcal{B} \) with \( \text{card}(\mathcal{B}_C) \leq \lambda \) such that

\[ f_0^{-}(C) \in \text{cl}_{\alpha(X, S)} \{f^{-}(C) : f \in \mathcal{B}_C\} \]

thus \( f_0^{-}(C) \in \text{cl}_{\alpha(X, S)} \{f^{-}(C) : f \in \mathcal{B}_0\} \), where \( \mathcal{B}_0 := \bigcup_{C \in \mathcal{C}} \mathcal{B}_C \). Th. 3.1 implies that \( f_0 \in \text{cl}_{\alpha(X, R)} \mathcal{B}_0 \) and \( \text{card}(\mathcal{B}_0) \leq \lambda \).

**Corollary 3.3.** The (infinite) character of \( \alpha(X, R) \) is not greater than that of \( \alpha(X, S) \).

**Proof.** Suppose that the character of \( \alpha(X, S) \) be \( \lambda \) and let \( \mathcal{C} \) be a countable base of closed subsets of \( \mathbb{R} \). Then \( f \in \lim_{\alpha \in (X, R)} F \) if and only if \( f^{-}(C) \in \lim_{\alpha \in (X, S)} F^{-}(C) \) for each element \( C \in \mathcal{C} \). By the assumption, for each \( C \in \mathcal{C} \) there is a filter \( \mathcal{E}_C \leq F^{-}(C) \) of character not greater than \( \lambda \) such that \( f^{-}(C) \in \lim_{\alpha \in (X, S)} \mathcal{E}_C \). Let \( \mathcal{F}_C \subset F \) be a filter on \( C(X, R) \) such that \( F \in \mathcal{F}_C \) whenever there is \( E \in \mathcal{E}_C \) for which \( E \subset F^{-}(C) \). Let \( \mathcal{C} \) be ranged in a sequence \( \{C_n : n < \omega\} \). Then there is a sequence \( \{\mathcal{F}_{C_n}\}_n \) such that \( \mathcal{F}_{C_n} \subset \mathcal{F}_{C_{n+1}} \subset F \) and \( f^{-}(C_n) \in \lim_{\alpha \in (X, S)} \mathcal{F}^{-}_{C_k}(C_n) \) for each \( k \leq n \). Consequently \( (\bigcup_{k < \omega} \mathcal{F}_{C_k})^{-}(C_n) \) converges to \( f^{-}(C_n) \) in \( \alpha(X, S) \) for each \( n < \omega \), and the character of \( \bigcup_{k < \omega} \mathcal{F}_{C_k} \) is not greater than \( \lambda \). By Th. 3.1, \( f \in \lim_{\alpha \in (X, R)} \bigcup_{k < \omega} \mathcal{F}_{C_k} \). \( \diamond \)
As we have seen, no assumptions on $X$ or $\alpha$ were needed to get the corollaries above. The converse inequality will be established in the case of compact webs in completely regular spaces.

4. Compact families

An openly isotone family $\mathcal{A}$ is compact if each family $\mathcal{P}$ of open sets such that $\bigcup \mathcal{P} \in \mathcal{A}$ has a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup \mathcal{P}_0 \in \mathcal{A}$.

We denote by $\kappa(X)$ the collection of all compact families on $X$. Here are fundamental examples:

- $K$ compact $\Rightarrow \mathcal{O}_X(K) \in \kappa(X)$;
- $x \in \lim_X \mathcal{F} \Rightarrow \mathcal{O}_X(\mathcal{F} \cap \{x\}) \in \kappa(X)$,

where $\mathcal{F} \cap \{x\} := \{\{F \cup \{x\} : F \in \mathcal{F}\}$.

The collection of (openly isotone) compact families fulfill the following properties:

$\emptyset, \mathcal{O}_X \in \kappa(X)$;

$\alpha \subset \kappa(X) \Rightarrow \bigcup_{\mathcal{A} \in \alpha} \mathcal{A} \in \kappa(X)$;

$\mathcal{A}_0, \mathcal{A}_1 \in \kappa(X) \Rightarrow \mathcal{A}_0 \cap \mathcal{A}_1 \in \kappa(X)$.

Therefore

**Corollary 4.1.** $\kappa(X)$ is the collection of open sets of a topology on $\mathcal{O}_X = C(X, \mathcal{S}^*)$.

The topology of Cor. 4.1 is called the Scott topology (see [11], [3]).

**Example 4.2.** If $\kappa = \kappa(X)$ is the collection of (openly isotone) compact families on $X$, then

$\{[\mathcal{A}, O] : \mathcal{A} \in \kappa(X), O \in \mathcal{O}_Z\}$

is a subbase of the Isbell topology on $C(X, Z)$. In particular, $\kappa(X)$ is the collection of open sets of $C_\kappa(X, \mathcal{S}^*)$.

**Lemma 4.3.** If $\mathcal{A} = \mathcal{O}(\mathcal{A})$ is a compact family of subsets of a completely regular topological space $X$, and $F$ is a closed subset of $X$ with $F^c \in \mathcal{A}$, then there is $A \in \mathcal{A}$ and $h \in C(X, [0, 1])$ such that $h(A) = \{0\}$ and $h(F) = \{1\}$.

**Proof.** By complete regularity, for every $x \notin F$, there is an open neighborhood $O_x$ of $x$ and $f_x \in C(X, [0, 1])$ such that $f_x(O_x) = \{0\}$ and $f_x(F) = \{1\}$. Therefore $F^c = \bigcup_{x \notin F} O_x \in \mathcal{A}$, so that by the compactness
of $\mathcal{A}$ there is $n < \omega$ and $x_1, \ldots, x_n \notin F$ such that $A = \bigcup_{i \leq n} O_{x_i} \in \mathcal{A}$. The continuous function $\min_{1 \leq i \leq n} f_{x_i}$ is 0 on $A$ and 1 on $F$. ♦

If $\mathcal{A}$ is an openly isotone family on $X$ and $C$ is a subset of $X$, then $\mathcal{A} \vee C := O_X (\{ A \cap C : A \in \mathcal{A} \})$.

**Lemma 4.4.** If $\mathcal{A}$ is a compact openly isotone family on $X$ and $C$ is a closed subset of $X$, then $\mathcal{A} \vee C$ is compact.

**Proof.** Indeed, if $P$ is a family of open sets such that $\bigcup P \in \mathcal{O} (\{ A \cap C : A \in \mathcal{A} \})$, then $\bigcup P \cup (X \setminus C) \in \mathcal{A}$, hence there exists a finite subfamily $P_0$ of $P$ such that $\bigcup P_0 \cup (X \setminus C) \in \mathcal{A}$, thus $\bigcup P_0 \in \mathcal{O} (\{ A \cap C : A \in \mathcal{A} \})$. ♦

The concept of network is well-known. Here we introduce a notion of web that extends and weakens that of network. A collection $\alpha$ of openly isotone families is a web in $X$ if for every $x \in X$ and each $O \in O_X (x)$ there is $A \in \alpha$ such that $A$ is generated by a filter on $O$. In particular, $\alpha_D (2.1)$ is a web if for each $x \in X$ and every $O \in O_X (x)$ there is $D \in D$ such that $D \subset O$. This is a weaker property than that of $D$ being a network. A collection of openly isotone families is called a compact web if it is a web consisting of compact families.

**Proposition 4.5.** If $D$ is a compact network, then $\alpha_D$ is a compact web.

Indeed, in this case, $\alpha_D$ is a collection of compact families. It is a web, because it includes $\{ O_X (\{x\}) : x \in X \}$. For instance, $\{ O_X (F) : F \in [X]^{<\aleph_0} \}$ and $\{ O_X (K) : K \in \mathcal{K}(X) \}$ are compact webs. Therefore,

**Corollary 4.6.** $\kappa(X)$ is a compact web on $X$.

In fact, $\kappa(X)$ is a web, because it includes a web, for example, $\{ O_X (K) : K \in \mathcal{K}(X) \}$. The following result extends [17, Th. 1.1.5].

**Lemma 4.7.** If $Z$ is Hausdorff and $\alpha$ is a web, then $C_\alpha (X, Z)$ is Hausdorff.

**Proof.** If $f_0 \neq f_1$ then there is $x \in X$ such that $f_0 (x) \neq f_1 (x)$, and because $Z$ is Hausdorff, there exist disjoint open sets $O_0$ and $O_1$ such that $f_0 (x) \in O_0$ and $f_1 (x) \in O_1$. Therefore $W := f_0^-(O_0) \cap f_1^-(O_1) \in O_X (x)$, and since $\alpha$ is a web, there exists $\mathcal{A} \in \alpha$ such that $\mathcal{A}$ is generated by a filter on $W$. Therefore $f_0 \in [\mathcal{A}, O_0], f_1 \in [\mathcal{A}, O_1]$ and $[\mathcal{A}, O_1] \cap [\mathcal{A}, O_0]$ is empty, for if $f \in [\mathcal{A}, O_1] \cap [\mathcal{A}, O_0]$ then there exist $W \supset A_0, A_1 \in \mathcal{A}$ such that $A_0 \subset f^-(O_0), A_1 \subset f^-(O_1)$ and $A := A_0 \cap A_1 \in \mathcal{A}$, hence $f(A) \subset O_0 \cap O_1 = \emptyset$. ♦

A family $\mathcal{D}$ of closed subsets of $X$ is called hereditarily closed pro-
vided that \( F \subset D \in \mathcal{D} \) and \( F \) is closed implies that \( F \in \mathcal{D} \).

It is proved in [17, Th. 1.1.7] that

**Theorem 4.8.** If \( \mathcal{D} \) is a hereditarily closed compact network and \( Z \) is a topological group, then \( C_\mathcal{D}(X, Z) \) is a topological group.

In particular, the topology \( \alpha_\mathcal{D} \) of Th. 4.8 is homogeneous. Of course, families of all closed compact subsets and of all finite subsets of \( T_1 \) topologies are hereditarily closed compact networks, so that, in particular, \( C_p(X, \mathbb{R}) \) and \( C_k(X, \mathbb{R}) \) are topological groups, in fact, topological vector spaces.

Nevertheless, there exists a topological space \( X \) (satisfying high separation axioms) and a collection \( \alpha \) of compact families including all families generated by compact sets, for which \( C_\alpha(X, \mathbb{R}) \) is not a translation invariant. Of course, such a space \( X \) must be dissonant.

**Example 4.9.** Consider the Arens topology on \( X := \{x_\infty\} \cup \bigcup_{n<\omega} X_n \) where \( X_n := \{x_{n,k} : k < \omega\} \): each \( x \neq x_\infty \) is isolated, and \( O \in \mathcal{O}_X(x_\infty) \) whenever there is \( n_O \) and a map \( h : \omega \to \omega \) such that

\[
\{x_\infty\} \cup \{x_{n,k} : n \geq n_O, k \geq h(n)\} \subset O.
\]

The Arens topology is a prime topology, that is, all the elements but possibly one are isolated. Each prime topology has strong separation properties, in particular, is zero-dimensional and paracompact. Every compact subset of the Arens space is finite. A compact family \( \mathcal{S} \) is simple if either \( \mathcal{S} = \mathcal{O}_X(F) \) where \( F \) is a compact (hence, finite) subset of \( X \), or \( \mathcal{S} \subset \mathcal{O}_X(x_\infty) \). Every compact family on the Arens space is a union of simple families. It is known [6] that the Arens topology is dissonant, in other words, there exists a compact family \( \mathcal{S} \) that is not of the form \( \mathcal{O}_X(F) \) with compact set \( F \), hence \( \mathcal{S} \not\subset \mathcal{O}_X(x_\infty) \).

Let \( \mathcal{D} \subset \mathcal{O}_X(x_\infty) \) be the compact family such that \( D \cap X_n \neq \emptyset \) for each \( n < \omega \) and every \( D \in \mathcal{D} \), and let \( \alpha := \{\mathcal{D}\} \cup \{\mathcal{O}_X(F) : F \in [X]^{<\omega_0}\} \).

Then \( C_\alpha(X, \mathbb{R}) \) is not translation invariant.

Indeed, let \( D_0 \in \mathcal{D} \) be such that \( X_n \setminus D_0 \neq \emptyset \) for each \( n < \omega \).

Define \( f(D_0) = \{0\} \) and \( f(X \setminus D_0) = \{1\} \). Then the translation \( g \mapsto f + g \) is not continuous at \( \emptyset \). Indeed, \( f + 0 \in [\mathcal{D}, B(0, \varepsilon)] \) where \( \varepsilon = \frac{1}{2} \).

Take any finite set \( F \) and \( 0 < \delta < \varepsilon \), and consider a neighborhood

\[
W_\delta := [\mathcal{D}, B(0, \delta)] \cap [\mathcal{O}_X(F), B(0, \delta)]
\]

of the zero function \( \emptyset \). Then there is \( n_F < \omega \) such that \( X_{n_F} \cap F = \emptyset \).

Let \( D_1 \in \mathcal{D} \) be such that \( X_{n_F} \cap D_1 \cap D_0 = \emptyset \). On the other hand, \( X_{n_F} \cap D_0 \neq \emptyset \) and \( X_{n_F} \cap D_1 \neq \emptyset \) by the definition of \( \mathcal{D} \). Set \( g(D_1 \cup F) = \{0\} \).

\(^{3}\)By analogy to openly isotone one could call this property closedly antitone.
and \( g(x) = 1 \) elsewhere, so that \( g \in W_\delta \) for each \( \delta > 0 \). Notice that \( f(x) + g(x) \in \{1, 2\} \) for each \( x \in X_n \), and since \( X_n \cap D \neq \emptyset \) for every \( D \in \mathcal{D} \), \((f + g)(D) \cap \{1, 2\} \neq \emptyset \) and thus \((f + g) \notin [\mathcal{D}, B(0, \varepsilon)]\).

5. Polar topologies

Recall that if \( \Omega \subset V \times W \), then the \( \Omega\text{-polar} \) \( \Omega^*A \) of a subset \( A \) of \( V \) is the greatest subset \( B \) of \( W \) such that \( A \times B \subset \Omega \). Dual topologies can be represented in terms of polarity.

For every open subset \( O \) of \( \mathbb{R} \) we define a relation \( \Omega^*_O := \{(x, f) : f(x) \in O\} \). Accordingly, for each \( A \in C(X, \$^*) \), the set \([A, O]\) is the \( \Omega^*_O \)-polar of \( A \). Indeed,

\[
[A, O] = \{ f : A \subset f^-(O) \} = \Omega^*_O A.
\]

On the other hand, \( \Omega^*_O \) is a relation on \( C(X, \$^*) \times C(X, \mathbb{R}) \), namely

\[
\Omega^*_O = \{(A, f) : A \subset f^-(O) \},
\]

so that if \( A \) is a subset of \( C(X, \$^*) \), then \( \Omega^*_O A = \bigcup_{A \in A} [A, O] = [A, O] \). Hence for a filter (base) \( \alpha \) on \( C(X, \$^*) \), our convention yields

\[
\Omega^*_O \alpha \approx \{ [A, O] : A \in \alpha \}.
\]

Finally

\[
\mathcal{N}_{\alpha(X, \mathbb{R})}(\tilde{0}) \approx \bigvee_{O \in \mathcal{N}_{\mathbb{R}}(0)} \Omega^*_O \alpha \approx \{ [A, O] : A \in \alpha, O \in \mathcal{N}_{\mathbb{R}}(0) \}.
\]

In case of homogeneity, it is enough to establish a property of \( \mathcal{N}_{\alpha(X, \mathbb{R})}(\tilde{0}) \) in order to prove that property for every neighborhood filter of \( C_\alpha(X, \mathbb{R}) \) (for \( \alpha = \alpha_\mathcal{D} \) with a compact network \( \mathcal{D} \) on a completely regular space \( X \)).

On the other hand, it follows from Th. 3.1 that the function \( \tilde{0} \in \lim_{\alpha(X, \mathbb{R})} \mathcal{F} \) implies, in particular, \( \tilde{0}^-(C) \in \lim_{\alpha(X, \$)} \mathcal{F}^-(C) \) for each closed subset \( C \) of \( \mathbb{R} \). If \( 0 \in C \) then \( 0^-(C) = X \), hence \( \tilde{0}^-(C) \in \lim_{\alpha(X, \$)} \mathcal{F}^-(C) \) for every \( \mathcal{F} \). Hence the only case to consider is that of \( 0 \notin C \) that is equivalent to \( \tilde{0}^-(C) = \emptyset \).

This observation implies that properties of \( \mathcal{N}_{\alpha(X, \$)}(\emptyset) \) are intimately related to properties of \( \mathcal{N}_{\alpha(X, \mathbb{R})}(\tilde{0}) \), hence to local properties of \( C_\alpha(X, \mathbb{R}) \), thanks to homogeneity (for \( \alpha = \alpha_\mathcal{D} \) with a compact network \( \mathcal{D} \) on a completely regular space \( X \)). As \( \alpha(X, \$) \) and \( \alpha(X, \$^*) \) are homeomorphic by complementation, the properties of \( \mathcal{N}_{\alpha(X, \$)}(\emptyset) \) and \( \mathcal{N}_{\alpha(X, \$^*)}(X) \) are the same. On the other hand, \( \mathcal{N}_{\alpha(X, \$^*)}(X) \) has a filter subbase \( \alpha \).
Properties transfer between topologies

If \( \Gamma \subset X_1 \times \ldots \times X_m \) is a relation, then for \( 1 \leq k \leq m \), let \( \Gamma_k : \Gamma \to X_k \) be the restriction to \( \Gamma \) of the \( k \)-th projection of \( X_1 \times \ldots \times X_m \).

Consider the fundamental relation \( \Gamma \subset C(X, \mathbb{R}) \times C(X, \mathbb{R}^*) \times C(\mathbb{R}, \mathbb{R}^*) \) defined by

\[
\Gamma := \{(f, A, O) : f \in [A, O]\}.
\]

The last component of \( \Gamma \) is valued in (open) subsets of \( \mathbb{R} \), and not in \( \mathbb{R} \), because \( \Gamma \) results from a polarity. Therefore, we need to define a filter on \( \mathcal{O}_\mathbb{R}(0) \) such that its projection on \( \mathbb{R} \) coincides with \( \mathcal{N}_\mathbb{R}(0) \).

A base for such a filter (denoted by \( \mathcal{\tilde{N}}_\mathbb{R}(0) \)) is given by \( \{P \in \mathcal{O}_\mathbb{R}(0) : P \subset O\} \) with \( O \in \mathcal{O}_\mathbb{R}(0) \).

**Theorem 5.1.** \( \mathcal{N}_{\alpha(X,\mathbb{R})}(0) \cong \Gamma_1(\mathcal{\Gamma}_2^\alpha \vee \mathcal{\Gamma}_3^\mathcal{\rho}_R(0)) \).

**Proof.** By definition, \( \mathcal{\Gamma}_2^\alpha A = \{(f, A, O) : f \in [A, O], A \in \mathcal{A}\} \), and \( \mathcal{\Gamma}_3 O = \{(f, A, O) : f \in [A, O]\} \). Hence \( \mathcal{\Gamma}_1(\mathcal{\Gamma}_2^\alpha \vee \mathcal{\Gamma}_3^\mathcal{\rho}_R(0)) = [A, O] \), so that \( \mathcal{N}_{\alpha(X,\mathbb{R})}(0) = \mathcal{\Gamma}_1(\mathcal{\Gamma}_2^\alpha \vee \mathcal{\Gamma}_3^\mathcal{\rho}_R(0)) \).

Let \( \Delta \) be the following subset of \( C(X, \mathbb{R}^*) \times C(X, \mathbb{R}^*) \):

\[
\Delta := \{(A, G) : \exists \theta \in C(X, [0, 1]) \theta(A) = \{0\} , \theta(X \setminus G) = \{1\}\}. 
\]

Let \( \Theta : \Delta \to C(X, [0, 1]) \) be such that

\[
\Theta(A, G)(\gamma) = \{0\} \text{ and } \Theta(A, G)(X \setminus G) = \{1\}.
\]

Denote by \( \mathcal{\Delta}_2 \) the projection of \( \Delta \) on the second component.

**Theorem 5.2.** If \( \alpha \) is a compact web, and \( X \) is completely regular, then

\[
\alpha \cong \mathcal{\Delta}_2(\mathcal{\Theta}^-\mathcal{N}_{\alpha(X,\mathbb{R})}(0)).
\]

**Proof.** If \( G \in \mathcal{\Delta}_2(\mathcal{\Theta}^-[A, (\frac{1}{n+\frac{1}{n})})] \) then there is an open subset \( D \) of \( X \) such that \( \Theta(D, G) \in [A, (\frac{1}{n}, \frac{1}{n})] \), that is, there \( A \in \mathcal{A} \) such that \( \Theta(D, G)(\gamma) \subset (\frac{1}{n}, \frac{1}{n}) \) hence \( A \subset G \), and thus \( G \in \mathcal{A} \). It follows that \( \mathcal{\Delta}_2(\mathcal{\Theta}^-[A, (\frac{1}{n}, \frac{1}{n})]) \subset \mathcal{A} \) for each \( n < \omega \).

Conversely, if \( G \in \mathcal{A} \) then, by Lemma 4.3, there is \( A \in \mathcal{A} \) such that \( (A, G) \in \Delta \), thus \( \Theta(A, G)(\gamma) = \{0\} , \Theta(A, G)(X \setminus G) = \{1\} \). Hence \( \Theta(A, G) \in [A, (\frac{1}{n}, \frac{1}{n})] \) for every \( n < \omega \). In other words, \( (A, G) \in \mathcal{\Theta}^-[A, (\frac{1}{n}, \frac{1}{n})] \) and so \( G \in \mathcal{\Delta}_2(\mathcal{\Theta}^-[A, (\frac{1}{n}, \frac{1}{n})]) \), showing that \( \mathcal{A} \subset \mathcal{\Delta}_2(\mathcal{\Theta}^-[A, (\frac{1}{n}, \frac{1}{n})]) \) for every \( n < \omega \). \( \diamondsuit \)

6. Transfer of properties

Let \( \mathcal{B} \) be a class of filters. A topology is \( \mathcal{B} \)-based if and only if each neighborhood filter is in \( \mathcal{B} \). For each class \( \mathcal{B} \), the \( \mathcal{B} \)-based topologies form a concretely coreflective subcategory of topologies. For example, classes of topologies of a given character, or of a given tightness, can
be represented as those of $\mathcal{B}$-based topologies for appropriate classes $\mathcal{B}$. Other instances of classes of $\mathcal{B}$-based topologies for appropriate classes of filters $\mathcal{B}$ are sequentiality, Fréchetness, strong Fréchetness, productive Fréchetness, bisequentiality, and others (see, e.g., [4]).

Thms. 5.1 and 5.2 enable us to transfer some such coreflective properties from $C_{\alpha}(X,\mathbb{R})$ to $C_{\alpha}(X,\mathbb{S})$ and vice versa.

If $H \subset X \times Y$, then $H_x := \{y \in Y : (x,y) \in H\}$, and if $A \subset X$ then $HA := \bigcup_{x \in A} H_x$. If now $\mathcal{F}$ and $\mathcal{H}$ are families of subsets of $X$ and $X \times Y$ respectively, then

$$HF := \{HF : F \in \mathcal{F}, H \in \mathcal{H}\}$$

is a family of subsets of $Y$. If $\mathcal{F}$ and $\mathcal{H}$ are filters, then, by a handy abuse of notation, $HF$ stands also for the filter it generates.

Let $F_\lambda$ denote the class of filters admitting a filter base of cardinality less than $\aleph_\lambda$. In particular, $F_0$ is the class of principal filters, and $F_1$ is the class of countably based filters. The class of all filters is denoted by $F$.

A class $\mathcal{B}$ of filters is $\mathcal{H}$-composable if $HF \in \mathcal{B}$ for each $F \in \mathcal{B}$ and every $H \in \mathcal{H}$ (see [8], [13], [16]). A class $\mathcal{B}$ of filters is $\mathcal{H}$-steady if $H \lor F \in \mathcal{B}$ for each $F \in \mathcal{B}$ and each $H \in \mathcal{H}$ (see [13], [16]).

If $\mathcal{H}$ is a class of filters and $\gamma$ is a filter subbase, then $\gamma \in \mathcal{H}$ means that the filter generated by $\gamma$ belongs to $\mathcal{H}$.

By Th. 5.1,

**Proposition 6.1.** Let $\mathcal{B}$ be $F_0$-composable and $F_1$-steady. If $X$ is completely regular, $\alpha$ is a compact web, and $\alpha \in \mathcal{B}$, then $C_{\alpha}(X,\mathbb{R})$ is $\mathcal{B}$-based at $\tilde{0}$. If moreover $D$ is a hereditarily closed compact network, then $C_{\alpha_D}(X,\mathbb{R})$ is $\mathcal{B}$-based.

**Proof.** If $\alpha \in \mathcal{B}$ then $\Gamma_2^{-} \alpha \in \mathcal{B}$, because $\mathcal{B}$ is $F_0$-composable. On the other hand, $\Gamma_3^{-} \mathcal{N}_\mathbb{R}(0)$ is a countably based filter, because $\mathcal{N}_\mathbb{R}(0)$ is countably based. Therefore, $\Gamma_2^{-} \alpha \lor \Gamma_3^{-} \mathcal{N}_\mathbb{R}(0) \in \mathcal{B}$, because $\mathcal{B}$ is $F_1$-steady. Finally, $\mathcal{N}_{\alpha(X,\mathbb{R})}(0) \in \mathcal{B}$ as the image by a map of a filter from $\mathcal{B}$. Therefore $C_{\alpha_D}(X,\mathbb{R})$ is $\mathcal{B}$-based because $C_{\alpha_D}(X,\mathbb{R})$ is homogeneous by Th. 4.8. $\Diamond$

**Proposition 6.2.** Let $\mathcal{B}$ be $F_0$-composable. If $\alpha$ is a compact web, $X$ is completely regular, and $C_{\alpha}(X,\mathbb{R})$ is $\mathcal{B}$-based, then $\alpha \in \mathcal{B}$.

**Proof.** If $C_{\alpha}(X,\mathbb{R})$ is $\mathcal{B}$-based, $\mathcal{N}_{\alpha(X,\mathbb{R})}(0) \in \mathcal{B}$, hence by Th. 5.2, $\alpha \in \mathcal{B}$, because $\mathcal{B}$ is $F_0$-composable. $\Diamond$

**Theorem 6.3.** Let $\mathcal{B}$ be $F_0$-composable and $F_1$-steady, and let $D$ be a hereditarily closed compact network on a completely regular space $X$. Then $C_{\alpha_D}(X,\mathbb{R})$ is $\mathcal{B}$-based if and only if $\alpha_D \in \mathcal{B}$. 
Properties transfer between topologies

F. Jordan established in [13, Th. 3] a special case of Th. 6.3 for
\( \alpha = \{ \mathcal{O}(D) : D \in [X]^{<\aleph_0} \} \), hence concerning \( C_{\alpha}(X, \mathbb{R}) \), in terms of \( \gamma \)-
connection (see Rem. 2.3). It is enough to replace in his proofs \([X]^{<\aleph_0}\) by any (additively stable) family \( \mathcal{D} \) of compact sets, in order that the
proofs remain valid for \( \alpha = \{ \mathcal{O}(D) : D \in \mathcal{D} \} \) and \( C_{\mathcal{D}}(X, \mathbb{R}) \).

Since \( \alpha \) is a filter subbase of \( \mathcal{N}_{\alpha(X, \ast)}(X) \), and \( \alpha(X, \ast) \) is homeo-
morphic to \( \alpha(X, \$) \) by complementation, we have

**Corollary 6.4.** Let \( \mathbb{B} \) be \( \mathbb{F}_0 \)-composable and \( \mathbb{F}_1 \)-steady, and let \( \mathcal{D} \) be
a hereditarily closed compact network on a completely regular space \( X \). Then \( C_{\alpha_{\mathcal{D}}}(X, \mathbb{R}) \) is \( \mathbb{B} \)-based if and only if \( \mathcal{N}_{\alpha_{\mathcal{D}}(X, \$)}(\emptyset) \in \mathbb{B} \).

7. Transferable properties

We shall review several \( \mathbb{F}_0 \)-composable \( \mathbb{F}_1 \)-steady classes of filters,
in other words, transferable local properties. Several results on compos-
ability and steadiness can be found in [13], [16].

We say that a property of topological spaces is **local** if there is a
class \( \mathbb{P} \) of filters\(^4\) such that a topology has the property whenever each
neighborhood filter belongs to \( \mathbb{P} \). Character and tightness are examples
of local properties.

Two families \( \mathcal{A} \) and \( \mathcal{B} \) of subsets of a given set **mesh** (in symbols,
\( \mathcal{A} \# \mathcal{B} \)) if \( A \cap B \neq \emptyset \) for each \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \). The **grill** \( \mathcal{A}^\# \) of a family
\( \mathcal{A} \) of subsets of \( X \) is defined as \( \{ H \subset X : H \# A \} \), where \( H \# A \) is an
abbreviation for \( \{ H \} \# \mathcal{A} \). The **character** \( \chi(\mathcal{F}) \) of a filter \( \mathcal{F} \) is the least
cardinal \( \tau \) such that \( \mathcal{F} \) has a filter base of cardinality \( \tau \). The **tightness**
\( t(\mathcal{F}) \) of a filter \( \mathcal{F} \) is the least cardinal \( \tau \) for which if \( A \in \mathcal{F}^\# \) then there
is \( B \subset A \) of cardinality \( \tau \) such that \( B \in \mathcal{F}^\# \). It was proved in [15] that

**Proposition 7.1.** (Infinite) character and tightness are \( \mathbb{F}_0 \)-composable
and \( \mathbb{F}_1 \)-steady.

A filter \( \mathcal{F} \) is \( \mathbb{G} \) to \( \mathbb{E} \) **refinable** [14] \( (\mathcal{F} \in (\mathbb{G}/\mathbb{E})_\geq) \) if for each filter
\( \mathcal{G} \in \mathbb{G} \) with \( \mathcal{G} \# \mathcal{F} \) there exists a filter \( \mathcal{E} \in \mathbb{E} \) such that \( \mathcal{E} \geq \mathcal{F} \cap \mathcal{G} \); a filter
\( \mathcal{F} \) is \( \mathbb{G} \) to \( \mathbb{E} \) **me-refinable** [14] \( (\mathcal{F} \in (\mathbb{G}/\mathbb{E}) \#_\geq) \) if for each filter \( \mathcal{G} \in \mathbb{G} \)
with \( \mathcal{G} \# \mathcal{F} \) there exists a filter \( \mathcal{E} \in \mathbb{E} \) such that \( \mathcal{E} \geq \mathcal{F} \) and \( \mathcal{E} \# \mathcal{G} \). The
following two facts were observed in [14] in special cases of countably
based filters.

**Lemma 7.2.** The property \( (\mathbb{F}_\kappa/\mathbb{F}_\lambda)_\geq \) is \( \mathbb{F}_\mu \)-steady if \( \mu \leq \kappa \).

\(^4\)possibly depending on the topology.
Proof. Let $\mathcal{F} \in (\mathcal{F}_\kappa/\mathcal{F}_\lambda)_{\geq}$, $\mathcal{E} \in \mathcal{F}_\kappa$ and $\mathcal{D} \in \mathcal{F}_\mu$ be such that $\mathcal{D} \# (\mathcal{E} \vee \mathcal{F})$. Then $(\mathcal{D} \vee \mathcal{E}) \# \mathcal{F}$ and $\mathcal{D} \vee \mathcal{E} \in \mathcal{F}_\kappa$, because $\mu \leq \kappa$; thus there is $\mathcal{G} \in \mathcal{F}_\lambda$ such that $\mathcal{G} \geq \mathcal{D} \vee \mathcal{E} \vee \mathcal{F}$. $\Box$

Lemma 7.3. The property $(\mathcal{F}_\kappa/\mathcal{F}_\lambda)_{\geq}$ is $\mathcal{F}_\mu$-composable if $\mu \leq \kappa \wedge \lambda$.

Proof. If $\mathcal{F} \in (\mathcal{F}_\kappa/\mathcal{F}_\lambda)_{\geq}$, $\mathcal{E} \in \mathcal{F}_\kappa$ and $\mathcal{M} \in \mathcal{F}_\mu$ be such that $\mathcal{E} \# (\mathcal{M} \mathcal{F})$. Then $\mathcal{M}^{-} \mathcal{E} \# \mathcal{F}$ and $\mathcal{M}^{-} \mathcal{E} \in \mathcal{F}_\kappa$ provided that $\mu \leq \kappa$. As $\mathcal{F} \in (\mathcal{F}_\kappa/\mathcal{F}_\lambda)_{\geq}$ there is $\mathcal{G} \in \mathcal{F}_\lambda$ such that $\mathcal{G} \geq \mathcal{M}^{-} \mathcal{E} \vee \mathcal{F}$. Thus $\mathcal{M} \mathcal{G} \geq (\mathcal{M}^{-} \mathcal{E} \vee \mathcal{F}) \geq \mathcal{E} \vee \mathcal{M} \mathcal{F}$ and $\mathcal{M} \mathcal{G} \in \mathcal{F}_\lambda$ provided that $\mu \leq \lambda$. $\Box$

Fréchetness, strong Fréchetness, productive Fréchetness and bisequentiality are other examples of local properties that can be expressed in terms of refinable and me-refinable filters with respect to various classes (see [15] and a pioneering paper [5]). A filter $\mathcal{F}$ is

1. Fréchet $\iff \mathcal{F} \in (\mathcal{F}_0/\mathcal{F}_1)_{\geq}$: A filter $\mathcal{F}$ is Fréchet if for each set $A$ such that $A \# \mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $A \in \mathcal{E} \geq \mathcal{F}$.

2. strongly Fréchet $\iff \mathcal{F} \in (\mathcal{F}_1/\mathcal{F}_1)_{\geq}$: A filter $\mathcal{F}$ is strongly Fréchet if for each countably filter $\mathcal{G}$ such that $\mathcal{G} \# \mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $\mathcal{E} \geq \mathcal{F} \vee \mathcal{G}$.

3. productively Fréchet $\iff \mathcal{F} \in (((\mathcal{F}_1/\mathcal{F}_1)_{\geq})/\mathcal{F}_1)_{\geq}$: A filter $\mathcal{F}$ is productively Fréchet if for each strongly Fréchet filter $\mathcal{G}$ such that $\mathcal{G} \# \mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $\mathcal{E} \geq \mathcal{F} \vee \mathcal{G}$.

4. bisequential $\iff \mathcal{F} \in (\mathcal{F}/\mathcal{F}_1)_{\#}$: A filter $\mathcal{F}$ is bisequential if for each filter $\mathcal{G}$ such that $\mathcal{G} \# \mathcal{F}$ there exists a countably based filter $\mathcal{E}$ such that $\mathcal{E} \geq \mathcal{F}$ and $\mathcal{E} \# \mathcal{G}$.

Of course, in the first three conditions (but not in the fourth) the existence of a countably based filter $\mathcal{E}$ is equivalent to the existence of a sequential filter\footnote{A filter is sequential if it is generated by the queues of a sequence.} $\mathcal{E}$. All these properties are $\mathcal{F}_0$-composable. Not all are $\mathcal{F}_1$-steady.

Proposition 7.4. Classes of strongly Fréchet, productively Fréchet and bisequential filters are $\mathcal{F}_1$-steady; the class of Fréchet filters is not $\mathcal{F}_1$-steady. All the listed properties are $\mathcal{F}_0$-composable.

Proof. All the cases are proved in [16] except for bisequential filters. Let $\mathcal{F}$ be bisequential and $\mathcal{E} \in \mathcal{F}_1$. If $\mathcal{D}$ is any filter such that $\mathcal{D} \# (\mathcal{E} \vee \mathcal{F})$, then $(\mathcal{D} \vee \mathcal{E}) \# \mathcal{F}$, hence there is $\mathcal{G} \in \mathcal{F}_1$ such that $\mathcal{G} \geq \mathcal{F}$ and $\mathcal{G} \# (\mathcal{D} \vee \mathcal{E})$. The filter $\mathcal{G} \vee \mathcal{E} \in \mathcal{F}_1$ and $\mathcal{G} \vee \mathcal{E}$ meshes with $\mathcal{D}$ and $\mathcal{G} \vee \mathcal{E} \geq \mathcal{G} \geq \mathcal{F}$. Let $\mathcal{F}$ be bisequential and $A$ a relation. If $\mathcal{D}$ is a filter such that $\mathcal{D} \# A \mathcal{F}$,
then $A^{-D}\#F$, hence there is $H \in F_1$ such that $H\#A^{-D}$ and $H \geq F$. Thus $AH\#D$ and $AH \geq AF$.

If $F$ is Fréchet but not strongly Fréchet, then there is $E \in F_1$ such that $G \geq E \lor F$ for no $G \in F_1$. Hence $E \lor F$ is not Fréchet. ◇

8. Dictionary $X \leftrightarrow O_X$

Here there is a list of elementary equivalences that will be used to establish equivalences of more convoluted equivalences between properties of $C_\alpha(X,\$^\ast\!)$ and $X$. We consider only those collections $\alpha$ that are finitely stable, that is, $A_0, A_1 \in \alpha$ implies that $A_0 \cap A_1 \in \alpha$.

Let $Y \subset X$. A family $B$ of (open) subsets of $X$ is called an $\alpha$-cover of $Y$ if $B \cap A \neq \emptyset$ for every $A \in \alpha$ such that $Y \in A$. In particular, if $\alpha = \{O(D) : D \in [X]^{<\omega}\}$, then an $\alpha$-cover is an $\omega$-cover, that is, for each finite set $D$ there is $B \in B$ such that $D \subset B$.

**Lemma 8.1.** A family $B$ meshes with $N_{\alpha(X,\$^\ast\!)}(Y)$ if and only if $B$ is an $\alpha$-cover of $Y$.

**Proof.** A family $B$ meshes with $N_{\alpha(X,\$^\ast\!)}(Y)$ if and only if $B \cap A \neq \emptyset$ for each $A \in \alpha$ such that $Y \in A$. This means exactly that $B$ is an $\alpha$-cover of $Y$. ◇

Let $A, B$ be families of subsets of a given set. We say that $A$ is coarser than $B$ (equivalently, $B$ is finer than $A$)

$$A \leq B$$

if for every $A \in A$ there is $B \in B$ such that $B \subset A$. A collection of families of subsets of $X$ can be considered as a family of subsets of $2^X$. In this sense, we say that a collection is finer (coarser) than another collection. The following facts are just rewording of definitions, but we formulate them as lemmas for easy reference.

**Lemma 8.2.** A collection $\gamma$ is finer than $N_{\alpha(X,\$^\ast\!)}(Y)$ if and only if for each $A \in \alpha$ such that $Y \in A$ there is $G \in \gamma$ such that $G \subset A$.

**Lemma 8.3.** A collection $\gamma$ is coarser than $N_{\alpha(X,\$^\ast\!)}(Y)$ if and only if for each $G \in \gamma$ there is $A \in \alpha$ such that $Y \in A \subset G$.

In particular, a sequence $(G_n)_{n=1}^\omega$, that is, a family $\gamma := \{G_n : n \geq m \geq m \geq m \geq m \geq m \geq m \geq m \geq m \}$ is finer than $N_{\alpha(X,\$^\ast\!)}(Y)$ if for every $A \in \alpha$ with $Y \in A$ there is $n_A < \omega$ such that $G_n \in A$ for each $n \geq n_A$. 
8.1. Tightness

Recall that (see e.g., [17]) the \(\alpha\)-Lindelöf number of a topological space \(X\) is the least cardinal \(\tau\) such that for each \(\alpha\)-cover there exists an \(\alpha\)-subcover of cardinality less than or equal to \(\tau\).\(^6\)

By Lemma 8.1,\(^7\)

**Theorem 8.4.** The tightness of \(C_\alpha(X, \mathcal{S})\) is equal to the supremum of the \(\alpha\)-Lindelöf numbers of open subsets of \(X\).

Hence, by Cor. 3.2 and Th. 6.3,

**Theorem 8.5.** If \(\alpha\) is a compact web on a completely regular space \(X\), then \(C_\alpha(X, \mathbb{R})\) is \(\tau\)-tight if and only if the \(\alpha\)-Lindelöf number of \(X\) is \(\tau\).

These facts specialize, in an obvious way, to compact-open topologies \(C_k(X, Z)\), when \(\alpha = \{O(K) : K \in \mathcal{K}\}\) where \(\mathcal{K}\) is the family of compact subsets of \(X\), to Isbell topologies \(C_\alpha(X, Z)\), when \(\alpha = \kappa(X)\) is the collection of compact families. If \(\alpha = \{O_X(D) : D \in [X]^{\leq \aleph_0}\}\) then Th. 8.5 specializes with \(\tau = \aleph_0\) to

**Proposition 8.6.** If \(X\) is completely regular, then \(C_p(X, \mathbb{R})\) is countably tight if and only if each open \(\omega\)-cover of \(X\) has a countable \(\omega\)-subcover of \(X\).

Recall that a family \(\mathcal{P}\) is an \(\omega\)-cover of \(X\) if for each finite subset \(F\) of \(X\) there is \(P \in \mathcal{P}\) such that \(F \subseteq P\).

The following theorem is due to Arhangel’skii [1] and Pytkeev [19]:

**Theorem 8.7.** If \(X\) is completely regular, then \(C_p(X, \mathbb{R})\) is countably tight if and only if \(X^n\) is Lindelöf for every \(n < \omega\).

8.2. Character

A subset \(\gamma\) of a collection \(\alpha\) (of openly isotone families) is a base of \(\alpha\) if for each \(A \in \alpha\) there is \(G \in \gamma\) such that \(G \subseteq A\). The least cardinality \(\tau\) such \(\alpha\) has a base of cardinality \(\tau\) is called the character \(\chi(\alpha)\) of \(\alpha\).

\(^6\)More generally, if \(\kappa \leq \lambda\) are cardinals, then we say that \(X\) is \(\lambda/\kappa[\alpha]\)-compact if for every open \(\alpha\)-cover of \(X\) of cardinality \(< \lambda\) there is an \(\alpha\)-subcover of cardinality \(< \kappa\) of \(X\). In particular, a topological space is \([\alpha]\)-compact if it is \(\lambda/\aleph_0[\alpha]\)-compact for each cardinal \(\lambda\), countably \([\alpha]\)-compact if it is \(\aleph_1/\aleph_0[\alpha]\)-compact, \([\alpha]\)-Lindelöf if it is \(\lambda/\aleph_1[\alpha]\)-compact for every \(\lambda\).

\(^7\)Similar characterizations can be formulated for \(\lambda/\kappa\)-tightness with \(\kappa \geq \aleph_0\). We say that a filter \(\mathcal{F}\) is \(\lambda/\kappa\)-tight if for each \(H \in \mathcal{F}^\#\) with card \(H < \lambda\) there is \(B \subseteq H\) such that card \(B < \kappa\) and \(B \in \mathcal{F}^\#\). A topological space is \(\lambda/\kappa\)-tight if its every neighborhood filter is \(\lambda/\kappa\)-tight.
Because the character of $\alpha$ is a hereditary property, Lemma 8.2 implies that

**Theorem 8.8.** The character of $C_\alpha(X, \mathcal{S})$ is equal to the character of $\alpha$.

It follows from Cor. 3.3 and Prop. 6.2 that

**Theorem 8.9.** If $\alpha$ is a compact web on a completely regular space $X$, then the character of $C_\alpha(X, \mathbb{R})$ is equal to the character of $\alpha$.

**Corollary 8.10.** If $X$ is $T_1$, then $C_p(X, \mathcal{S})$ is of countable character if and only if $X$ is countable.

**Proof.** By Th. 8.8, the character of $C_p(X, \mathcal{S})$ is countable, if and only if for every open subset $Y$ of $X$ there is a sequence $(x_n)_n \subset Y$ such that $\{O_X(\{x_1, \ldots, x_n\}) : n < \omega\}$ is finer than $\{O_X(F) : F \in [X]^{<\aleph_0}\}$, that is, for every finite subset $F$ of $Y$ there is $n < \omega$ such that $\{x_1, \ldots, x_n\} \subset O$ implies $F \subset O$ for each open set $O$. Since $X$ is $T_1$, this means that $F \subset \{x_1, \ldots, x_n\}$. ♦

**Corollary 8.11.** If $X$ is $T_1$, then $C_k(X, \mathcal{S})$ is of countable character if and only if $X$ is hereditarily hemicompact.

**Proof.** Let $Y$ be an open subset of $X$. The neighborhood filter $N^K(X, \mathcal{S}^*)_Y$ is countably based if and only if there exists a sequence $(K_n)_n$ of compact subsets of $Y$ such that for every $n < \omega$ there exists $n$ such that $\mathcal{O}_X(K_n) \subset \mathcal{O}_X(K)$, which, for a $T_1$-topology, is equivalent $K \subset K_n$. ♦

It is well-known that a (Hausdorff) topological vector space is metrizable if and only if it is of countable character. Therefore, we recover [17, p. 60]

**Corollary 8.12.** If $X$ is completely regular, then $C_p(X, \mathbb{R})$ is metrizable if and only if it is of countable character if and only if $X$ is countable.

**Corollary 8.13.** If $X$ is completely regular, then $C_k(X, \mathbb{R})$ is metrizable if and only if it is of countable character if and only if $X$ is hemicompact.

### 8.3. Variants of Fréchetness

Here we characterize some of the properties $(\mathbb{H}/\mathbb{E})_{\geq}$ of hyperspaces in terms of their underlying spaces.

**Proposition 8.14.** $C_\alpha(X, \mathcal{S})$ is $(\mathbb{F}_\kappa/\mathbb{F}_\lambda)_{\geq}$-based if and only if $X$ enjoys the following property: For each open subset $Y$ of $X$, for every collection $\gamma$ of $\alpha$-covers of $Y$ with $\text{card} (\gamma) \leq \aleph_\kappa$, there exists a collection $\zeta$ of families of open sets with $\text{card} (\zeta) \leq \aleph_\lambda$ such that for every $A \in \alpha$ with $Y \in A$, and each $G \in \gamma$ there exists $Z \in \zeta$ such that $Z \subset A \cap G$. 
As we have observed in preliminary considerations, the property exhibited in the proposition above is necessarily hereditary for open sets. The class \((\mathbb{F}_0/\mathbb{F}_1)\geq\) is that of Fréchet filters, and \((\mathbb{F}_1/\mathbb{F}_1)\geq\) that of strongly Fréchet filters. In the following corollaries we will use sequence characterizations of these properties: a filter \(\mathcal{F}\) is Fréchet if for each \(H \in \mathcal{F}^\#\) there is a sequence \((x_n)_n \subset H\) that is finer than \(\mathcal{F}\); a filter \(\mathcal{F}\) is strongly Fréchet if for each decreasing sequence \((H_n)_n\) such that \(H_n \in \mathcal{F}^\#\) for each \(n\), there is a sequence \((x_n)_n\) finer than \(\mathcal{F}\) and such that \(x_n \in H_n\) for each \(n\). Prop. 8.14 specializes as follows:

**Corollary 8.15.** \(C_\alpha(X, \mathcal{G})\) is Fréchet at \(X_0 \in C(X, \mathcal{G})\) if and only if for each family \(\mathcal{G}\) of open sets such that \(\mathcal{G} \cap A \neq \emptyset\) for each \(X \setminus X_0 \in A \in \alpha\), there exists a sequence \((G_n)_n \subset \mathcal{G}\) such that for each \(X \setminus X_0 \in A \in \alpha\), there is \(n_A < \omega\), for which \(G_n \in A\) for every \(n \geq n_A\).

**Corollary 8.16.** \(C_\alpha(X, \mathcal{G})\) is strongly Fréchet at \(X_0 \in C(X, \mathcal{G})\) if and only if for each decreasing sequence \((G_n)_n\) of families of open sets such that \(G_n \cap A \neq \emptyset\) for each \(X \setminus X_0 \in A \in \alpha\), there exists a sequence \((G_n)_n\) with \(G_n \in G_n\) such that for each \(X \setminus X_0 \in A \in \alpha\), there is \(n_A < \omega\), for which \(G_n \in A\) for every \(n \geq n_A\).

Of course, the sequence \((G_n)_n\) fulfills the condition above if and only if it converges to \(X \setminus X_0\) in \(C_\alpha(X, \mathcal{G}')\). In the case of \(\alpha = \alpha_D\), where \(\mathcal{D} = [X]^{<\aleph_0}\), it is equivalent to \(X \setminus X_0 \subset \liminf_{n<\omega} G_n := \bigcup_{n<\omega} \bigcap_{k>n} G_k\) (the set-theoretic lower limit). In particular, for \(X_0 = \emptyset\) the condition above is the condition \((\gamma)\) of Gerlits and Nagy [10]: if \(\mathcal{G}\) is an open \(\omega\)-cover of \(X\), then there is a sequence \(G_n \in \mathcal{G}\) with \(\liminf_{n<\omega} G_n = X\).

As we have seen in Prop. 7.4, Fréchetness is not \(\mathbb{F}_1\)-steady. Nevertheless, it is known that a Fréchet topological group is strongly Fréchet (see [18]). Therefore

**Theorem 8.17.** If \(\mathcal{D}\) is a compact network on a completely regular space \(X\), then \(C_{\alpha_D}(X, \mathbb{R})\) is Fréchet if and only if it is strongly Fréchet if and only if for every \(\mathcal{D}\)-cover \(\mathcal{P}\) of \(X\) there is a sequence \((P_n)_n \subset \mathcal{P}\) such that for each \(D \in \mathcal{D}\) there is \(n_D < \omega\) such that \(P_n \in A\) for each \(n \geq n_D\).

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References
