AN INITIATION INTO CONVERGENCE THEORY

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In memory of Gustave Choquet (1915 - 2006)

Abstract. Convergence theory offers a versatile and effective framework to topology and analysis. Yet, it remains rather unfamiliar to many topologists and analysts. The purpose of this initiation is to provide, in a hopefully comprehensive and easy way, some basic ideas of convergence theory, which would enable one to tackle convergence-theoretic methods without much effort.

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I sent a preliminary version of this initiation to Professor Gustave Choquet who, with his habitual exquisite kindness, expressed his appreciation in a postcard of October 2005.

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1. Introduction

Convergence theory offers a versatile and effective framework for topology and analysis. Yet, it remains rather unfamiliar to many topologists and analysts. The purpose of this initiation is to provide, in a hopefully comprehensive and easy way, some basic ideas of convergence theory, which would enable one to tackle convergence-theoretic methods without much effort. Of course, the choice of what is essential, reflects the research experience of the author.

A relation between the filters on a set X and the elements of X, denoted by

$$x \in \lim \mathcal{F},$$

is called a *convergence* on X, provided that $\mathcal{F} \subset \mathcal{G}$ implies $\lim \mathcal{F} \subset \lim \mathcal{G}$, and that the principal ultrafilter of every element is in this relation with the element. Each topological space defines a convergence space (on the same set) to the effect that $x \in \lim \mathcal{F}$ whenever $\mathcal{F}$ contains every open set containing x. A convergence defined in this way is said to be *topological.*

Non-topological convergences arise naturally in analysis, measure theory, optimization and other branches of mathematics: in topological vector spaces,[1] there is in general no coarsest topology on the space of continuous linear forms, for which the coupling function is continuous; convergence in measure and convergence almost everywhere are, in general, not topological; stability of the minimizing set is, in general, non-topological.

A non-topological convergence can be a natural formalism of a stability concept. On the other hand, often a non-topological convergence arises as a solution of a problem formulated in purely topological terms.

A crucial example is the convergence structure resulting from the search for a *power* with respect to topologies τ and σ, that is, the coarsest topology θ on $C(\tau, \sigma)$ such that the natural coupling[2]

$$\langle x, f \rangle = f(x)$$

be (jointly) continuous from $\tau \times \theta$ to σ. It turned out that this problem has no solution unless τ is locally compact [4], but there is always a convergence, denoted by $[\tau, \sigma]$, which solves the problem. In other terms, the category of topological spaces is not *exponential* (in a predominant terminology, *Cartesian closed*) but it can be extended to an exponential category, that of convergence spaces. Actually there exist strict subcategories of the category of convergence spaces that include all topologies and are exponential (for example, that of *pseudotopologies*). [3]

In a fundamental paper [8] Gustave Choquet studies natural convergences on *hyperspaces*[4], and concludes that some of them are not topological unless the underlying topology is locally compact, but are always pseudotopological. The notion of pseudotopology was born. From a perspective of posterior research, these non-topological convergences were power convergences with respect to a special coupling topology.

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[1]See [5] of R. Beattie and H.-P. Butzmann.

[2]$C(\tau, \sigma)$ stands for the set of maps, which are continuous from a topology τ on X to a topology σ on Z.

[3]G. Bourdaud showed in [6] that the least exponential reflective subcategory of convergences, which includes topologies, is that of *epitopologies* of P. Antoine [3].

[4]A *hyperspace* is a set of closed subsets of a topological space (more generally, of a convergence space).

I do not intend to give here a historical account of convergence theory. Let me only mention that non-topological convergences called *pretopologies*[5] were already studied by F. Hausdorff [31], W. Sierpiński [40] and E. Čech [7]. An actual turning point however was, in my opinion, the emergence of pseudotopologies in [8] of Gustave Choquet.

2. Filters

A fundamental concept of convergence theory is that of *filter*. A non-empty family $\mathcal{F}$ of subsets of a set X is called a *filter* on X if

$$\varnothing \notin \mathcal{F}; \tag{1}$$

$$G \supset F \in \mathcal{F} \Longrightarrow G \in \mathcal{F}; \tag{2}$$

$$F_0, F_1 \in \mathcal{F} \Longrightarrow F_0 \cap F_1 \in \mathcal{F}. \tag{3}$$

Remark 1. *The family 2^X (of all subsets of X) is the only family that fulfills (2) and (3) but not (1). We call 2^X the* degenerate filter *on X. As 2^X does not fulfill all the assumptions above, it is not considered, despite its name, as a (full right) filter. So by a* filter *I mean a non-degenerate filter, unless I explicitly admit a degenerate filter.*

A subfamily $\mathcal{B}$ of $\mathcal{F}$ such that for every $F \in \mathcal{F}$ there is $B \in \mathcal{B}$ with $B \subset F$ is called a *base* of $\mathcal{F}$. We say that $\mathcal{B}$ *generates* $\mathcal{F}$; if $\mathcal{B}_0$ and $\mathcal{B}_1$ generate the same filter, then we write $\mathcal{B}_0 \approx \mathcal{B}_1$; in particular if $\mathcal{B}$ is a base of $\mathcal{F}$ then $\mathcal{B} \approx \mathcal{F}$.

If you study topologies, you study filters, whether you like it or not.

In fact, for every filter $\mathcal{F}$ on a given set X there is a unique topology on either X or on $\{\infty\} \cup X$ determined by $\mathcal{F}$. Let me explain this statement.

2.1. Neighborhood filters.

If X is a topological space, then for every $x \in X$, the set $\mathcal{N}(x)$ (of neighborhoods of x) is a filter.[6]

Recall that a topology is *prime* if it has at most one non-isolated point.[7] If $\mathcal{H}$ is a filter on Y and $X \subset Y$, then the *trace* $\mathcal{H}|_X$ of $\mathcal{H}$ on X is defined by

$$\mathcal{H}|_X = \{H \cap X : H \in \mathcal{H}\}.$$

This is a (non-degenerate) filter provided that $H \cap X \neq \varnothing$ for every $H \in \mathcal{H}$. A filter $\mathcal{F}$ is *free* whenever $\bigcap_{F \in \mathcal{F}} F = \varnothing$. If π is a Hausdorff prime topology on Y and $\infty \in Y$ is not isolated, then the trace of $\mathcal{N}_\pi(\infty)$ on X is a free filter on X. Conversely

Proposition 2. *For every free (possibly degenerate) filter $\mathcal{F}$ on X there exists a (unique) Haudorff prime topology on $\{\infty\} \cup X$ such that the trace of the neighborhood filter of ∞ is equal to $\mathcal{F}$.*

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[5] by Choquet in [8]

[6] A set V is a *neighborhood* of x if there exists an open set O such that $x \in O \subset V$.

[7] An element x of a topological space is *isolated* if $\{x\}$ is a neighborhood of x.

Proof. If $\mathcal{F}$ is a free filter on X, then define a topology π on the disjoint union $Y = \{\infty\} \cup X$ so that every $x \in X$ is isolated, and $\mathcal{N}_\pi(\infty) = \{\{\infty\} \cup F : F \in \mathcal{F}\}$.[8] Then the trace of $\mathcal{N}_\pi(\infty)$ on X is $\mathcal{F}$. ∎

On the other hand,

Proposition 3. *If $\mathcal{F}$ is a filter on X and $\bigcap_{F\in\mathcal{F}} F \neq \varnothing$, then there is a unique topology π* on X such *that $\mathcal{F} = \mathcal{N}_\pi(x)$ for every $x \in \bigcap_{F\in\mathcal{F}} F$, and x is isolated for each $x \notin \bigcap_{F\in\mathcal{F}} F$.*[9]

2.2. **Principal filters.** If $A \subset X$ then the family $(A)_\bullet = \{B \subset X : A \subset B\}$ is a filter, called the *principal filter* of A. This filter is degenerate if and only if $A = \varnothing$, because $A \in (A)_\bullet$, and A is the least element of it. The family $\{A\}$ is the smallest base of $(A)_\bullet$.

2.3. **Cofinite filters.** If $A \subset X$ then the family $(A)_\circ = \{B \subset X : \mathrm{card}(A \backslash B) < \infty\}$ is a filter, called the *cofinite filter* of A. This filter is degenerate if and only if A is finite, because $A\backslash\varnothing$ is finite whenever A is finite. The family $\{A\backslash F : F \subset A, \mathrm{card}\, F < \infty\}$ is a base of $(A)_\circ$.

2.4. **Sequential filters.** We say that a sequence $(x_n)_n$ of elements of X *generates* a filter $\mathcal{S}$ on X (in symbols $\mathcal{S} \approx (x_n)_n$) if the family $\{\{x_n : n \geq m\} : m < \infty\}$ is a base of $\mathcal{S}$. A filter on X is called *sequential* if there exists a sequence $(x_n) = (x_n)_n$ that generates it.[10]

2.5. **Countably based filters.** A filter is said to be *countably based* if it admits a countable base. Principal filters and sequential filters are special cases of countably based filters.

Example 4. *Let $(A_n)_n$ be a descending sequence of sets such that* $\mathrm{card}(A_n \backslash A_{n+1}) = \infty$ *and $\bigcap_{n<\infty} A_n = \varnothing$. Then $\{A_n : n < \infty\}$ is a base of a free countably based filter, which is not sequential.*

2.6. **Grills.** Two families $\mathcal{A}, \mathcal{B}$ of subsets of X *mesh* (in symbols, $\mathcal{A}\#\mathcal{B}$) if $A \cap B \neq \varnothing$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$; the *grill* of a family $\mathcal{A}$ of subsets of X is

$$\mathcal{A}^\# = \{B \subset X : \forall_{A\in\mathcal{A}} \ A \cap B \neq \varnothing\}.$$

A family $\mathcal{A}$ is *isotone* if $B \supset A \in \mathcal{A}$ implies $B \in \mathcal{A}$. If $\mathcal{A}$ isotone then[11]

$$H \notin \mathcal{A}^\# \Leftrightarrow H^c \in \mathcal{A}. \tag{4}$$

Let $\Omega \subset X \times Y$ be a relation. Then the *image* by Ω of x is given by

$$\Omega x = \{y \in Y : (x, y) \in \Omega\}.$$

[8]This topology is discrete if and only if $\mathcal{F}$ is degenerate.

[9]In fact, this topology π is defined so that a set A is open if either $A \cap \bigcap_{F\in\mathcal{F}} F = \varnothing$ or $A \in \mathcal{F}$.

[10]A filter $\mathcal{E}$ is sequential if and only if $\mathcal{E}$ contains a countable set, and admits a countable base $\mathcal{B}$ such that $B_1 \backslash B_0$ is finite for every $B_0, B_1 \in \mathcal{B}$.

[11]By definition, $H \notin \mathcal{A}^\#$ whenever there is $A \in \mathcal{A}$ such that $H \cap A = \varnothing$, equivalently $H^c \supset A$, that is, $H^c \in \mathcal{A}$, because $\mathcal{A}$ is istone.

Consequently the *image* ΩA of $A \subset X$ by Ω is

$$\Omega A = \bigcup\nolimits_{x \in A} \Omega x,$$

and the *preimage* $\Omega^{-} B$ of $B \subset Y$ by Ω is the image of B by the inverse relation of Ω. Thus $\Omega^{-} B = \{x \in X : \Omega x \cap B \neq \varnothing\}$. We notice the following useful equivalence

$$\Omega A \# B \Leftrightarrow A \# \Omega^{-} B \Leftrightarrow (A \times B) \# \Omega.$$

In particular, if $f : X \to Y$ then $f(A)$ and $f^{-}(B)$ are respectively the image and the preimage by the graph relation $\{(x, y) : y = f(x)\}$. In particular, I denote by $f^{-}(y)$ the preimage of y by f.

If $\mathcal{F}$ is a filter, then $\mathcal{F} \subset \mathcal{F}^{\#}$. Notice also that if $\mathcal{F}$ is a filter, then[12]

$$H_0 \cup H_1 \in \mathcal{F}^{\#} \Rightarrow (H_0 \in \mathcal{F}^{\#} \text{ or } H_1 \in \mathcal{F}^{\#}). \tag{5}$$

2.7. **Order, ultrafilters.** We say that a filter $\mathcal{F}$ is *coarser* than a filter $\mathcal{G}$ ($\mathcal{G}$ is *finer* than $\mathcal{F}$) if $\mathcal{F} \subset \mathcal{G}$. If $\mathcal{B}$ is a base of $\mathcal{F}$ and $\mathcal{B} \subset \mathcal{G}$, then $\mathcal{F} \subset \mathcal{G}$.

Remark 5. *We can loosely say that the smaller the sets belonging to a filter, the finer is the filter. Formally, if $\mathcal{B}_0$ is a base of a filter $\mathcal{F}_0$, and $\mathcal{B}_1$ of $\mathcal{F}_1$, and for every $B \in \mathcal{B}_0$ there is $D \in \mathcal{B}_1$ with $D \subset B$, then $\mathcal{F}_0 \subset \mathcal{F}_1$.*

Of course, this partial order is induced on the set φX (of all filters on X) from that of all the families of subsets of X. We denote by $\mathcal{F} \vee \mathcal{G}$ the supremum and by $\mathcal{F} \wedge \mathcal{G}$ the infimum of two filters $\mathcal{F}$ and $\mathcal{G}$.

Remark 6. *The supremum of filters $\mathcal{F}, \mathcal{G}$ in the ordered set of filters exists if and only if $\mathcal{F} \# \mathcal{G}$. If it does not exist, then the supremum of $\mathcal{F}$ and $\mathcal{G}$ in the complete lattice of all the families of subsets is, of course, equal to the degenerate filter.*

Remark 7. *Occasionally I use the symbols $\vee$ and $\wedge$ (respectively) for the supremum and the infimum in the partially ordered set of filter bases. This notation is not ambiguous, because in the case when the considered filter bases are filters, their supremum and infimum are filters too.*

The infimum $\bigwedge_{j \in J} \mathcal{F}_j$ exists for an arbitrary set $\{\mathcal{F}_j : j \in J\}$ of filters on X, and

$$\bigwedge_{j \in J} \mathcal{F}_j = \bigcap_{j \in J} \mathcal{F}_j$$

(the coarsest filter is $\{X\}$, the principal filter of X), while $\bigvee_{j \in J} \mathcal{F}_j$ exists whenever $\varnothing \notin \mathcal{B} = \{\bigcap_{j \in J_0} F_j : J_0 \subset J, \operatorname{card} J_0 < \infty\}$, and in this case $\mathcal{B}$ is a base of $\bigvee_{j \in J} \mathcal{F}_j$. By the Zorn-Kuratowski lemma, for every filter $\mathcal{F}$ on X there exists a maximal filter $\mathcal{U}$, which is finer than $\mathcal{F}$, called an *ultrafilter.* The set of all the ultrafilters finer than $\mathcal{F}$ is denoted by $\beta \mathcal{F}$.

[12] In fact, if $H_0 \notin \mathcal{F}^{\#}$ and $H_1 \notin \mathcal{F}^{\#}$, then $H_0^c \in \mathcal{F}$ and $H_1^c \in \mathcal{F}$, hence $H_0^c \cap H_1^c \in \mathcal{F}$ thus $H_0 \cup H \notin \mathcal{F}^{\#}$.

If $f : X \to Y$ and $\mathcal{F}$ is a filter on X, then $f(\mathcal{F}) = \{f(F) : F \in \mathcal{F}\}$ is a filter base on Y. We shall use the symbol $f(\mathcal{F})$ also for the filter it generates.[13]

A filter $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}^{\#} = \mathcal{F}$.[14] It follows that

If $\mathcal{U}$ is an ultrafilter, then $f(\mathcal{U})$ is an ultrafilter.

Moreover if $\mathcal{W} \in \beta f(\mathcal{F})$, then there exists $\mathcal{U} \in \beta\mathcal{F}$ such that $\mathcal{W} = f(\mathcal{U})$.[15]

2.8. **Decomposition.** A filter $\mathcal{F}$ is called *free* if $\bigcap_{F \in F} F = \varnothing$. The cofinite filter is free. In fact, if $\mathcal{F}$ is a free filter and $A \in \mathcal{F}$ then $\mathcal{F} \supset (A)_\circ$.[16]

Proposition 8 (Filter decomposition)**.** [15] *For every filter $\mathcal{F}$ on X, there exists a unique pair of (possibly degenerate) filters $\mathcal{F}^\circ, \mathcal{F}^\bullet$ such that $\mathcal{F}^\circ$ is free, $\mathcal{F}^\bullet$ is principal, and*[17]

$$\mathcal{F} = \mathcal{F}^\circ \wedge \mathcal{F}^\bullet \text{ and } \mathcal{F}^\circ \vee \mathcal{F}^\bullet = 2^X. \tag{6}$$

In particular, every sequential filter admits such a decomposition. Notice that

Proposition 9. *A filter is sequential and free if and only if it is the cofinite filter of a countably infinite set.*

Proof. If $(x_n)_n$ is a sequence, then we set $A_n = \{x_k : k \geq n\}$. Accordingly, $(x_n)_n$ is free if and only if $\bigcap_{n<\infty} A_n = \varnothing$. The sequential filter generated by $(x_n)_n$ is the cofinite filter of A_m for each m, because $A_m \backslash A_n$ is finite for every $n < \infty$ and $(A_n)_n$ is a base of it. Conversely, if $\mathcal{S}$ is the cofinite filter of a countably infinite set A, then represent $A = \{x_k : k < \infty\}$, where $x_m \neq x_k$ if $m \neq k$. Then $\{A_n : n < \infty\}$ is a base of $\mathcal{S}$, so that $\mathcal{S} \approx (x_n)_n$. ∎

Proposition 10. *A filter is sequential and principal if and only if it is the principal filter of a countable set.*

Proof. Indeed, if A is finite (of cardinality $0 < m < \infty$) then we can represent $A = \{a_1, a_2, \ldots, a_m\}$ and $(A)_\bullet$ is sequential, because the sequence

$$a_1, a_2, \ldots, a_m, a_1, a_2, \ldots, a_m, a_1, a_2, \ldots, a_m, \ldots$$

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[13]This abuse of notation should not lead to any confusion.

[14]If a filter $\mathcal{F}$ is not an ultrafilter, then there is a filter $\mathcal{G} \supsetneq \mathcal{F}$. Therefore $\mathcal{F} \subsetneq \mathcal{G} \subset \mathcal{G}^{\#} \subset \mathcal{F}^{\#}$. Conversely, if there is $H \in \mathcal{F}^{\#} \setminus \mathcal{F}$, then $\mathcal{F} \vee H$ is a filter that is obviously finer than $\mathcal{F}$ and that contains H. Since $H \notin \mathcal{F}$, the filter $\mathcal{F}$ is not maximal.

[15]Let $\mathcal{W}$ be an ultrafilter finer than $f(\mathcal{F})$. Equivalently, $f^-(\mathcal{W})\#\mathcal{F}$, hence there exists an ultrafilter $\mathcal{U}$ finer than $\mathcal{F} \vee f^-(\mathcal{W})$, so that $\mathcal{U} \in \beta\mathcal{F}$ and $\mathcal{U}\#f^-(\mathcal{W})$. The latter is equivalent to $f(\mathcal{U})\#\mathcal{W}$ thus $f(\mathcal{U}) = \mathcal{W}$.

[16]Indeed, if $x \in A$ and $\mathcal{F}$ is free, then there is $F \in \mathcal{F}$ such that $x \notin F$, that is, $F \subset \{x\}^c$ thus $\{x\}^c \in \mathcal{F}$. As each finite intersection of elements of a filter belongs to that filter, $A \backslash D \in \mathcal{F}$ for every finite subset D of A. Hence $(A)_\circ \in \mathcal{F}$.

[17]Let $\mathcal{F}^\bullet$ be the principal filter of $F_\bullet = \bigcap_{F \in \mathcal{F}} F$. Then $\mathcal{F}^\circ = \mathcal{F} \vee F_\bullet^c$ is free, and obviously (6) holds. If now $(A)_\bullet$ is a principal filter finer than $\mathcal{F}$, then $A \subset F_\bullet$, hence $\mathcal{F} \vee A^c$ is free if and only if $A = F_\bullet$, which shows the uniqueness of the decomposition.

generates $(A)_\bullet$; if A is countably infinite, then we can represent $A = \{a_n : n \in \mathbb{N}\}$ where $a_n \neq a_k$ for $n \neq k$. Then the sequence

$$a_1, a_1, a_2, a_1, a_2, a_3, \ldots, a_1, a_2, \ldots, a_n \ldots$$

generates $(A)_\bullet$. ∎

2.9. **Stone transform.** Let me shortly mention the *Stone space* of a given set X, that is, the set βX of all ultrafilters on X endowed with the *Stone topology*. The *Stone transform* β associates to every filter $\mathcal{F}$ on X, the set $\beta\mathcal{F}$ of ultrafilters that are finer than $\mathcal{F}$. It is clear that $\mathcal{F}_0 \subset \mathcal{F}_1$ implies $\beta\mathcal{F}_0 \supset \beta\mathcal{F}_1$.[18] A base for the open sets of the Stone topology consists of the Stone transforms of principal filters, that is, $\{\beta A : A \subset X\}$. This topology is compact.[19]

It is well-known that

Proposition 11. *A subset of the Stone space is closed if and only if it is of the form $\beta\mathcal{F}$, where $\mathcal{F}$ is a (possibly degenerate) filter.*[20]

3. Basic classes of convergences

If ξ is an arbitrary relation between the non-degenerate filters $\mathcal{F}$ on X and the elements x of X, then we write

$$x \in \lim_\xi \mathcal{F} \tag{7}$$

whenever $(x, \mathcal{F}) \in \xi$ and say that the filter $\mathcal{F}$ *converges* to x with respect to ξ (equivalently, x is a *limit* of $\mathcal{F}$ with respect to ξ). A relation ξ is a *convergence* if

$$\mathcal{F} \subset \mathcal{G} \Longrightarrow \lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G}, \tag{8}$$

$$\forall_{x \in X}\ x \in \lim_\xi (x)_\bullet, \tag{9}$$

where $(x)_\bullet$ is the principal ultrafilter determined by x.[21]

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[18] Moreover, $\beta(\mathcal{F}_0 \vee \mathcal{F}_1) = \beta\mathcal{F}_0 \cap \beta\mathcal{F}_1$ and $\beta(\mathcal{F}_0 \wedge \mathcal{F}_1) = \beta\mathcal{F}_0 \cup \beta\mathcal{F}_1$.

[19] If $\{\beta D : D \in \mathcal{D}\}$ is a cover of βX, then there is a finite subfamily $\mathcal{A}$ of $\mathcal{D}$ such that $\beta X \subset \bigcup_{A \in \mathcal{A}} \beta A$, for otherwise $\beta(\bigcap_{A \in \mathcal{A}} A^c) = \bigcap_{A \in \mathcal{A}} \beta(A^c) \neq \varnothing$ for each finite subfamily $\mathcal{A}$ of $\mathcal{D}$. In other words, $\{\bigcap_{A \in \mathcal{A}} A^c : \mathcal{A} \subset \mathcal{D}, \operatorname{card} \mathcal{A} < \infty\}$ is a filter base. If now $\mathcal{U}$ is an ultrafilter that includes it, then *a fortiori* $D^c \in \mathcal{U}$ for each $D \in \mathcal{D}$, hence $D \notin \mathcal{U}$ (equivalently $\mathcal{U} \notin \beta D$) for each $D \in \mathcal{D}$, which is a contradiction.

[20] If $\mathcal{F}$ is a filter and $\mathcal{U} \notin \beta\mathcal{F}$ then there is $H \in \mathcal{F} \backslash \mathcal{U}$ so that the Stone open (and closed) set βH contains $\mathcal{U}$ and is disjoint from $\beta\mathcal{F}$. Conversely, if A is a Stone closed set, then $\mathcal{F} = \bigcap_{\mathcal{U} \in A} \mathcal{U}$ is a filter such that $\beta\mathcal{F} = A$. By construction each $\mathcal{U} \in A$ belongs to $\beta\mathcal{F}$. If $\mathcal{W} \notin A$ there is an open, closed set of the form βW such that $\mathcal{W} \in \beta W$ and $\beta W \cap A = \varnothing$, that is, $\beta(W^c) \supset A$, that is $W^c \in \mathcal{U}$ for each $\mathcal{U} \in A$, hence $W^c \in \mathcal{F}$ and thus $\mathcal{W} \notin \beta\mathcal{F}$.

[21] There are several definitions of convergence. Many authors add to the set of our axioms, a third one like $\lim \mathcal{F}_0 \cap \lim \mathcal{F}_1 \subset \lim(\mathcal{F}_0 \cap \mathcal{F}_1)$ (H. J. Kowalsky [36]), or $x \in \lim \mathcal{F} \Rightarrow x \in \lim((x)_\bullet \cap \mathcal{F})$ (e.g., D. C. Kent, G D. Richardson [34]).

Remark 12. *If ξ is a convergence on a set X, then the couple (X, ξ) is called a* convergence space*. The set, on which a convergence ξ is defined, is called the* underlying set *of ξ. Notice that an underlying set is determined by a convergence thanks to (9); we denote the underlying set of a convergence ξ by $|\xi|$. Consequently a convergence determines the corresponding convergence space. Therefore I will use the terms* convergence *and* convergence space *interchangeably.*

Example 13. *If τ is a topology on a set X then we define the* associated convergence *by $x \in \lim_\tau \mathcal{F}$ whenever $\mathcal{N}_\tau(x) \subset \mathcal{F}$. This relation fulfills (8) and (9),*[22] *so that we can identify each topology with its associated* (topological) *convergence. Notice that a topological convergence fulfills*

$$x \in \lim \mathcal{N}(x)$$

for every $x \in X$, and the neighborhood filter $\mathcal{N}(x)$.

Here is a basic example of a non-topological convergence.

Example 14. *If ν is the natural topology on $\mathbb{R}$, then we define a convergence* $\operatorname{Seq}\nu$ *on X by setting $x \in \lim_{\operatorname{Seq}\nu} \mathcal{F}$, whenever there exists a sequential filter $\mathcal{E}$ such that $\mathcal{E} \subset \mathcal{F}$ and $x \in \lim_\nu \mathcal{E}$. This defines a convergence, which is not a topology. Indeed, $\mathcal{N}_\nu(0) = \bigcap\{\mathcal{E} : 0 \in \lim_{\operatorname{Seq}\nu} \mathcal{E}\}$ but there is no sequential filter which is coarser than $\mathcal{N}_\nu(0)$, hence $0 \notin \lim_{\operatorname{Seq}\nu} \mathcal{N}_\nu(0)$.*[23]

We say that a convergence ξ is *finer* than a convergence θ (in symbols, $\xi \geq \theta$) if $\lim_\xi \mathcal{F} \subset \lim_\theta \mathcal{F}$ for every filter $\mathcal{F}$. If Ξ is a set of convergences on X, then the supremum and the infimum of Ξ are given by

$$\lim_{\vee\Xi} \mathcal{F} = \bigcap_{\xi \in \Xi} \lim_\xi \mathcal{F}, \quad \lim_{\wedge\Xi} \mathcal{F} = \bigcup_{\xi \in \Xi} \lim_\xi \mathcal{F}.$$

The greatest element of the set of all convergences on X is the *discrete topology* $\iota = \iota_X$ of X, which is defined by $x \in \lim_\iota \mathcal{F}$ whenever $\mathcal{F}$ is the principal ultrafilter of x.[24] The least element is the *chaotic topology* (called also *indiscrete topology*) $o = o_X$ on X, defined by $\lim_o \mathcal{F} = X$ for every filter $\mathcal{F}$ on X.

3.1. **Topologies.** A subset O of a convergence space is *open* if $\lim \mathcal{F} \cap O \neq \varnothing$ implies that $O \in \lim \mathcal{F}$. The family $\mathcal{N}_\xi(x)$ of all the sets V such that there exists a ξ-open set O such that $x \in O \subset V$, is a filter, called the *neighborhood filter* of x for ξ and is denoted by $\mathcal{N}_\xi(x)$. The family $\mathcal{O}_\xi$ of all the open sets of a convergence ξ fulfills all the axioms of open sets of a topology.

[22] In fact, we have defined a functor, which embeds the category of topologies in that of convergences.

[23] Indeed, $\mathcal{N}_\nu(0) \subset \bigcap\{\mathcal{E} : 0 \in \lim_{\operatorname{Seq}\nu} \mathcal{E}\}$ by the definition of $\operatorname{Seq}\nu$, and if $A \notin \mathcal{N}_\nu(0)$ then $A^c \in \mathcal{N}_\nu(0)^\#$ hence for every $n < \infty$ there is $x_n \in A^c \cap \{x : |x| < \frac{1}{n}\}$. In other words A belongs to a sequential filter (generated by) (x_n), which is finer than $\mathcal{N}_\nu(0)$.

[24] In fact, the convergence of the principal ultrafilter of x to x is postulated by the definition of convergence.

Conversely, if $\mathcal{T}$ is the family of open sets of a topological space, then

$$x \in \lim_{\mathcal{T}} \mathcal{F} \Leftrightarrow (x \in T \in \mathcal{T} \Rightarrow T \in \mathcal{F}) \tag{10}$$

defines a convergence. Convergences constructed in this way are called topologies. For every convergence ξ there exists the finest topology that is coarser than ξ called the *topologization* of ξ and denoted by $T\xi$. This the convergence constructed with the aid of (10) with $\mathcal{T} = \mathcal{O}_\xi$. It is straightforward that $\mathcal{O}_{T\xi} = \mathcal{O}_\xi$. Therefore T fulfills

$$\begin{gathered} \zeta \leq \xi \Rightarrow T\zeta \leq T\xi; \\ T\xi \leq \xi; \\ T(T\xi) = T\xi. \end{gathered}$$

for every ζ and ξ. It follows that the set of all topologies (on a given set) is closed for arbitrary suprema. Moreover, the coarsest convergence on X is the chaotic topology on X.

Example 15. *Take the convergence* $\operatorname{Seq}\nu$ *of Example 14. By definition of open set for a convergence, a set is open for* $\operatorname{Seq}\nu$ *if and only if it is sequentially open, hence if it is open for* ν*, because* ν *is a sequential topology.*[25] *Therefore* $T(\operatorname{Seq}\nu) = \nu$*.*

The complement of an open set is said to be *closed*. The least ξ-closed set, which includes A is called the *ξ-closure* of A and is denoted by $\operatorname{cl}_\xi A$. We notice that

$$x \in \operatorname{cl}_\xi A \Leftrightarrow A \in \mathcal{N}_\xi^{\#}(x). \tag{11}$$

Proof. We have that $x \notin \operatorname{cl}_\xi A$ whenever there is an open set O such that $x \in O$ and $O \cap A = \varnothing$, that is, whenever $A \notin \mathcal{N}_\xi^{\#}(x)$. ∎

3.2. **Sequentially based convergences.** A convergence ξ is *sequentially based* if whenever $x \in \lim_\xi \mathcal{F}$, then there exists a sequential filter $\mathcal{E}$ such that $\mathcal{E} \subset \mathcal{F}$ and $x \in \lim_\xi \mathcal{E}$. If $(x_n)_n$ is a sequence that generates $\mathcal{E}$ then we write $x \in \lim_\xi (x_n)_n$ whenever $x \in \lim_\xi \mathcal{E}$.[26]

Denote by εX the set of sequential filters on X. If θ is an arbitrary convergence of X,

$$x \in \lim_{\operatorname{Seq}\theta} \mathcal{F} \Longleftrightarrow \exists_{\mathcal{E} \in \varepsilon X}\ (x \in \lim_\theta \mathcal{E} \text{ and } \mathcal{E} \subset \mathcal{F})$$

defines a sequentially based convergence $\operatorname{Seq}\theta$. It is straightforward that for every ζ and ξ,

$$\begin{gathered} \zeta \leq \xi \Rightarrow \operatorname{Seq}\zeta \leq \operatorname{Seq}\xi; \\ \xi \leq \operatorname{Seq}\xi; \\ \operatorname{Seq}(\operatorname{Seq}\xi) = \operatorname{Seq}\xi. \end{gathered} \tag{12}$$

[25] A topology is *sequential* if every set closed for convergent sequences is closed.

[26] This is an example of the obvious and natural extension of convergences from filters to filter bases.

A topology is a sequentially based convergence, if and only if each neighborhood filter is generated by a sequence. Therefore the *sequential modifier* Seq can be used to produce numerous examples of non-topological convergences, as in Example 14.

3.3. **Convergences of countable character.** A convergence ξ is of *countable character* provided that if $x \in \lim_\xi \mathcal{F}$ then there exists a countably based filter $\mathcal{E}$ such that $\mathcal{E} \subset \mathcal{F}$ and $x \in \lim_\xi \mathcal{E}$. Each sequentially based convergence is of countable character; Example 14 is that of a topology of countable character, which is not sequentially based. The *modifier (of countable character)* $\mathrm{First}\,\theta$ of an arbitrary convergence θ is defined similarly to $\mathrm{Seq}\,\theta$; the map First has the same properties as Seq in (12).

3.4. **Pretopologies.** A convergence ξ is called a *pretopology* if[27]

$$\bigcap_{\mathcal{F}\in\mathbb{F}} \lim_\xi \mathcal{F} \subset \lim_\xi (\bigcap_{\mathcal{F}\in\mathbb{F}} \mathcal{F}) \tag{13}$$

for every set $\mathbb{F}$ of filters. It follows that for every element x of a pretopological space, there exists a coarsest filter that converges to x. For every element x let

$$\mathcal{V}_\xi(x) = \bigcap_{x\in\lim_\xi \mathcal{F}} \mathcal{F} \tag{14}$$

be the *vicinity filter* of x for a convergence ξ; the elements of the vicinity filter of x are called *vicinities* of x. By (13) ξ is a pretopology if and only if $x \in \lim_\xi \mathcal{V}_\xi(x)$ for every x. If A is a non-empty subset of a pretopological space (X,ξ) then $\mathcal{V}_\xi(A) = \bigcap_{x\in A} \mathcal{V}_\xi(x)$ is the *vicinity* of A.

A set is open if it is a vicinity of each of its elements.[28]

Every topology is a pretopology; if ξ is a topology, then $\mathcal{V}_\xi(x) \subset \mathcal{N}_\xi(x)$ for every x, and since the inverse inclusion always holds,

$$\xi = T\xi \Rightarrow \mathcal{V}_\xi(x) = \mathcal{N}_\xi(x) \tag{15}$$

Let us give an example of a non-topological pretopology.

Example 16. *Let* $X = \{x_\infty\} \cup \{x_n : n < \infty\} \cup \{x_{n,k} : n,k < \infty\}$, *where all the elements are distinct. We define a convergence* π *by* $x_{n,k} \in \lim_\pi \mathcal{F}$ *whenever* $\mathcal{F} = (x_{n,k})_\bullet$, $x_n \in \lim_\pi \mathcal{F}$ *whenever* $(x_n)_\bullet \wedge (x_{n,k})_k \subset \mathcal{F}$, *and* $x_\infty \in \lim_\pi \mathcal{F}$ *provided that* $(x_\infty)_\bullet \wedge (x_n)_n \subset \mathcal{F}$. *The convergence* π *is a pretopology and a sequentially based convergence. Indeed, for every element, there is a coarsest filter converging to that element, that is, a vicinity filter. The elements of the form* $x_{n,k}$ *are isolated, hence* $\{x_{n,k}\} \in \mathcal{V}_\pi(x_{n,k})$ *for each* $n,k < \infty$; $\mathcal{V}_\pi(x_n) \approx (x_n)_\bullet \wedge (x_{n,k})_k$ *for every* $n < \infty$; $\mathcal{V}_\pi(x_\infty) \approx (x_\infty)_\bullet \wedge (x_n)_n$. *It is not a topology, because if* O *such that* $x_\infty \in O$ *is an*

[27]This definition of pretopology is due to G. Choquet [8]. However an equivalent concept was considered by F. Hausdorff [31], W. Sierpiński [40] and E. Čech [7].

[28]If A is open and $x \in A$, then $A \in \mathcal{V}(x)$, because every filter, which converges to x, contains A, and conversely.

open set, then there is n_0 such that $x_n \in O$ for every $n \geq n_0$ and there is $\kappa(n) < \infty$ such that $x_{n,k} \in O$ for each $k > \kappa(n)$. The neighborhood filter $\mathcal{N}_\pi(x_\infty)$, which is generated by

$$\{x_\infty\} \cup \{x_n : n > n_0\} \cup \{x_{n,k} : k > \kappa(n), n > n_0\},$$

where $n_0 < \infty$ and $\kappa : \mathbb{N} \to \mathbb{N}$, does not converge to x_∞. Actually we have already described the topologization $T\pi$ of π, namely $\mathcal{N}_{T\pi}(x_\infty) = \mathcal{N}_\pi(x_\infty)$ was given above, and we have described $\mathcal{N}_{T\pi}(x) = \mathcal{V}_\pi(x)$ if $x \neq x_\infty$.[29]

It can be easily seen that the set of all pretopologies (on a given set) is closed for arbitrary suprema, and that the coarsest convergence on a given set is the chaotic topology on that set. This is equivalent to the existence of a map P associating with every convergence ξ the finest pretopology $P\xi$ that is coarser than ξ. This map is called the *pretopologizer* and fulfills

$$\begin{gathered} \zeta \leq \xi \Rightarrow P\zeta \leq P\xi \\ P\xi \leq \xi \\ P(P\xi) = P\xi \end{gathered}$$

for every ζ and ξ. The pretopologizer can be easily written explicitly. Therefore $x \in \lim_{P\zeta} \mathcal{F}$ if and only if $\mathcal{V}_\zeta(x) \subset \mathcal{F}$.

Remark 17. *A* network *of a topological space (X, τ) is a family $\mathcal{P}$ of subsets of X such that for each $x \in X$ and $O \in \mathcal{N}_\tau(x)$ there is $P \in \mathcal{P}$ such that $x \in P \subset O$. A network is called a* weak base *whenever each subset B of X, with the property that for every $x \in B$ there is $P \in \mathcal{P}$ such that $x \in P \subset B$, is open. For example, the family of all singletons is a network, which is not a weak base unless the topology is discrete. Let $\mathcal{P}$ be a family of subsets of X, which covers X. Then the family of finite intersections of $\{P \in \mathcal{P} : x \in P\}$ is a filter base; the filter $\mathcal{V}_\mathcal{P}(x)$ it generates is a vicinity filter of a pretopology, which we denote by $\pi_\mathcal{P}$. It follows immediately from the definitions that*

Proposition 18. *A family $\mathcal{P}$ is a network of τ if and only if $\pi_\mathcal{P} \geq \tau$; a family $\mathcal{P}$ is a weak base of τ if and only if $T\pi_\mathcal{P} = \tau$.*

4. Continuity

Let ξ be a convergence on X and τ be a convergence on Y. A map $f : X \to Y$ is *continuous* (from ξ to τ) if for every filter $\mathcal{F}$ on X,

$$f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F}). \tag{16}$$

It follows that the composition of continuous maps is continuous. A bijective map f such that both f and f^- are continuous is called a *homeomorphism.*

[29]Notice that $\mathcal{N}_{T\xi}(x) = \mathcal{N}_\xi(x)$ for every convergence ξ and each $x \in |\xi|$.

4.1. **Initial convergences.** For every map $f : X \to Y$ and each convergence τ on Y, there exists the coarsest among the convergences ξ on X for which f is continuous (from ξ to τ). It is denoted by $f^{-}\tau$ and called the *initial* convergence for (f, τ).[30] If $V \subset X$ and θ is a convergence on X, then the initial convergence such that the embedding $i : V \to X$ is continuous is called a *subconvergence* of θ on V and is denoted by $\theta \vee V$.

Let τ_i be a convergence and $f_i : X \to |\tau_i|$ be a map for every $i \in I$. Then the coarsest convergence on X, for which f_i is continuous for each $i \in I$, is called the *initial convergence* with respect to $\{f_i : i \in I\}$. Of course, it is equal to $\bigvee_{i\in I} f_i^{-}\tau_i$. It is straightforward that

Proposition 19. *If ξ is the initial convergence with respect to $\{f_i : i \in I\}$ then $x \in \lim_\xi \mathcal{F}$ if and only if $f_i(x) \in \lim f_i(\mathcal{F})$ for every $i \in I$.*

4.2. **Final convergences.** For every map $f : X \to Y$ and each convergence ξ on X, there exists the finest among the convergences τ on Y for which f is continuous (from ξ to τ). It is denoted by $f\xi$ and called the *final* convergence for (f, ξ) (or the *quotient* of ξ by f).[31]

Let ξ_i be a convergence and $f_i : |\xi_i| \to Y$ be a map for every $i \in I$. Then the finest convergence on Y, for which f_i is continuous for each $i \in I$, is called the *final convergence* with respect to $\{f_i : i \in I\}$. Of course, it is equal to $\bigwedge_{i\in I} f_i\xi_i$.

It is good to have in mind this immediate observation.

Proposition 20. *The following statements are equivalent:*

$$f \text{ is continuous from } \xi \text{ to } \tau;$$

$$f\xi \geq \tau;$$

$$\xi \geq f^{-}\tau.$$

4.3. **Continuity in subclasses.** One easily sees that the preimage of an open set by a continuous map is open.[32] Hence if τ is a topology, then $f^{-}\tau$ is a topology.[33] Similarly, if τ is a pretopology, then $f^{-}\tau$ is a pretopology.[34] Therefore if f is continuous from ξ to τ, then it is continuous also from $P\xi$ to $P\tau$, and from $T\xi$ to $T\tau$.[35] It is also easy to notice that if ξ is a sequentially

[30]Indeed, it follows from (16) that if f is continuous from ξ to τ, then $\lim_\xi \mathcal{F} \subset f^{-}(\lim_\tau f(\mathcal{F}))$. Therefore $\lim_{f^{-}\tau} \mathcal{F} = f^{-}(\lim_\tau f(\mathcal{F}))$.

[31]It is straightforward that $\lim_{f\xi} \mathcal{G} = \bigcup_{f(\mathcal{F})\leq\mathcal{G}} f(\lim_\xi \mathcal{F})$. Indeed $y \in \lim_{f\xi} \mathcal{G}$ whenever there exists a filter $\mathcal{F}$ such that $\lim_\xi \mathcal{F} \cap f^{-}(y) \neq \varnothing$ and $\mathcal{G} \geq f(\mathcal{F})$.

[32]Let f be continuous from ξ to τ, let $O \in \mathcal{O}(\tau)$ and let $x \in \lim_\xi \mathcal{F} \cap f^{-}(O)$. It follows that $f(x) \in \lim_\tau f(\mathcal{F})$ and $f(x) \in O$, hence $O \in f(\mathcal{F})$. Therefore $f^{-}(O) \in \mathcal{F}$.

[33]Let $f^{-}(O) \in \mathcal{F}$ for every τ-open set O such that $x \in f^{-}(O)$. It follows that $O \supset ff^{-}(O) \in f(\mathcal{F})$ and $f(x) \in O$ and thus $f(x) \in \lim_\tau f(\mathcal{F})$, hence $x \in f^{-}(f(x)) \subset f^{-}(\lim_\tau f(\mathcal{F})) = \lim_{f^{-}\tau} \mathcal{F}$.

[34]From the category theory point of view, topologies and pretopologies are concrete reflective subcategories of the category of convergences with continuous maps as morphisms.

[35]By Proposition 20, $\xi \geq f^{-}\tau$, hence $T\xi \geq T(f^{-}\tau)$. On the other hand $f^{-}\tau \geq f^{-}(T\tau)$ and the latter convergence is a topology. Therefore $T\xi \geq f^{-}(T\tau)$.

based convergence, then $f\xi$ is also sequential.[36] It follows (by Proposition 20 for instance) that if f is continuous from ξ to τ, then it is continuous also from $\operatorname{Seq}\xi$ to $\operatorname{Seq}\tau$.

4.4. **Products.** If ξ and υ are convergences on X and Y respectively, then the *product convergence* $\xi \times \upsilon$ on $X \times Y$ is defined by

$$(x, y) \in \lim_{\xi\times\upsilon} \mathcal{F}$$

whenever there exist filters $\mathcal{G}$ on X and $\mathcal{H}$ on Y such that $x \in \lim_\xi \mathcal{G}, y \in \lim_\upsilon \mathcal{H}$ and $\mathcal{G} \times \mathcal{H} \leq \mathcal{F}$.[37]

In other words, a filter converges to (x, y) in the product convergence if and only if its projections converge to x and y respectively.

More generally, let Ξ be a set of convergences such that ξ is a convergence on X_ξ for $\xi \in \Xi$. The *product convergence* $\prod \Xi = \prod_{\xi\in\Xi} \xi$ is the coarsest convergence on $\prod_{\xi\in\Xi} X_\xi$, for which each *projection* $p_\theta : \prod_{\xi\in\Xi} X_\xi \to X_\theta$ is continuous. In other words, $\prod \Xi = \bigvee_{\xi\in\Xi} p_\xi^- \xi$.

In particular, each (convergence) product of topologies (respectively, of pretopologies) is a topology (respectively, a pretopology).

4.5. **Powers.** If X and Z are sets, hence Z^X is the set of all maps from X to Z, then the map

$$e = \langle \cdot, \cdot \rangle : X \times Z^X \to Z$$

defined by $e(x, f) = \langle x, f \rangle = f(x)$ is called the *evaluation map.* If ξ is a convergence on X and σ on Z, then $C(\xi, \sigma)$ stands for the subset of Z^X consisting of all the maps continuous from ξ to σ. The *power (convergence)* $[\xi, \sigma]$ (of ξ with respect to σ) is the coarsest among the convergences τ on $C(\xi, \sigma)$ for which the evaluation is continuous from $\xi \times \tau$ to σ.

The power $[\xi, \sigma]$ *exists for arbitrary convergences* ξ *and* σ.[38]

Let us describe explicitly the power convergence. If $\mathcal{G}$ is a filter on X, ξ is a convergence on X, and $\mathcal{F}$ is a filter on $C(\xi, \sigma)$, then $\langle \mathcal{G}, \mathcal{F} \rangle$ stands for the filter generated by $\{\bigcup_{f\in F} f(G) : G \in \mathcal{G}, F \in \mathcal{F}\}$. Then

$$f \in \lim_{[\xi,\sigma]} \mathcal{F}$$

if and only if $f(x) \in \lim_\sigma \langle \mathcal{G}, \mathcal{F} \rangle$ for every $x \in |\xi|$ and filter $\mathcal{G}$ on $|\xi|$ such that $x \in \lim_\xi \mathcal{G}$.

The definition above was already given by H. Hahn [30] for sequential filters $\mathcal{F}$. As mentioned in the introduction, power convergences constituted

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[36] From the category theory point of view, sequential convergences constitute a concrete coreflective subcategory of the category of convergences with continuous maps as morphisms.

[37] The product filter $\mathcal{G} \times \mathcal{H}$ is the filter generated by $\{G \times H : G \in \mathcal{G}, H \in \mathcal{H}\}$.

[38] Indeed, if ι is the discrete topology on $C(\xi, \sigma)$, then e is continuous from $\xi \times \iota$ to σ if and only if $e(\cdot, f)$ is continuous from ξ to σ for every f. Now, if T is a set of convergences on X such that $\xi \times \tau \geq e^-\sigma$ for each $\tau \in T$, then $\xi \times \bigwedge T\tau \geq e^-\sigma$, because $(x, f) \in \lim_{\xi\times\bigwedge_{\tau\in T}\tau} \mathcal{H}$ if and only if there exist filters $\mathcal{F}$ and $\mathcal{G}$ such that $x \in \lim_\xi \mathcal{F}$ and $f \in \bigcup_{\tau\in T} \lim_\tau \mathcal{G}$.

a decisive point in the development of convergence theory. And they remain a most important object of study till today.

5. Adherences

An important notion in convergence theory is that of adherence. If ξ is a convergence on X and $\mathcal{H}$ is a family of subsets of X, then

$$\operatorname{adh}_\xi \mathcal{H} = \bigcup\nolimits_{\mathcal{F}\#\mathcal{H}} \lim\nolimits_\xi \mathcal{F}$$

is the *adherence* of $\mathcal{H}$. Therefore if $\mathcal{U}$ is an ultrafilter, then $\operatorname{adh}_\xi \mathcal{U} = \lim_\xi \mathcal{U}$. Clearly, $\operatorname{adh}_\xi \mathcal{A} \subset \operatorname{adh}_\theta \mathcal{A}$ if $\xi \geq \theta$.

Recall that a family $\mathcal{A}$ is isotone if $B \supset A \in \mathcal{A}$ implies $B \in \mathcal{A}$. If $\mathcal{A}, \mathcal{B}$ are isotone families, then[39]

$$\operatorname{adh} 2^X = \varnothing;$$
$$\operatorname{adh}(\mathcal{A} \cap \mathcal{B}) = \operatorname{adh} \mathcal{A} \cup \operatorname{adh} \mathcal{B}.$$

It follows that $\mathcal{G} \supset \mathcal{F}$ implies $\operatorname{adh} \mathcal{G} \subset \operatorname{adh} \mathcal{F}$.

If A is a subset of a convergence space (X, ξ), then $\operatorname{adh}_\xi A$ (an abbreviation for $\operatorname{adh}_\xi\{A\} = \operatorname{adh}_\xi (A)_\bullet$) is the *adherence* of a set A. It follows that the operation of adherence of sets fulfills

$$\operatorname{adh} \varnothing = \varnothing;$$
$$\operatorname{adh}(A \cup B) = \operatorname{adh} A \cup \operatorname{adh} B;$$
$$A \subset \operatorname{adh} A.$$

for every A and B. Therefore $A \subset B$ implies $\operatorname{adh} A \subset \operatorname{adh} B$.

Remark 21. *The vicinity filter was defined in (14) for an arbitrary convergence. Notice that*

$$x \in \operatorname{adh}_\xi A \Leftrightarrow A \in \mathcal{V}_\xi^{\#}(x). \tag{17}$$

If X is a fixed set, then I denote $A^c = X \backslash A$ for each $A \subset X$.

Proposition 22. *If ξ is a topology, then the (set) adherence* adh_ξ *is idempotent and equal to the closure* cl_ξ.

Proof. If $x \notin \operatorname{adh}_\xi A$ then there is $V \in \mathcal{V}_\xi(x)$ such that $V \cap A = \varnothing$, and if ξ is a topology, then by (15) there is an open set O such that $x \in O$ and $O \cap A = \varnothing$. Therefore $x \notin O^c \supset \operatorname{cl}_\xi A$. Because $\operatorname{cl}_\xi A$ is closed, and $\operatorname{adh}_\xi A \subset \operatorname{cl}_\xi A$, also $\operatorname{adh}_\xi^2 A = \operatorname{adh}_\xi(\operatorname{adh}_\xi A) \subset \operatorname{cl}_\xi A$. ∎

[39] Indeed, a filter $\mathcal{F}$ does not mesh neither $\mathcal{A}$ nor $\mathcal{B}$ if and only if there exist $F_0, F_1 \in \mathcal{F}$ and $A \in \mathcal{A}, B \in \mathcal{B}$ such that $F_0 \cap A = \varnothing$ and $F_1 \cap B = \varnothing$, equivalently $F_0 \cap F_1 \cap (A \cup B) = \varnothing$. Because $\mathcal{A}, \mathcal{B}$ are isotone, the elements of $\mathcal{A} \cap \mathcal{B}$ are of the form $A \cup B$ with $A \in \mathcal{A}, B \in \mathcal{B}$. Thus $\mathcal{F}$ does not mesh with $\mathcal{A} \cap \mathcal{B}$, because $F_0 \cap F_1 \in \mathcal{F}$.

Remark 23. *If ξ is a convergence on X and $\mathcal{F}$ is a filter on X, then we denote by* $\mathrm{adh}_\xi^\natural \mathcal{F}$ *the filter generated by* $\{\mathrm{adh}_\xi F : F \in \mathcal{F}\}$. *Therefore we distinguish between the set* $\mathrm{adh}_\xi \mathcal{F}$ *and the filter* $\mathrm{adh}_\xi^\natural \mathcal{F}$. *Similarly* $\mathrm{cl}_\xi^\natural \mathcal{F}$ *denotes the filter generated by* $\{\mathrm{cl}_\xi F : F \in \mathcal{F}\}$.

Dual notions of adherence and of closure are those of respectively *inherence* and *interior*, namely

$$\mathrm{inh}\, A = (\mathrm{adh}\, A^c)^c, \ \mathrm{int}\, A = (\mathrm{cl}\, A^c)^c.$$

Notice that $x \in \mathrm{inh}\, A$ if and only if $A \in \mathcal{V}(x)$, and $x \in \mathrm{int}\, A$ if and only if $A \in \mathcal{N}(x)$.

6. Covers

Let (X, ξ) be a convergence space. A family $\mathcal{P}$ of subsets of X is a *cover* of $B \subset X$ if $\lim_\xi \mathcal{F} \cap B \neq \varnothing$ implies that $\mathcal{F} \cap \mathcal{P} \neq \varnothing$. As for every convergence each principal ultrafilter converges to its defining point, each cover $\mathcal{P}$ of B is a set-theoretic cover of B, that is, $B \subset \bigcup \mathcal{P}$.[40] Let us investigate the notion of cover in special cases.

Example 24. *If ξ is a pretopology, then the coarsest filter that converges to x is the vicinity filter $\mathcal{V}_\xi(x)$. Therefore $\mathcal{P}$ is a cover of B in ξ if and only if for every $x \in B$ there exists $P \in \mathcal{P}$ with $P \in \mathcal{V}_\xi(x)$, equivalently $x \in \mathrm{inh}_\xi P$. In other words, $\mathcal{P}$ is a cover of B in ξ if and only if $B \subset \bigcup_{P \in \mathcal{P}} \mathrm{inh}_\xi P$. In particular, if ξ is a topology, then this becomes $B \subset \bigcup_{P \in \mathcal{P}} \mathrm{int}_\xi P$. In other words, $\mathcal{P}$ is a cover of a subset B of a topological space if and only if $\{\mathrm{int}\, P : P \in \mathcal{P}\}$ is an (open) set-theoretic cover of B.*[41]

We denote $\mathcal{P}_c = \{P^c : P \in \mathcal{P}\}$.

Theorem 25. [13] *A family $\mathcal{P}$ is a cover of B if and only if*

$$\mathrm{adh}\, \mathcal{P}_c \cap B = \varnothing. \tag{18}$$

Proof. By definition, (18) means that if a filter $\mathcal{F}$ converges to an element of B then $\mathcal{F}$ does not mesh with $\mathcal{P}_c$, that is, there exist $F \in \mathcal{F}$ and $P \in \mathcal{P}$ such that $F \cap P^c = \varnothing$, equivalently $F \subset P$, that is, $\mathcal{F} \cap \mathcal{P} \neq \varnothing$, which means that $\mathcal{P}$ is a cover of A. ∎

Notice that in general $\mathcal{P}_c$ in (18) is not a filter, even not a filter base. A family $\mathcal{R}$ is an *ideal* if $S \subset R \in \mathcal{R}$ implies $S \in \mathcal{R}$, and if $\bigcup \mathcal{T} \in \mathcal{R}$ for each finite $\mathcal{T} \subset \mathcal{R}$. Clearly, $\mathcal{R}$ is an ideal if and only if $\mathcal{R}_c$ is a (possibly degenerate) filter. Denote by $\tilde{\mathcal{P}}$ the least ideal including $\mathcal{P}$. Then $(\tilde{\mathcal{P}})_c$ is the

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[40] If $\xi \le \zeta$ then each ξ-cover of B is a ζ-cover of B. Then the last statement follows from the observation that $\mathcal{P}$ is a set-theoretic cover of B if and only if $\mathcal{P}$ is a cover of B for the discrete topology ι.

[41] In each convergence space, a family of open sets is a cover if and only if it is a set-theoretic cover.

(possibly degenerate) filter generated by the finite intersections of elements of $\mathcal{P}_c$.[42]

Remark 26. *In a topological space, if $\mathcal{P}$ is a family of open sets and $\tilde{\mathcal{P}}$ is a cover of B, then $\mathcal{P}$ is also a cover of B,[43] and on the other hand, for each cover $\mathcal{P}$ of B the family $\{\operatorname{int} P : P \in \mathcal{P}\}$ is an open cover of B.*

7. Compactness

If A and B are subsets of a topological space X, then A is called (relatively) *compact* at B if for every open cover of B there exists a finite subfamily, which is a cover of A.[44] It is known that

A is *compact at* B if and only if for every filter $\mathcal{H}$,

$$A \in \mathcal{H}^{\#} \Rightarrow \operatorname{adh} \mathcal{H} \cap B \neq \varnothing. \tag{19}$$

If A and B are subsets of a convergence space X, then we take the characterization above for the definition.[45]

Proposition 27. *A set A is compact at B if and only if $A \in \mathcal{P}$ for every ideal cover $\mathcal{P}$ of B.*

Proof. Formula (19) means that $\operatorname{adh} \mathcal{H} \cap B = \varnothing$ implies that $A \notin \mathcal{H}^{\#}$, and because $\mathcal{H}$ is isotone, $A^c \in \mathcal{H}$ by (4), hence by Theorem 25, if $\mathcal{H}_c$ is a cover of B then $A \in \mathcal{H}_c$. As $\mathcal{H}$ is a filter, $\mathcal{H}_c$ is an ideal. ∎

In general convergence spaces there exists a notion of cover-compactness, which is (in general, stricly) stronger than that of compactness.[46]

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[42]Notice that if $\mathcal{B}$ is base of a filter $\mathcal{F}$ then $\operatorname{adh} \mathcal{B} = \operatorname{adh} \mathcal{F}$. However $\operatorname{adh} \mathcal{H}$ is (in general, strictly) bigger than $\operatorname{adh}\{\bigcap \mathcal{G} : \mathcal{G} \subset \mathcal{H}, \operatorname{card} \mathcal{G} < \infty\}$.

For example, if $\mathcal{H} = \{H_0, H_1\}$ then $\operatorname{adh} \mathcal{H} = \operatorname{adh} H_0 \cap \operatorname{adh} H_1$ while the adherence of the (filter generated by) finite intersections of elements of $\mathcal{H}$ is $\operatorname{adh}(H_0 \cap H_1)$.

[43]Indeed, for every $x \in B$ there is a finite subset $\mathcal{T}$ of $\mathcal{P}$ such that $x \in \bigcup \mathcal{T}$, hence there is $P \in \mathcal{T} \subset \mathcal{P}$ such that $x \in P$.

[44]Many authorssay that a topological space X is *compact* if it is Hausdorff and if is compact at X (in the sense of our definition).

[45]If A is compact at B in the topological sense, and $\mathcal{P}$ is an ideal cover of B, then by Remark 26 $\{\operatorname{int} P : P \in \mathcal{P}\}$ is an open cover of B, hence there is a finite subfamily $\mathcal{R}$ of $\mathcal{P}$ such that $\{\operatorname{int} P : P \in \mathcal{R}\}$ is a cover of A, so that $A \subset \bigcup \mathcal{R} \in \mathcal{P}$, thus by Propostion 27 A is compact at B in the convergence sense. Conversely, if A is compact at B in the convergence sense and $\mathcal{P}$ is an open cover of B then by Remark 26 $\tilde{\mathcal{P}}$ is an ideal cover of B, thus by Propostion 27 there is a finite subfamily $\mathcal{R}$ of $\mathcal{P}$ such that $A \subset \bigcup \mathcal{R}$.

[46]A is *cover-compact* at B if for each cover $\mathcal{P}$ of B there is a finite subfamily $\mathcal{R}$ of $\mathcal{P}$ which is a cover of A. If A is cover-compact at B then A is compact at B. Indeed, the condition holds in particular for ideal covers, and a finite family $\mathcal{R}$ is a cover of A, then *a fortiori* $A \subset \bigcup \mathcal{R}$. It suffices to use Proposition 27 to conclude. The converse is not true in general. Take the pretopology from Example 16. Let $A = \{x_\infty\} \cup \{x_n : n < \infty\}$ and $A_n = \{x_n\} \cup \{x_{n,k} : k < \infty\}$. The set A is compact at itself but not cover-compact at itself. In fact, every free ultrafilter on A converges to x_∞. On the other hand, the family $\mathcal{P} = \{A\} \cup \{A_n : n < \infty\}$ is a cover of A but no finite subfamily is a cover of A. The subfamily $\{F\} \cup \{F_n : n < m\}$ is not a cover of F, because each vicinity of x_{m+1} includes

If a subset A of a convergence space X is compact at X, then I call it *relatively compact.* A subset of a convergence space is *compact* if it is compact at itself.

7.1. **Compact families.** Our definitions have an obvious natural extension to families of sets [16]. Let $\mathcal{A}, \mathcal{B}$ be families of subsets of X. Then $\mathcal{A}$ is *compact at* $\mathcal{B}$ if for every filter $\mathcal{H}$,

$$\mathcal{A}\#\mathcal{H} \Rightarrow \operatorname{adh}\mathcal{H} \in \mathcal{B}^{\#}. \tag{20}$$

A family $\mathcal{A}$ is *relatively compact* if it is compact at (the whole space) X, and *compact* if it is compact at itself.[47,48] These notions generalize not only that of (relatively) compact sets, but also of convergent filters. In fact,

Every convergent filter is relatively compact.[49]

More precise relationship between convergence and compactness will be given in Proposition 34. It is immediate that the image of a compact filter by a continuous map is compact.

Theorem 28 (Tikhonov theorem)**.** *A filter (on a product of convergence spaces) is relatively compact if and only if its every projection is relatively compact.*

Proof. The necessity follows from the preceding remark. As for the sufficiency, let $\mathcal{F}$ be a filter on $\prod \Xi$. Let $\mathcal{U}$ be an ultrafilter with $\mathcal{U}\#\mathcal{F}$. This implies $p_\xi(\mathcal{U})\# p_\xi(\mathcal{F})$ for each $\xi \in \Xi$, and since $p_\xi(\mathcal{F})$ is ξ-relatively compact there is $x_\xi \in X_\xi$ such that $x_\xi \in \lim_\xi p_\xi(\mathcal{U})$, which means that $(x_\xi)_\xi \in \lim_{\prod \Xi} \mathcal{U}$. ∎

No separation condition has been required in the definition of compactness.

7.2. **Weaker versions of compactness.** I will now weaken the definition (20) of compactness by restricting the class of filters $\mathcal{H}$. Let $\mathbb{H}$ be a class of filters. A family $\mathcal{A}$ (of subsets of a convergence space) is $\mathbb{H}$-*compact at* $\mathcal{B}$ (another family of subsets of that space) if

$$\forall_{\mathcal{H}\in\mathbb{H}}\ \mathcal{H}\#\mathcal{A} \ \Rightarrow\ \operatorname{adh}\mathcal{H} \in \mathcal{B}^{\#}.$$

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all but finitely elements of F_{m+1}. The ideal $\tilde{\mathcal{P}}$ is *a fortiori* a cover of A, for which no element is a cover of A.

[47] This is a terminological turnover with respect to the previous papers of mine and of my collaborators, where the term *compactoid* was used for all the sorts of relative compactness. The present choice is done for the sake of simplicity, and follows that of Professor Iwo Labuda of the University of Mississippi. The term *compactoid space* was introduced by Gustave Choquet [8] for compact space without any separation axiom.

[48] If $\mathcal{B} = \{B\}$ then we say *compact at* B instead of *compact at* $\mathcal{B}$; if moreover, $B = \{x\}$ then we say *compact at* x.

[49] Actually if $x \in \lim \mathcal{F}$ then $\mathcal{F}$ is compact at x. Indeed, if $\mathcal{H}\#\mathcal{F}$ then there is an ultrafilter $\mathcal{U}$ finer than $\mathcal{H} \vee \mathcal{F}$, hence $x \in \lim \mathcal{U} = \operatorname{adh}\mathcal{U} \subset \operatorname{adh}\mathcal{H}$.

If $\mathbb{H}$ is the class of all filters, then $\mathbb{H}$-compactness is equivalent to compactness.[50]

7.3. **Countable compactness.** If $\mathbb{H}$ is the class of countably based filters, then $\mathbb{H}$-compactness is equivalent to *countable compactness*.

7.4. **Finite compactness.** If $\mathbb{H}$ is the class of principal filters, then $\mathbb{H}$-compactness is called *finite compactness*. This property is very broad (and useless) in the case of sets. Indeed, a subset A in a Hausdorff topological space is finitely compact at a set B if and only if $A \subset B$. However the notion is far from being trivial and useless in the context of filters [11].

Proposition 29. *A filter $\mathcal{F}$ is finitely compact at a set B if and only if $\mathcal{V}(B) \subset \mathcal{F}$.*

Proof. By definition, $\mathcal{F}$ is finitely compact at B if $\operatorname{adh} H \cap B \neq \varnothing$ for every $H \in \mathcal{F}^{\#}$. Equivalently, if $\operatorname{adh} H \cap B = \varnothing$, that is, if $H^c \in \mathcal{V}(B)$ then $H^c \in \mathcal{F}$. ∎

7.5. **Sequential compactness.** By definition, a convergence ξ is *sequentially compact* if for every sequential filter (equivalently, for every countably based filter) $\mathcal{E}$ there exists a sequential filter $\mathcal{F} \supset \mathcal{E}$ such that $\lim_\xi \mathcal{F} \neq \varnothing$. Notice that

$$\operatorname{adh}_{\operatorname{Seq}\xi} \mathcal{E} = \bigcup_{\mathcal{F} \in \varepsilon\mathcal{E}} \lim_\xi \mathcal{F},$$

where $\varepsilon\mathcal{E}$ stands for the set of sequential filters finer than $\mathcal{E}$. In other words,[51]

Proposition 30. *A T_1 convergence ξ is sequentially compact if and only if* $\operatorname{Seq}\xi$ *is countably compact.*

8. Adherence-determined convergences

8.1. **Pseudotopologies.** A convergence ξ is a *pseudotopology* if $x \in \lim_\xi \mathcal{F}$ whenever $x \in \lim_\xi \mathcal{U}$ for every ultrafilter $\mathcal{U}$ finer than $\mathcal{F}$, that is, if

$$\lim_\xi \mathcal{F} \supset \bigcap_{\mathcal{U} \in \beta\mathcal{F}} \lim_\xi \mathcal{U}. \tag{21}$$

[50] A family $\mathcal{A}$ is relatively $\mathbb{H}$-*compact* if it is $\mathbb{H}$-compact at the whole space. $\mathcal{A}$ is $\mathbb{H}$-*autocompact* if is $\mathbb{H}$-compact at itself. So far I used the term $\mathbb{H}$-compact for the property above, but Iwo Labuda convinced me that that terminology was not appropriate. In fact, if $\mathcal{A}$ is a family of subsets of X such that $\mathcal{A} = \{A\}$ with $A \subsetneq X$, then it is $\mathbb{H}$-*compact* if A (with the convergence induced from X) is $\mathbb{H}$-compact. This property is, in general, different from that of $\mathbb{H}$-autocompactness of A. Of course, the two notions coincide in case when $\mathbb{H}$ is the class of all filters. In other words, compactness of sets is absolute (that is, independent of environment). I prefer however the term $\mathbb{H}$-*autocompact* to Labuda's $\mathbb{H}$-*selfcompact*, as the latter has a mixed (English-Latin) origin.

[51] This result is due to Ivan Gotchev [26, Theorem 3.6] for topologies T_0.

This means that each pseudotopology is determined by the limits of all ultrafilters.[52]

Example 31 (non-pseudotopological convergence)**.** *In an infinite set X distinguish an element ∞, and define a convergence on X as follows: the principal ultrafilter $(x)_\bullet$ converges to x, and for each finite subset F of $\beta_\circ X$, the set of all free ultrafilters on X, one has $\{\infty\} = \lim \bigcap_{\mathcal{U}\in F} \mathcal{U}$. This convergence is not a pseudotopology, because if $\mathcal{F}$ is a free filter such that $\beta\mathcal{F}$ is infinite,*[53] *then $\infty \notin \lim \mathcal{F}$ but $\infty \in \lim \mathcal{U}$ for each $\mathcal{U} \in \beta\mathcal{F}$.*[54]

The set of pseudotopologies on a given set is stable for arbitrary suprema and contains the chaotic topology. As a result, for every convergence ζ there exists the finest among coarser pseudotopologies, the *pseudotopologization* $S\zeta$ of ζ. It is straightforward that

$$\lim_{S\zeta} \mathcal{F} = \bigcap_{\mathcal{U}\in\beta\mathcal{F}} \lim_\zeta \mathcal{U}.$$

The *pseudotopologizer* is isotone, expansive and idempotent. As we have seen, this property holds also for the topologizer and the pretopologizer. The following property is particular for the pseudotopologizer:

Proposition 32. *If Θ is a set of convergences on X, then*

$$S(\bigvee \Theta) = \bigvee_{\theta\in\Theta} S\theta.$$

This proposition is very important for the sequel. Therefore, I shall provide its proof, even though it is straightforward and simple.

Proof. By definition, $\lim_{\bigvee_{\theta\in\Theta} S\theta} \mathcal{F}$

$$= \bigcap_{\theta\in\Theta} \bigcap_{\mathcal{U}\in\beta\mathcal{F}} \lim_\theta \mathcal{U} = \bigcap_{\mathcal{U}\in\beta\mathcal{F}} \bigcap_{\theta\in\Theta} \lim_\theta \mathcal{U} = \bigcap_{\mathcal{U}\in\beta\mathcal{F}} \lim_{\vee\Theta} \mathcal{U} = S(\lim_{\vee\Theta} \mathcal{F}).$$

∎

As for the topologizer and the pretopologizer, the pseudotopologizer fulfills $S(f^-\tau) \geq f^-(S\tau)$ for every convergence τ. The pseudotopologizer has another particular property (with important implications in topology). Namely,

$$S(f^-\tau) = f^-(S\tau) \tag{22}$$

for every convergence τ and each map f.

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[52] Each pseudotopology ξ on X can be characterized with the aid of the Stone transform. For every x let $\mathbb{V}_\xi(x)$ be the set of all ultrafilters which converge to x in ξ. Then by (21) $x \in \lim_\xi \mathcal{F}$ if and only if $\beta\mathcal{F} \subset \mathbb{V}_\xi(x)$. It follows that each map $\mathbb{V} : X \to \beta X$ such that $(x)_\bullet \in \mathbb{V}(x)$ for each x defines a pseudotopology.

[53] It is known (e.g., [23, Theorems 3.6.11 and 3.6.14]) that if $\mathrm{card}(\beta_\circ\mathcal{F})$ is infinite, then it is at least $2^{2^{\aleph_0}}$.

[54] This convergence is a a *prototopology*, that is, fulfilling $\lim \mathcal{F}_0 \cap \lim \mathcal{F}_1 \subset \lim(\mathcal{F}_0 \cap \mathcal{F}_1)$.

Proof. Indeed, if $x \in \lim_{f^-(S\tau)} \mathcal{F}$ then equivalently $f(x) \in \lim_{S\tau} f(\mathcal{F})$, that is, $f(x) \in \lim_\tau \mathcal{U}$ for every $\mathcal{U} \in \beta f(\mathcal{F})$. If now $\mathcal{W} \in \beta\mathcal{F}$, then $f(\mathcal{W}) \in \beta f(\mathcal{F})$ and thus $f(x) \in \lim_\tau f(\mathcal{W})$, equivalently $x \in \lim_{f^-\tau} \mathcal{W}$, which means that $x \in \lim_{S(f^-\tau)} \mathcal{F}$. ∎

Because the product is the supremum of initial convergences with respect to the projections on component spaces, we get an important

Theorem 33 (prototheorem of Tikhonov)**.**

$$S(\prod \Xi) = \prod_{\xi \in \Xi} S\xi \tag{23}$$

for every set of convergences Ξ*.*

The relationship between compactness and pseudotopological convergence is very close. In fact,

Proposition 34. *A filter* $\mathcal{F}$ *is* ξ*-compact at* x *if and only if* $x \in \lim_{S\xi} \mathcal{F}$*.*

Proof. Indeed, $x \in \lim_{S\xi} \mathcal{F}$ if and only if $x \in \lim_\xi \mathcal{U}$ for every $\mathcal{U} \in \beta\mathcal{F}$, which is equivalent to the compactness of $\mathcal{F}$ at x. ∎

We notice that the generalization of the classical Tikhonov Theorem 28 can be easily deduced from the Tikhonov prototheorem (Theorem 33).

8.2. **Narrower classes of adherence-determined convergences.** If $\mathbb{H}$ is a class of filters, then

$$\lim_{\mathrm{A}_{\mathbb{H}}\,\xi} \mathcal{F} = \bigcap_{\mathbb{H} \ni \mathcal{H} \# \mathcal{F}} \mathrm{adh}_\xi \mathcal{H} \tag{24}$$

defines a convergence $\mathrm{A}_{\mathbb{H}}\,\xi$ obtained from the original convergence ξ. Of course, if $\mathbb{H}$ is the class of all filters, then $\mathrm{A}_{\mathbb{H}}$ is the pseudotopologizer. More generally,

Theorem 35. [10] *An* $\mathrm{A}_{\mathbb{H}}$*-convergence is a*

$$\textit{pseudotopology} \Leftrightarrow \mathbb{H} \textit{ is the class of all filters;} \tag{25}$$

$$\textit{paratopology} \Leftrightarrow \mathbb{H} \textit{ is the class of countably based filters;} \tag{26}$$

$$\textit{pretopology} \Leftrightarrow \mathbb{H} \textit{ is the class of principal filters;} \tag{27}$$

Actually, *paratopologies* were defined in [10] as the convergences fulfilling (26) of Theorem 35.

Proof. (25) For each convergence ξ, if $\mathcal{H}\#\mathcal{F}$ then $\lim_\xi \mathcal{F} \subset \mathrm{adh}_\xi \mathcal{H}$, hence $\lim_\xi \mathcal{F} \subset \bigcap_{\mathcal{H}\#\mathcal{F}} \mathrm{adh}_\xi \mathcal{H}$. If ξ is a pseudotopology, then $\bigcap_{\mathcal{H}\#\mathcal{F}} \mathrm{adh}_\xi \mathcal{H} \subset \bigcap_{\mathcal{U}\in\beta\mathcal{F}} \lim_\xi \mathcal{U} \subset \lim_\xi \mathcal{F}$. Conversely, if $\bigcap_{\mathcal{H}\#\mathcal{F}} \mathrm{adh}_\xi \mathcal{H} \subset \lim_\xi \mathcal{F}$ then $\bigcap_{\mathcal{U}\in\beta\mathcal{F}} \lim_\xi \mathcal{U} \subset \lim_\xi \mathcal{F}$, because for every filter $\mathcal{H}\#\mathcal{F}$ there is an ultrafilter $\mathcal{U} \geq \mathcal{H} \vee \mathcal{F}$, that is, $\mathcal{U} \in \beta\mathcal{F}$ and $\mathrm{adh}_\xi \mathcal{H} \supset \lim_\xi \mathcal{U}$.

(27) Suppose that ξ is a pretopology and let $x \in \mathrm{adh}_\xi H$ for every $H \in \mathcal{F}^\#$. Since $x \in \mathrm{adh}_\xi H$ amounts to $H \in \mathcal{V}_\xi^\#(x)$, we infer that $\mathcal{F}^\# \subset \mathcal{V}_\xi^\#(x)$, that is,

$\mathcal{F} \supset \mathcal{V}_\xi(x)$, that is $x \in \lim_\xi \mathcal{F}$. Conversely, suppose that $\bigcap_{H\in\mathcal{F}^\#} \mathrm{adh}_\xi H \subset \lim_\xi \mathcal{F}$ and $\mathcal{F} \supset \mathcal{V}_\xi(x)$, but $x \notin \lim_\xi \mathcal{F}$. Hence there exists $H \in \mathcal{F}^\#$ such that $x \notin \mathrm{adh}_\xi H$. The latter means that $H \notin \mathcal{V}_\xi^\#(x)$, that is, $\mathcal{F}^\#$ is not a subfamily of $\mathcal{V}_\xi^\#(x) \neq \varnothing$, equivalently $\mathcal{V}_\xi(x)$ is not a subfamily of $\mathcal{F}$, which yields a contradiction. ∎

It turns out that (22) extends to the discussed modifiers. Indeed, if $\mathbb{H}$ is an $\mathbb{F}_0$-*composable* class of filters (that is, if a $\mathcal{H} \in \mathbb{H}$ is a filter on X and $\Omega \subset X \times Y$, then $\Omega\mathcal{H} \in \mathbb{H}$) and $A_\mathbb{H}$ is given by (24), then [21, Theorem 19] states that

$$A_\mathbb{H}(f^-\tau) = f^-(A_\mathbb{H}\tau)$$

for every convergence τ and each map f. In particular, the formula above holds for the pretopologizer and the paratopologizer.

The topologizer can be also described by a formula of the type (24), but with a class $\mathbb{H}$ which depends on topologies. I prefer instead to give another, more direct, formula

$$\lim_{T\xi} \mathcal{F} = \bigcap_{H\in\mathcal{F}^\#} \mathrm{cl}_\xi H. \tag{28}$$

Proof. One has $x \notin \lim_{T\xi} \mathcal{F}$ if and only if there exists a ξ-open set O such that $x \in O \notin \mathcal{F}$, equivalently $x \notin O^c \in \mathcal{F}^\#$, that is, there is $H = \mathrm{cl}_\xi H \in \mathcal{F}^\#$ such that $x \notin H$. As $H \in \mathcal{F}^\#$ implies $\mathrm{cl}_\xi H \in \mathcal{F}^\#$, we infer (28). ∎

It turns out that each class of adherence-determined convergences corresponds to a version of compactness. Namely,

Theorem 36. [12] *Let $\mathbb{H}$ be a class of filters. A filter $\mathcal{F}$ is $\mathbb{H}$-compact at x for ξ if and only if $x \in \lim_{\mathrm{A}_\mathbb{H}\xi} \mathcal{F}$.*

Proof. A filter $\mathcal{F}$ is $\mathbb{H}$-compact at x for ξ whenever $x \in \mathrm{adh}_\xi \mathcal{H}$ for every filter $\mathcal{H} \in \mathbb{H}$ such that $\mathcal{H}\#\mathcal{F}$, that is, whenever $x \in \lim_{\mathrm{A}_\mathbb{H}\xi} \mathcal{F}$. ∎

We conclude that compactness is of pseudotopological nature, countable compactness of paratopological and finite compactness of pretopological.

Adherence-determined	Compactness variant
Pseudotopologies	Compactness
Paratopologies	Countable compactness
Pretopologies	Finite compactness

It is easy to construct pseudotopologies, which are not pretopologies, using the following

Remark 37. *Recall that if ξ is a pseudotopology, and $\mathbb{V}_\xi(x)$ stands for the set of ultrafilters that converge to x in ξ, then*

$$x \in \lim_\xi \mathcal{F} \Leftrightarrow \beta\mathcal{F} \subset \mathbb{V}_\xi(x). \tag{29}$$

A pseudotopology ξ is a pretopology if and only if $\mathbb{V}_\xi(x)$ is closed with respect to the Stone topology for each x.[10, Proposition A.1][55]

The following remark enables one to construct paratopologies which are not pretopologies.

Remark 38. *Let $G_\delta\beta$ stand for the topology on βX such that a neighborhood base of $\mathcal{U} \in \beta X$ consists of G_δ subsets (with respect to the Stone topology of βX) which contain $\mathcal{U}$. A pseudotopology ξ is a paratopology if and only if $\mathbb{V}_\xi(x)$ is $G_\delta\beta$-closed for each x.*[10, Proposition A.2][56]

Here is an example of a pseudotopology τ such that $\tau > P_\omega\tau > P\tau = T\tau$.[57]

Example 39. *This is a pseudotopology τ on a countably infinite set X, in which all elements but one are isolated, that is, if x is not equal to a distinguished element ∞, then $(x)_\bullet$ is the only filter that converges to x. To define τ at ∞, let B be a subset of $\beta_\circ X$ (the set of all free ultrafilters on X), which is $G_\delta\beta$-closed and is not Stone-closed.*[58] *Let $\mathcal{U} \in B$ and set $B_0 = B\backslash\{\mathcal{U}\}$. Then $B_0 \neq \mathrm{cl}_{G_\delta\beta} B_0 \neq \mathrm{cl}_\beta B_0$, where the latter stands for the Stone closure of B_0. If we set $\mathbb{V}_\tau(\infty) = B_0 \cup \{\infty\}$, then by virtue of the preceding remarks $\mathbb{V}_{P_\omega\tau}(\infty) = \mathrm{cl}_{G_\delta\beta} B_0$ and $\mathbb{V}_{P\tau}(\infty) = \mathrm{cl}_\beta B_0$. Because all other points are isolated, $P\tau = T\tau$.*

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[55]Indeed, $\beta(\mathcal{V}_\xi(x)) = \mathrm{cl}_\beta \mathbb{V}_\xi(x)$, where $\mathcal{V}_\xi(x)$ is the vicinity filter of x for ξ by virtue of (29). Therefore ξ is a pretopology if and only if $x \in \lim_\xi \mathcal{V}_\xi(x)$, that is, whenever $\mathbb{V}_\xi(x)$ is β-closed.

Actually, if $\mathbb{V}_{P\xi}(x)$ is the set of all ultrafilters that converge to x in $P\xi$, then

$$\mathbb{V}_{P\xi}(x) = \mathrm{cl}_\beta \mathbb{V}_\xi(x).$$

[56]If ξ is a paratopology and an ultrafilter $\mathcal{U} \notin \mathbb{V}_\xi(x)$, that is, $x \notin \lim_\xi \mathcal{U}$ by virtue of (29), then by (24) there is a countably based filter $\mathcal{H}$, coarser than $\mathcal{U}$ and such that $x \notin \mathrm{adh}_\xi \mathcal{H}$. Let $(H_n)_n$ be a decreasing sequence that generates $\mathcal{H}$. Then $\beta\mathcal{H} = \bigcap_{n<\omega} \beta H_n$. No $\mathcal{W} \in \beta\mathcal{H}$ converges to x in ξ, that is, $\beta\mathcal{H}$ is a G_δ set disjoint from $\mathbb{V}_\xi(x)$, which proves that $\mathbb{V}_\xi(x)$ is $G_\delta\beta$ -closed. If ξ is a pseudotopology, but not a paratopology, then there exist x, an ultrafilter $\mathcal{U}$ such that $x \notin \lim_\xi \mathcal{U}$ but $x \in \mathrm{adh}_\xi \mathcal{H}$ for each countably based filter $\mathcal{H}$ coarser than $\mathcal{U}$. Therefore $\mathcal{U} \notin \mathbb{V}_\xi(x)$ but $\beta\mathcal{H} \cap \mathbb{V}_\xi(x) \neq \varnothing$ for each $\beta\mathcal{H}$ from $\{\beta\mathcal{H} : \mathcal{H} \in \mathbb{F}_\omega, \mathcal{H} \geq \mathcal{U}\}$, which is a neighborhood base of $\mathcal{U}$ in $G_\delta\beta$. This shows that $\mathbb{V}_\xi(x)$ is not $G_\delta\beta$ -closed. Actually,

$$\mathbb{V}_{P_\omega\xi}(x) = \mathrm{cl}_{G_\delta\beta} \mathbb{V}_\xi(x).$$

[57]A slight modification of this example [22, Example 5] provides a pseudotopology ξ such that $\xi > P_\omega\xi > P\xi > T\xi$.

[58]Such sets exist, because if $(A_n)_n$ is a descending sequence of infinite subsets of the set $\mathbb{N}$ of natural numbers such that $A_n\backslash A_{n+1}$ is infinite, then the supremum of cofinite filters $(A_n)_\circ$ is not cofinite but admits a finer cofinite filter (of an infinite set). In terms of the Stone transform, the intersection of Stone open (and closed) sets $A = \bigcap_{n<\infty} \beta_\circ A_n$ is not open (is G_δ of course), but $\mathrm{int}\, A \neq \varnothing$.

9. Diagonality and regularity

If $\mathcal{N}(x)$ is a neighborhood filter of a topology on X, then $\mathcal{N}(A) = \bigcap_{x\in A}\mathcal{N}(x)$ is the neighborhood filter of a subset A of X. If $\mathcal{A}$ is a family of subsets, then let

$$\mathcal{N}(\mathcal{A}) = \bigcup_{A\in\mathcal{A}} \mathcal{N}(A). \tag{30}$$

In other words, $B \in \mathcal{N}(\mathcal{A})$ whenever there is $A \in \mathcal{A}$ such that $B \in \mathcal{N}(x)$ for each $x \in A$.

Example 40. *In particular, if $\mathcal{A} = \mathcal{N}(x_0)$ then*

$$\mathcal{N}(\mathcal{N}(x_0)) = \mathcal{N}(x_0).$$

Indeed, $\mathcal{N}(\mathcal{N}(x_0)) \subset \mathcal{N}(x_0)$ because if $B \in \mathcal{N}(\mathcal{N}(x_0))$ then there is $A \in \mathcal{N}(x_0)$ such that $B \in \mathcal{N}(x)$ for each $x \in A$, in particular $B \supset A$ hence $B \in \mathcal{N}(x_0)$. Conversely, if $B \in \mathcal{N}(x_0)$, that is, B is a neighborhood, then by a fundamental property of neighborhoods of a topological space, there is a neighborhood A of x_0 such that B is a neighborhood of every $x \in A$, that is, $B \in \mathcal{N}(\mathcal{N}(x_0))$.

Formula (30) defines a special contour. The example above shows that in topological spaces, the contour of the family of neighborhoods along a neighborhood of x_0, converges to x_0. We shall see that this means that topologies are diagonal convergences.

Regularity can be also expressed in terms of contours, and is in some sense (that we will make precise in a moment) inverse to diagonality. It turns out that for Hausdorff compact pseudotopologies, diagonality and regularity coincide.[59]

9.1. Contours.

Consider a family $\mathcal{F}$ of subsets of a set X and for every $x \in X$, let $\mathcal{G}(x)$ be a family of subsets of Y. The *contour* of $\mathcal{F}$ over $\mathcal{G}(\cdot)$ (or of $\mathcal{G}(\cdot)$ along $\mathcal{F}$) is the following family of subsets of Y:

$$\mathcal{G}(\mathcal{F}) = \bigcup_{F\in\mathcal{F}} \bigcap_{x\in F} \mathcal{G}(x). \tag{31}$$

When $\mathcal{F} = \{F\}$, then we abridge $\mathcal{G}(F) = \bigcap_{x\in F}\mathcal{G}(x)$. Consequently, $\mathcal{G}(\mathcal{F}) = \bigcup_{F\in\mathcal{F}}\mathcal{G}(F)$. If $\mathcal{E}$ is a filter generated by $(x_n)_n$ and $\mathcal{F}(n)$ is a filter generated by $(x_{n,k})_k$ for every n, then the contour $\mathcal{F}(\mathcal{E})$ is denoted by

$$\int_{(n)} (x_{n,k})_k.$$

If $\mathcal{F}$ is a filter on the underlying set $|\xi|$ of a convergence ξ, then the symbol $\mathrm{adh}_\xi^\natural \mathcal{F}$ denotes the filter generated by $\{\mathrm{adh}_\xi F : F \in \mathcal{F}\}$. In the particular

[59] This is due to the minimality of compact pseudotopologies.

case, where $\mathcal{G}(x) = \mathcal{V}_\theta(x)$ for every x,[60] we have the following extension of Remark 21:

$$\mathcal{A}\#\mathcal{V}_\theta(\mathcal{B}) \iff (\mathrm{adh}_\theta^\natural \mathcal{A})\#\mathcal{B}. \tag{32}$$

Proof. Indeed, let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be such that $A\#\bigcap_{x\in B}\mathcal{V}_\theta(x)$, equivalently $A \in \bigcup_{x\in B}\mathcal{V}_\theta^\#(x)$, that is, $A \in \mathcal{V}_\theta^\#(x)$ for some $x \in B$. This is tantamount to $x \in \mathrm{adh}_\theta A \cap B$ and the proof is complete. ∎

9.2. **Hausdorff convergences.** A convergence is said to be *Hausdorff* if every limit is at most a singleton. If a convergence ξ happens to be Hausdorff, then we will often write $x = \lim_\xi \mathcal{F}$ as an abbreviation for $\{x\} = \lim_\xi \mathcal{F}$.

Compact Hausdorff pseudotopologies are minimal among Hausdorff pseudotopologies,[61] namely

Proposition 41. *If $\zeta \geq \xi$ are pseudotopologies, ζ is compact and ξ is Hausdorff, then $\zeta = \xi$.*

Proof. Because ξ and ζ are pseudotopologies, it is enough to show that they coincide for ultrafilters. Let $\mathcal{U}$ be an ultrafilter and $x \in \lim_\xi \mathcal{U}$. By compactness, $\varnothing \neq \mathrm{adh}_\zeta \mathcal{U} = \lim_\zeta \mathcal{U} \subset \lim_\xi \mathcal{U} = \{x\}$ because ξ is Hausdorff, thus $x \in \lim_\zeta \mathcal{U}$. ∎

A convergence ξ is *topologically Hausdorff* if $T\xi$ is Hausdorff.

Proposition 42. *Each topologically Hausdorff compact pseudotopology is a topology.*

Proof. If ξ is a compact pseudotopology and $T\xi$ is Hausdorff with $\xi \geq T\xi$, then $\xi = T\xi$ by Proposition 41. ∎

9.3. **Diagonality.** A convergence ξ on X is *diagonal* provided that if $x_0 \in \lim_\xi \mathcal{F}$ and if $x \in \lim_\xi \mathcal{G}(x)$ for every $x \in X$, then $x_0 \in \lim_\xi \mathcal{G}(\mathcal{F})$. A convergence ξ on X is *pretopologically diagonal* if $x_0 \in \lim_\xi \mathcal{F}$, then $x_0 \in \lim_\xi \mathcal{V}_\xi(\mathcal{F})$.

Example 43. *The pretopology π of Example 16 is not diagonal; its topologization $T\pi$ is diagonal. Notice that $\int_{(n)}(x_{n,k})_k$ is $\mathcal{N}^\circ_{T\pi}(x_\infty)$ (the free part of the neighborhood filter of x_∞).*

Proposition 44. *A pretopology is a topology if and only if it is diagonal.*

Proof. By Example 40 each topology is a diagonal pretopology. To prove the converse implication it is enough to show that if π is a diagonal pretopology, then $\mathrm{adh}_\pi^2 A \subset \mathrm{adh}_\pi A$ because then $\mathrm{cl}_\pi A \subset \mathrm{adh}_\pi A$ (for every A). Indeed, if $x_0 \in \mathrm{adh}_\pi^2 A$ then there is a filter $\mathcal{F}$ on $\mathrm{adh}_\pi A$ with $x_0 \in \lim_\pi \mathcal{F}$. On the other hand, for each $x \in \mathrm{adh}_\pi A$ there is a filter $\mathcal{G}(x)$ on A with $\lim_\pi \mathcal{G}(x)$. Of course, $A \in \mathcal{G}(\mathcal{F})$, and by diagonality $x_0 \in \lim_\pi \mathcal{G}(\mathcal{F})$, hence $x_0 \in \mathrm{adh}_\pi A$. ∎

[60] $\mathcal{V}_\theta(x)$ is the vicinity filter of x with respect to θ.

[61] Like topologies.

9.4. **Regularity.** A convergence ξ on X is *regular* with respect to a convergence θ on X (in short, θ*-regular*) if for every filter $\mathcal{F}$,

$$\lim_\xi \mathcal{F} \subset \lim_\xi \operatorname{adh}_\theta^\natural \mathcal{F}. \tag{33}$$

A convergence ξ is *regular* if it is ξ-regular (Fischer [24]), *topologically regular* if it is $T\xi$-regular. For each convergence ζ there exists the finest among the regular convergences that are coarser than ζ. It is called the *regularization* of ζ and is denoted by $R\zeta$.

An element x of a convergence space ξ is called *irregular* if there exists a filter $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F} \setminus \lim_\xi \operatorname{adh}_\xi^\natural \mathcal{F}$.

Example 45. *Let $X = \{x_\infty\} \cup \{x_n : n < \infty\} \cup \{x_{n,k} : n,k < \infty\}$ be the set of Example 16. Define the convergence ζ by $x_{n,k} \in \lim_\zeta \mathcal{F}$ whenever $\mathcal{F} = (x_{n,k})_\bullet$, $x_n \in \lim_\zeta \mathcal{F}$ whenever $(x_n)_\bullet \wedge (x_{n,k})_k \subset \mathcal{F}$, and $x_\infty \in \lim_\zeta \mathcal{F}$ provided that $(x_\infty)_\bullet \wedge \int_{(n)} (x_{n,k})_k \subset \mathcal{F}$. This is a pretopology, which is not regular. Actually, the elements $x_{n,k}$ and x_n are regular for each $n, k < \infty$ and x_∞ is irregular. Notice that $x_\infty \in \lim_{R\zeta} \mathcal{F}$ whenever $(x_\infty)_\bullet \wedge (x_n)_n \wedge \int_{(n)} (x_{n,k})_k \subset \mathcal{F}$, that is, $R\zeta = T\pi$, where π is the convergence of Example 16.*

The following proposition [14, Proposition 7.4] is the essence of classical examples of irregular topologies.[62]

Proposition 46. *Let ξ be a pretopology of countable character. An element x is irregular with respect to ξ if and only if there exist a sequence $(x_n)_n$ and for every $n < \omega$ a free sequence $(x_{n,k})_k$ such that $x_n \in \lim_\xi (x_{n,k})_k$*

$$x \in \lim_\xi \int_{(n)} (x_{n,k})_k.$$

but $x \notin \operatorname{adh}_\xi (x_n)_n$.

Proof. An element x is irregular for ξ if and only if $\operatorname{adh}_\xi^\natural \mathcal{V}_\xi(x)$ does not converge to x, that is, whenever there is $V \in \mathcal{V}_\xi(x)$ and a decreasing filter base (V_n) of $\mathcal{V}_\xi(x)$ such that for every $n < \omega$ there is $x_n \in \operatorname{adh}_\xi V_n \setminus V$. Hence $x \notin \operatorname{adh}_\xi (x_n)_n$, and for each such n there exists a sequence $(x_{n,k})_k$ on V_n for which $x_n \in \lim_\xi (x_{n,k})_k$. Since $\int_{(n)} (x_{n,k})_k$ is finer than $\mathcal{V}_\xi(x)$, it converges to x in ξ. If $(x_{n,k})_k$ were not free for infinitely many n, then $\int_{(n)} (x_{n,k})_k$ would be coarser than a subsequence of $(x_n)_n$, which does not converge to x in ξ. Therefore, $(x_{n,k})_k$ is free for almost all n, hence for all n, after having possibly dropped a finite number of them. ∎

[62] E. g., [23, Example 1.5.6]. Consider the unit interval $[0,1]$ in which a basic family of closed sets consists of the closed sets for the natural topology and of $\{\frac{1}{n} : n < \omega\}$. In this topology $x = 0$ is irregular. Then $x_n = \frac{1}{n}$ and $x_{n,k} = \frac{1}{n} + \frac{1}{k}$ verify Proposition 46.

Example 47. *Consider a variant of σ of Example 45, where $x_\infty \in \lim_\sigma \mathcal{F}$ provided that $\mathcal{F}$ is finer than the filter generated by the family*

$$\{\{x_\infty\} \cup \{x_{n,k} : n \geq m\} : m < \infty\}. \tag{34}$$

This is a pretopology of countable character with x_∞ as a unique irregularity point. Notice that $R\sigma = R\zeta$, where ξ is defined in Example 45. Proposition 46 implies [14] *that if a point x of a Hausdorff pretopological space ξ of countable character is irregular, then there exists a homeomorphic embedding i of the pretopology σ in ξ such that $x = i(x_\infty)$.*[63]

In some cases, the definition of Fischer coincides with that of Grimeisen [28][29]: ξ is θ-regular if

$$\operatorname{adh}_\xi \mathcal{V}_\theta(\mathcal{H}) \subset \operatorname{adh}_\xi \mathcal{H} \tag{35}$$

for every filter $\mathcal{H}$.

Proposition 48. *A pseudotopology ξ is θ-regular if and only if (35) holds for every family $\mathcal{H}$.*

Proof. Indeed, $x \in \operatorname{adh}_\xi \mathcal{V}_\theta(\mathcal{H})$ whenever there exists an ultrafilter $\mathcal{F}\#\mathcal{V}_\theta(\mathcal{H})$ such that $x \in \lim_\xi \mathcal{F}$. By (32) $(\operatorname{adh}_\theta^\natural \mathcal{F})\#\mathcal{H}$ and by (33) $x \in \lim_\xi(\operatorname{adh}_\theta^\natural \mathcal{F})$, hence $x \in \operatorname{adh}_\xi \mathcal{H}$.

Conversely, if (35) holds, ξ is a pseudotopology and $x \notin \lim_\xi \operatorname{adh}_\theta^\natural \mathcal{F}$, then by (21) $x \notin \operatorname{adh}_\xi \mathcal{H}$ for some ultrafilter $\mathcal{H}\# \operatorname{adh}_\theta^\natural \mathcal{F}$, and thus by (35) $x \notin \operatorname{adh}_\xi \mathcal{V}_\theta(\mathcal{H})$. By virtue of (32) $\mathcal{F}\#\mathcal{V}_\theta(\mathcal{H})$ and thus $x \notin \lim_\xi \mathcal{F}$. ∎

9.5. **Interactions between regularity and topologicity.** As we have seen, regularity and diagonality are in some sense inverse properties. The minimality (under some uniqueness assumptions) of compact convergences in the complete lattice of convergences makes regularity and diagonality coincide. A convergence is *normal* if for any two disjoint closed sets A_0, A_1 there exist disjoint open sets O_0, O_1 such that $A_0 \subset O_0$ and $A_1 \subset O_1$.

Theorem 49. *Each compact topologically regular convergence is normal.*

Proof. Suppose that a convergence on X is not normal: there exist disjoint closed sets A_0, A_1 such that $\mathcal{N}(A_0)\#\mathcal{N}(A_1)$. Let $\mathcal{U}$ be an ultrafilter finer than $\mathcal{N}(A_0) \vee \mathcal{N}(A_1)$. By compactness, there exists $x \in \lim \mathcal{U}$. If $x \notin A_0$, then A_0^c is an open set that contains x, hence by topological regularity there exists $U \in \mathcal{U}$ such that $\operatorname{cl} U \cap A_0 = \varnothing$, that is, $U \notin \mathcal{N}^\#(A_0)$, which yields a contradiction. For the same reason, $x \in A_1$, and thus $A_0 \cap A_1 \neq \varnothing$ contrary to the assumption. ∎

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[63] Indeed, because ξ is of countable character, $x \in \lim_\xi \int_{(n)} (x_{n,k})_k$ implies that there exists a countably based filter $\mathcal{E}$ such that $x \in \lim_\xi \mathcal{E}$ and $\mathcal{E} \subset \int_{(n)} (x_{n,k})_k$. Then, if needed, we can pick a subsequence of $(x_n)_n$ and for each n present in this subsequence, a subsequence of $(x_{n,k})_k$ so that the filter of the type (34) restricted to these subsequences, converges to x.

A convergence ξ is a *quasi-topology* if $P\xi = T\xi$. By Proposition 22, this means that adh_ξ is idempotent.

Theorem 50. *Each Hausdorff regular compact convergence is a quasi-topology.*

Proof. Let $x \in \mathrm{adh}^2 A$. By definition, there exists an ultrafilter $\mathcal{U}$ such that $\{x\} = \lim \mathcal{U}$ (by Hausdorffness) and $\mathcal{U} \# \mathrm{adh}\, A$, hence $\mathcal{V}(\mathcal{U}) \# A$ by virtue of (32). Therefore there is an ultrafilter $\mathcal{W}$ on A such that $\mathcal{W} \# \mathcal{V}(\mathcal{U})$, equivalently $\mathcal{U} \# \mathrm{adh}^\natural \mathcal{W}$. Therefore, by regularity and by compactness,

$$\{x\} = \lim \mathcal{U} \supset \lim \mathrm{adh}^\natural \mathcal{W} = \lim \mathcal{W} \neq \varnothing,$$

which proves that $x \in \mathrm{adh}\, A$. ∎

Every regular quasi-topology is topologically regular. Indeed, the regularity of a convergence ξ is defined with the aid of set adherence adh_ξ, which depends only on $P\xi$; if ξ is a quasi-topology, $P\xi = T\xi$, thus the regularity of ξ amounts to the topological regularity of ξ. Therefore by Theorems 50 and 49, each Hausdorff regular compact convergence is normal, and in view of Proposition 42, we get [33][35][25]

Corollary 51. *Each Hausdorff regular compact pseudotopology is a topology.*

Unlike for topologies, a compact Hausdorff convergence need not be regular, because there exist non topological compact Hausdorff pseudotopologies (see e.g., the Kuratowski convergence in Subsection 15.3 and Proposition 93).

Diagonality and regularity are antithetic properties. Therefore, in case of Hausdorff compact pseudotopologies, each of them entails the other. This is due to the minimality of Hausdorff compact pseudotopologies in the class of Hausdorff pseudotopologies. Hence, the following extension of a classical result can be considered as a dual of Corollary 51.

Theorem 52. *Each Hausdorff pretopologically diagonal compact pseudotopology is regular.*

Proof. If not, then there is x and a filter $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F} \setminus \lim_\xi \mathrm{adh}_\xi^\natural \mathcal{F}$. Because ξ is Hausdorff, $\{x\} = \lim_\xi \mathcal{W}$ for every $\mathcal{W} \in \beta\mathcal{F}$, so that $\mathrm{adh}_\xi \mathcal{F} = \{x\}$. Let $\mathcal{U}$ be an ultrafilter such that $\mathcal{U} \# \mathrm{adh}_\xi^\natural \mathcal{F}$ and $x \notin \lim_\xi \mathcal{U}$. Consequently, $\mathcal{V}_\xi(\mathcal{U}) \# \mathcal{F}$. As ξ is compact, there is $y \in \lim_\xi \mathcal{U}$, thus $y \in \lim_\xi \mathcal{V}_\xi(\mathcal{U})$, because ξ is pretopologically diagonal, hence $y \in \mathrm{adh}_\xi \mathcal{F}$. Therefore $x = y$, which is a contradiction. ∎

By virtue of Corollary 51, under the hypotheses of Theorem 52, the resulting convergence is also topological.

10. Filter-determined convergences

Let $\mathbb{H}$ be a class of filters. A convergence ξ is said to be $\mathbb{H}$-*based* if $x \in \lim_\xi \mathcal{F}$ implies the existence of a filter $\mathcal{H} \in \mathbb{H}$ such that $x \in \lim_\xi \mathcal{H}$ and $\mathcal{H} \subset \mathcal{F}$.

Example 53. *If $\mathbb{H}$ is the class of sequential filters, then $\mathbb{H}$-based convergences are precisely sequentially based convergences.*

Example 54. *If $\mathbb{H}$ is the class of countably based filters, then $\mathbb{H}$-based convergences are convergences of* countable character*.*

Example 55. *A filter on a convergence space is said to be* locally compact *if it contains a compact set. The set $\mathbb{H}(X)$ is that of the filters on X, which are of the class $\mathbb{H}$. If $\mathbb{H}(X)$ is the set of locally compact filters on X, then $\mathbb{H}$-based convergences are called* locally compact*.*

Class $\mathbb{H}$ of filters	$\mathbb{H}$-based convergences
Sequential	sequentially based
Countably based	of countable character
Locally compact	locally compact

11. Categories of convergence spaces

We have seen that the sets of *topologies,pretopologies* and *pseudotopologies* on a given underlying set, is stable for arbitrary suprema, and contains the least convergence, that is, the chaotic topology. The initial convergence of a topology (respectively, a pretopology) is a topology (respectively, a pretopology).

The set of *sequentially based* convergences on a given underlying set, is stable for arbitrary infima, and contains the greatest convergence, that is, the discrete topology. The final convergence of a sequentially based convergence is a sequentially based convergence.

Such situations are well-known in category theory.[64] At this point a use of category theory is not only illuminating, but essential for the efficiency of our investigation. I keep this use at a minimal level, because, on one hand, a comfortable employment of category theory requires itself a patient apprenticeship, and on the other, convergences form quite a simple category. The book [1] of J. Adámek, H. Herrlich, and E. Strecker is a basic reference in category theory.

11.1. Abstract and concrete categories.

Objects and *morphisms* are primitive notions. A *category* **C** consists of a class of objects and of a class

[64] They correspond to concretely reflective and coreflective subcategories of a topological construct.

of morphisms such that for every couple (ξ, τ) of objects of $\mathbf{C}$, there exists a set $\hom_{\mathbf{C}}(\xi, \tau)$ of morphisms

$$f : \xi \to \tau,$$

so that for $f \in \hom_{\mathbf{C}}(\xi, \tau)$ and $g \in \hom_{\mathbf{C}}(\tau, \theta)$, the *composition* $g \circ f \in \hom_{\mathbf{C}}(\xi, \theta)$, the composition is associative, and for each object ξ, the set $\hom_{\mathbf{C}}(\xi, \xi)$ contains a neutral element 1_ξ of the composition (called *identity*).

A map F from the class of morphisms of a category $\mathbf{C}$ to the class of morphisms of a category $\mathbf{D}$ is called a *functor* (or, *covariant functor*) if

$$F(g \circ f) = Fg \circ Ff,\ F(1_\xi) = 1_{F\xi}$$

for every $\xi \in \mathbf{C}$. Therefore each functor F induces a map on objects, denoted also by F, to the effect that $F(\xi) = F(1_\xi)$.

Example 56. *In the category of sets* $\mathbf{Set}$*, the objects are sets, and the morphisms are maps. More precisely, for* $X, Y \in \mathbf{Set}$*, one defines the set of morphisms* $\hom_{\mathbf{Set}}(X, Y) = Y^X$ *(the set of all maps from* X *to* Y*).*

A category $\mathbf{C}$ is called *concrete* (over $\mathbf{Set}$) if there exists a functor $|\cdot| : \hom_{\mathbf{C}}(\xi, \tau) \to \hom_{\mathbf{Set}}(|\xi|, |\tau|)$, which is *faithful* (which, in the language of category theory, means *injective*). Therefore we can identify

$$|\hom_{\mathbf{C}}(\xi, \tau)| \subset \hom_{\mathbf{Set}}(|\xi|, |\tau|).$$

Consequently if $\mathbf{C}$ is concrete, then $|\xi|$ is a set for every $\mathbf{C}$-object ξ (the *underlying set* of ξ), and for every morphism $\varphi \in \hom_{\mathbf{C}}(\xi, \tau)$, the image $|\varphi|$ is a map from the set $|\xi|$ to the set $|\tau|$.

Example 57. *In the category of convergences* $\mathbf{C} = \mathbf{Conv}$*, the objects are convergences, and the morphisms are continuous maps. For every convergence* ξ *there is a unique set* $|\xi|$ *on which the convergence is defined, and every continuous map* $\varphi \in C(\xi, \tau)$ *defines* $|\varphi| : |\xi| \to |\tau|$*. On the other hand, if* $f : X \to Y$*, and* ξ, τ *are convergences respectively on* X *and* Y*, then there is at most one morphism* $\varphi \in \hom_{\mathbf{C}}(\xi, \tau)$ *such that* $|\varphi| = f$*. In other words,* $\varphi \in \hom_{\mathbf{C}}(\xi, \tau)$ *if and only if* $|\varphi| \in C(\xi, \tau)$*.*

We have seen that there always exist initial and final convergences associated with families of maps. In other words, the category of convergence spaces always admits initial and final objects. Such categories are called *topological constructs.*

The fiber of a set X of a concrete category $\mathbf{C}$ is $\{\xi \in \mathbf{C} : |\xi| = X\}$. Each concrete category induces a partial order on its fibers, that is, if $|\xi| = |\theta|$ then $\xi \geq \theta$ whenever there is $\varphi \in \hom_{\mathbf{C}}(\xi, \theta)$ such that $|\varphi| = i$, the identity map on X. If $\mathbf{C}$ is a topological construct, then each fiber endowed with this partial order constitutes a complete lattice.

In a topological construct $\mathbf{C}$ a map $f \in \hom_{\mathbf{Set}}(|\xi|, |\tau|)$ is a morphism of $\mathbf{C}$ whenever $f\xi \geq \tau$ (equivalently, $\xi \geq f^- \tau$). Let $\mathbf{C}$ be a topological construct. A functor $F : \mathbf{C} \to \mathbf{C}$ is *concrete* if $|F\xi| = |\xi|$ for every object ξ of $\mathbf{C}$. The following is a special case of a result in [21].

Theorem 58. *A map H on the class of convergences uniquely determines a concrete functor if and only if*

$$|H\xi| = |\xi|, \tag{36}$$

$$\zeta \geq \xi \Rightarrow H\zeta \geq H\xi, \tag{37}$$

$$f(H\xi) \geq H(f\xi) \tag{38}$$

for every ζ, ξ and every map f from $|\xi|$.

Proof. Let H be the restriction to objects of $\mathbf{C}$ of a concrete functor. If $\zeta \geq \xi$, that is, if the identity map i belongs to $C(\xi, \tau)$ then $i \in C(H\xi, H\tau)$, equivalently $H\xi \geq H\tau$. As $f \in C(\xi, f\xi)$, also $f \in C(H\xi, H(f\xi))$, which means that $f(H\xi) \geq H(f\xi)$.

Conversely, if $f \in C(\xi, \tau)$ then $f\xi \geq \tau$, hence $f(H\xi) \geq H(f\xi) \geq H\tau$ by (38) and (37), hence $f \in C(H\xi, H\tau)$. Therefore if $\varphi \in \hom_{\mathbf{C}}(\xi, \tau)$ then $\mathfrak{H}\varphi$ is the unique morphism in $\hom_{\mathbf{C}}(H\xi, H\tau)$ such that $|\varphi| = |\mathfrak{H}\varphi|$. If $\varphi \in \hom_{\mathbf{C}}(\xi, \tau)$ and $\psi \in \hom_{\mathbf{C}}(\tau, \theta)$ then $\psi \circ \varphi \in \hom_{\mathbf{C}}(\xi, \theta)$. Then $|\mathfrak{H}(\psi \circ \varphi)| = |\psi \circ \varphi| = |\psi| \circ |\varphi| = |\mathfrak{H}\psi| \circ |\mathfrak{H}\varphi|$, thus by construction $\mathfrak{H}(\psi \circ \varphi) = \mathfrak{H}\psi \circ \mathfrak{H}\varphi$. Also $|\mathfrak{H}(i_\xi)| = |i_\xi| = |i_{H\xi}|$, which shows that $\mathfrak{H}(i_\xi) = i_{H\xi}$. ∎

Notice that (36),(37) and (38) are equivalent to (36),(37) and

$$H(g^-\tau) \geq g^-(H\tau) \tag{39}$$

for every map f to $|\tau|$[65]).

11.2. **Subcategories of convergence spaces.** A category $\mathbf{D}$ is a subcategory of a category $\mathbf{C}$ if the objects and morphisms of $\mathbf{D}$ are also, respectively, objects and morphisms of $\mathbf{C}$.

We have seen that the category of convergence spaces is concrete. A class of convergence spaces is a (concretely) *reflective* subcategory (of the category of convergence spaces) if (on every fiber of $|\cdot|$) it

(1) is stable for arbitrary suprema,
(2) contains the least convergence,
(3) is preserved by initial convergences.

A class of convergence spaces (with continuous maps as morphisms) is a (concretely) *coreflective* subcategory (of the category of convergence spaces) if (on every fiber of $|\cdot|$) it

(1) is stable for arbitrary infima,
(2) contains the greatest convergence,
(3) is preserved by final convergences.

[65]Suppose (38) and let $\tau = f\xi$ to get $H\xi \geq H(f^-f\xi) \geq f^-(H(f\xi))$. Apply f to both the sides of the inequality to the effect that $f(H\xi) \geq ff^-(H(f\xi)) \geq H(f\xi)$. Conversely, if (39) holds, then set $\xi = f^-\tau$ and apply f^- to obtain $H(f^-\tau) \geq f^-f(H(f^-\tau)) \geq f^-H(ff^-\tau) \geq f^-(H\tau)$.

If $\mathbf{H}$ stands for the objects of a (concretely) reflective subcategory, then there exists a map H associating with each convergence ξ (on X) the finest convergence $H\xi$ (on X) from $\mathbf{H}$, which is coarser than ξ. The map H is the corresponding *reflector* on $\operatorname{fix} H = \mathbf{H}$, where $\operatorname{fix} H = \{\xi : H\xi = \xi\}$. The *coreflector* is defined analogously.

As in the sequel I will not consider non-concretely reflective and coreflective subcategories, the word *concretely* will be always omitted.

Therefore topologies and pretopologies are reflective subcategories of the category of convergence spaces. The *topologizer* T and the *pretopologizer* P are the corresponding *reflectors*. Similarly, *pseudotopologies* and *paratopologies* are reflective subcategories of the category of convergence spaces. The *pseudotopologizer* S and the *paratopologizer* P_ω are the corresponding reflectors. If W is a functor, then we say that a convergence ξ is W*-regular* if it is $W\xi$-regular. It can be easily checked that W-regular convergences form a reflective subcategory for every functor W.

Sequentially based convergences form a coreflective subcategory of the category of convergence spaces. The sequential modification Seq is the corresponding *coreflector*.

Proposition 59. [10] *If $\mathbb{H}$ is a class of filters (possibly depending on convergence) that includes all the principal ultrafilters, and such that $\mathbb{H}(\xi) \subset \mathbb{H}(\theta)$ if $\xi \geq \theta$ and such that $\mathcal{H} \in \mathbb{H}(\xi)$ implies that $f(\mathcal{H}) \in \mathbb{H}(f\xi)$, then the class of $\mathbb{H}$-based convergences is coreflective and*

$$\lim\nolimits_{\mathrm{B}_{\mathbb{H}}\xi} \mathcal{F} = \bigcup_{\mathbb{H}(\xi) \ni \mathcal{G} \subset \mathcal{F}} \lim\nolimits_\xi \mathcal{G}$$

is the coreflector.

Example 60. *Because the class of countably based filters does not depend on convergence, and the image of a countably based filter is countably based, the convergences of countable character form a coreflective subcategory of convergences. The corresponding coreflector will be denoted by* First.

Example 61. *If $\xi \geq \theta$, then each ξ-compact set is θ-compact, and thus each locally compact filter in ξ is locally compact in θ. As the continuous image of a compact set is compact, the continuous image of a locally compact filter is locally compact. Therefore, by Proposition 59, the locally compact convergences form a coreflective subcategory of convergences. The corresponding coreflector will be denoted by K.*

Reflectors and coreflectors are functors.

Theorem 62. [20] *For every functor H, the class of all convergences ξ such that*

$$\xi \leq H\xi \tag{40}$$

is reflective, and

$$H\xi \leq \xi \tag{41}$$

is coreflective.

Proof. As H is order-preserving, $o \leq Ho$. If (40) holds for each $\xi \in \Xi$, then $\bigvee \Xi \leq \bigvee_{\xi \in \Xi} H\xi$, and the latter is always less than $H(\bigvee \Xi)$. Finally (40) implies $f^{-}\xi \leq f^{-}(H\xi)$, which is less than $H(f^{-}\xi)$ by (39), showing that (40) is preserved by initial convergences. This proves that the class (40) is reflective. The coreflectivity of (41) can be proved analogously. ∎

A category-theory concepts of initial and final density are extremely useful in this quest. A class $\mathbb{D}$ of convergences is called *initially dense* in a subcategory $\mathbf{M}$ (of convergences) if for every (object) τ of $\mathbf{M}$ there exists a collection of maps $\{f_\iota : \iota \in I\}$ such that

$$\tau = \bigvee\nolimits_{\iota \in I} f_\iota^{-} \zeta_\iota$$

with $\zeta_\iota \in \mathbb{D}$ for each $\iota \in I$. A class $\mathbb{D}$ of convergences is called *finally dense* in a subcategory $\mathbf{M}$ if for every (object) ξ of $\mathbf{M}$ there exists a family of maps $\{f_\iota : \iota \in I\}$ such that

$$\xi = \bigwedge\nolimits_{i \in I} f_i \tau_i.$$

with $\tau_\iota \in \mathbb{D}$ for each $\iota \in I$.

12. Functorial inequalities

Some important classes of classical topologies can be characterized with the aid of convergence inequalities of the type

$$\xi \geq JE\xi, \tag{42}$$

where J is a reflector and E is a coreflector [15][10]. We shall call them *JE-convergences.* By Theorem 62 , classes of JE-convergences form a coreflective subcategory of the category of convergence spaces. Restricted to topologies, they form a coreflective subcategory of the category of topological spaces provided that $J \geq T$.

Example 63. *A topology is called* sequential *if each* sequentially closed[66] *set is closed. A topology ξ is sequential if and only if (42) holds with $J = T$ (the topologizer) and $E = \mathrm{Seq}$ (the sequential modifier). Indeed (for an arbitrary convergence ξ) $\xi \geq T\,\mathrm{Seq}\,\xi$ amounts to $T\xi \geq T\,\mathrm{Seq}\,\xi \geq T\xi$ (because T is order-preserving and $\mathrm{Seq}\,\xi \geq \xi$), which means that if a set is closed for $\mathrm{Seq}\,\xi$ then it is closed for ξ.*

Example 64. *A topology is called* Fréchet *if $x \in \mathrm{cl}_\xi H$ implies the existence of a sequential filter $\mathcal{E}$ on H that converges to x in ξ. It turns out that a topology ξ is Fréchet if and only if (42) holds with $J = P$ (the pretopologizer) and $E = \mathrm{Seq}$ (the sequential modifier). In fact, $\xi \geq P\,\mathrm{Seq}\,\xi$ for a topology ξ whenever $\mathrm{cl}_\xi H = \mathrm{adh}_\xi H \subset \mathrm{adh}_{\mathrm{Seq}\,\xi} H$ for every H, which means that*

[66] A set A is *sequentially closed* if the limit of every sequence with terms in A, is in A.

$x \in \mathrm{cl}_\xi H$ implies the existence of a sequential filter $\mathcal{E}$ on H that converges to x in ξ. Conversely, the latter statement implies that for every filter $\mathcal{F}$,

$$\bigcap_{H \in \mathcal{F}^\#} \mathrm{adh}_\xi H \subset \bigcap_{H \in \mathcal{F}^\#} \mathrm{adh}_{\mathrm{Seq}\,\xi} H,$$

which, in view of Theorem 35 (27) means that $\lim_{P\xi} \mathcal{F} \subset \lim_{P\,\mathrm{Seq}\,\xi} \mathcal{F}$, and thus $\xi \geq P\xi \geq P\,\mathrm{Seq}\,\xi$.

Example 65. *A topology ξ is a* k-topology *if $H \cap C$ is closed in $\xi \vee C$ (the restriction of ξ to C) for every ξ-compact set C, then H is ξ-closed. It turns out that a topology ξ is a k-topology if and only (42) holds with $J = T$ (the topologizer) and $E = K$ (compact localization), that is, $\xi \geq TK\xi$.*

JE-convergences for various special reflectors J and coreflectors E, are written in terms of standard classes of topologies, the definitions of which are furnished in the footnote below.

J/E		Seq	First	K
I	identity	sequentially based	countable character	locally compact
S	pseudo$\sim$	sequentially based	bisequential	locally compact
P_ω	para$\sim$	strongly Fréchet	strongly Fréchet	strongly k'
P	pre$\sim$	Fréchet	Fréchet	k'
T	$\sim$	sequential	sequential	k

In the second column of the table, the sign $\sim$ replaces *topologizer.* See [10] for a more complete table.[67]

13. Quotient maps

The objects of every reflective subcategory of convergence spaces can be represented as $\mathrm{fix}\,J$ (the collection of fixed points of a reflector J). If $f : X \to Y$ is a map, and $\xi \in \Xi$ is a convergence on X, then there exists on Y the finest convergence from $\mathrm{fix}\,J$, for which f is continuous.[68] This convergence is called the *J-quotient* of ξ by f. Actually, the J-quotient of ξ

[67] A topology ξ is called *strongly Fréchet* if for every descending sequence $(H_n)_n$ of sets such that $x \in \bigcap_{n<\infty} \mathrm{cl}_\xi H_n$ implies the existence of a sequence $(x_n)_n$ such that $x \in \lim_\xi (x_n)_n$ and $x_n \in H_n$ for each $n < \infty$. This is equivalent to the condition $\mathrm{adh}_\xi \mathcal{H} \subset \mathrm{adh}_{\mathrm{First}\,\xi} \mathcal{H}$ for each countably based filter $\mathcal{H}$.

A topology ξ is called *bisequential* if $\mathrm{adh}_\xi \mathcal{H} \subset \mathrm{adh}_{\mathrm{First}\,\xi} \mathcal{H}$ for every filter $\mathcal{H}$.

A topology ξ is called a *k'-topology* if $x \in \mathrm{cl}_\xi H$ implies that there is compact set C such that $x \in \mathrm{cl}_\xi (H \cap C)$.

A topology ξ is called a *strongly k'-topology* if $x \in \bigcap_{n<\infty} \mathrm{cl}_\xi H_n$ for a descending sequence $(H_n)_n$ of sets, then there exists a compact set C such that $x \in \mathrm{cl}_\xi (H_n \cap C)$ for each $n < \infty$.

[68] Indeed, the set Ξ_f of all the Ξ-convergences on Y for which f is continuous is non-empty, because it contains the chaotic topology. As the supremum of Ξ_f belongs to Ξ_f, there exists the finest Ξ-convergence for which f is continuous.

by f is equal to $J(f\xi)$. Therefore, we say that a map f from a convergence space (X,ξ) to a convergence space (Y,τ) is a J-*quotient* if

$$\tau \geq J(f\xi). \tag{43}$$

It turns out that many types of classical maps in topology, like quotient maps, hereditarily quotient maps, countably biquotient maps, biquotient maps, almost open maps are J-quotient maps with respect to a reflective subcategory fix J of convergence spaces. You will find classical definitions of quotient, hereditarily quotient and biquotient maps in the examples below.

Example 66. *A map between topological spaces $f : X \to Y$ is* quotient *if a subset F of Y is closed whenever $f^{-}(F)$ is closed.*

Example 67. *A map between topological spaces $f : X \to Y$ is* hereditarily quotient *if $y \in \operatorname{cl} B$ implies that* $\operatorname{cl} f^{-}(B) \cap f^{-}(y) \neq \varnothing$ *for each subset B of Y.*

Example 68. *A map between topological spaces $f : X \to Y$ is* biquotient *if $y \in \operatorname{adh} \mathcal{H}$ implies that* $\operatorname{adh} f^{-}(\mathcal{H}) \cap f^{-}(y) \neq \varnothing$ *for each filter $\mathcal{H}$ on Y.*

Theorem 69. [10] *Let $\mathbb{J}$ be a class of filters and let $J = \mathrm{A}_{\mathbb{J}}$ be the reflector on the subcategory of convergences, which are adherence determined by $\mathbb{J}$. Then a map $f : (X,\xi) \to (Y,\tau)$ is a J-quotient if and only if for every filter $\mathcal{J}$,*

$$\mathcal{J} \in \mathbb{J}, y \in \operatorname{adh}_\tau \mathcal{J} \Longrightarrow f^{-}(y) \cap \operatorname{adh}_\xi f^{-}(\mathcal{J}) \neq \varnothing. \tag{44}$$

Proof. Notice that (44) amounts to

$$\operatorname{adh}_\tau \mathcal{J} \subset f(\operatorname{adh}_\xi f^{-}(\mathcal{J}))$$

for every filter $\mathcal{J} \in \mathbb{J}$, and as $\operatorname{adh}_{f\xi} \mathcal{J} = f(\operatorname{adh}_\xi f^{-}(\mathcal{J}))$, we conclude that

$$\lim\nolimits_\tau \mathcal{F} \subset \bigcap_{\mathbb{J} \ni \mathcal{J} \# \mathcal{F}} \operatorname{adh}_\tau \mathcal{J} \subset \bigcap_{\mathbb{J} \ni \mathcal{J} \# \mathcal{F}} \operatorname{adh}_{f\xi} \mathcal{J} = \lim\nolimits_{\mathrm{A}_{\mathbb{J}} f\xi} \mathcal{F},$$

which is equivalent to $\tau \geq J(f\xi)$. ∎

The following observation is an initial step in the quest for preservation of various classes of spaces by variants of quotient maps.

Proposition 70. [10] *Let J be a reflector and E a coreflector. Then ξ is a JE-convergence if and only if the identity $i : (X, E\xi) \to (X,\xi)$ is J-quotient.*

Proof. By definition, i is J-quotient from $E\xi$ to ξ whenever $\xi \geq J(i(E\xi)) = JE\xi$, that is, whenever ξ is a JE-convergence. ∎

Theorem 71. [10] *The image of a JE-convergence by a continuous J-quotient is a JE-convergence.*

Proof. If $\xi \geq JE\xi$ and $\tau \geq J(f\xi)$ then $f\xi \geq f(JE\xi) \geq JE(f\xi)$ by (38). Therefore $\tau \geq J(f\xi) \geq JE(f\xi) \geq JE\tau$ because of the continuity. ∎

Theorem 71 is illustrated below for various fundamental variants of quotient maps (and which are quotient with respect to reflective classes of convergence spaces) and for most known local properties of topologies.[69]

	J	Seq	First	K
almost open	I	sequentially based	countable character	locally compact
biquotient	S	sequentially based	bisequential	locally compact
countably biquotient	P_ω	Fréchet	strongly Fréchet	strongly k'
hereditarily quotient	P	strongly Fréchet	Fréchet	k'
quotient	T	sequential	sequential	k

The reflector with respect to which the quotient is considered is in the second column, the classical name of the type of quotient is in the first column. The rows show which properties are preserved by which types of quotient maps.

If $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ then the *product map* $f_0 \times f_1 : X_0 \times X_1 \to Y_0 \times Y_1$ is defined by

$$(f_0 \times f_1)(x_0, x_1) = (f_0(x_0), f_1(x_1)).$$

The problem when the product of two quotient map is quotient has been studied by many authors. Special cases of it (when one of the maps is the identity) are answered by[70]

Theorem 72 (Whitehead-Michael). *A regular (Hausdorff) topological space is locally compact if and only if the product of its identity map with every quotient map is quotient.*

Theorem 73 (Michael). *A regular (Hausdorff) topological space is locally countably compact if and only if the product of its identity map with every quotient map from a sequential topological space is quotient.*

It turns out that there exists a simple convergence-theoretic scheme, which enables one to answer this question. The definition of J-quotient map (43) can be extended as follows. If M is a functor, then we say that a map f from a convergence space (X, ξ) to a convergence space (Y, τ) is an *M-quotient* if $\tau \geq M(f\xi)$.

Remark 74. *If $M = JE$ where J is a reflector and E a coreflector, then a map f, which is a JE-quotient from ξ to τ, is J-quotient from $JE\xi$ to τ.*[71]

We denote by $i_X : X \to X$ the *identity map* on X.

Proposition 75. [19] *Let M be a functor and L a reflector. For every convergence τ*

$$\xi \times M\tau \geq L(\xi \times \tau) \tag{45}$$

[69] A more extensive table can be found in [10].

[70] [23, Theorem 3.3.17],[37, Theorem 2.1 and 4.1]

[71] In fact, by (39) $\tau \geq JJE(f\xi) \geq J(f(JE\xi))$.

if and only if $i_{|\xi|} \times f$ *is an* L*-quotient for every* M*-quotient* f.

Proof. Let $f : \tau_0 \to \tau_1$ be an M-quotient, that is, $\tau_1 \geq M(f\tau_0)$. Then $\xi \times \tau_1 \geq \xi \times M(f\tau_0) \geq L(\xi \times f\tau_0) = L(i \times f)(\xi \times \tau_0)$. Conversely, if $f = i_{|\tau|}$, then $M\tau = M(i_{|\tau|}\tau)$ so that $i_{|\tau|}$ is an M-quotient. By (45) $i_{|\xi|} \times i_{|\tau|}$ is an L-quotient: $\xi \times M\tau \geq L(\xi \times \tau)$. ∎

In Section 17 we will see that Theorems 72 and 73 are special cases of Proposition 75.

14. Power convergences

One of the principal reasons for the occurrence of non-topological convergences was the fact that in general there exists no coarsest topology on the *space of continuous maps* (from one topological space to another) making the evaluation map continuous. As we shall see, closed subsets of a topological space can be identified with continuous maps (valued in the Sierpiński topology). Therefore, in general, there is no coarsest topology on a *hyperspace* (space of closed subsets of a topological space) making the natural evaluation continuous. These facts were at the origin of the introduction of pseudotopologies by Gustave Choquet in [8] in 1947-1948.

The space of continuous maps from a convergence ξ to a convergence σ is denoted by $C(\xi, \sigma)$. If M is a functor, then

$$C(\xi, \sigma) \subset C(M\xi, M\sigma) \tag{46}$$

by virtue of (39) (or equivalently of (38)).

Lemma 76. *If* J *is a reflector, and* ζ *and* ξ *are convergences, then*

$$\zeta \geq J\xi \Leftrightarrow \forall_{\sigma \in \operatorname{fix} J}\, C(\xi, \sigma) \subset C(\zeta, \sigma).$$

Proof. Let ζ, ξ be convergences on X. The inequality $\zeta \geq J\xi$ implies $C(J\xi, \sigma) \subset C(\zeta, \sigma)$ for every convergence σ, and if $\sigma = J\sigma$ then $C(\xi, \sigma) \subset C(J\xi, \sigma)$ by (46). Conversely, suppose that $C(\xi, \sigma) \subset C(\zeta, \sigma)$ for each $\sigma \in \operatorname{fix} J$, in particular $C(\xi, J\xi) \subset C(\zeta, J\xi)$. As $\xi \geq J\xi$, the identity i on X belongs to $C(\xi, J\xi)$, hence $i \in C(\zeta, J\xi)$, that is, $\zeta \geq J\xi$. ∎

Proposition 77. *Let* $\mathbb{D}$ *be initially dense in* $\operatorname{fix} J$. *If* $C(\xi, \sigma) \subset C(\zeta, \sigma)$ *for each* $\sigma \in \mathbb{D}$, *then this holds for each* $\sigma \in \operatorname{fix} J$.

Proof. Let $f \in C(\xi, \sigma)$ for some $\sigma \in \operatorname{fix} J$. By initial density there exists a class of maps $\{g_\iota : \iota \in I\}$ such that $\sigma = \bigvee_{\iota \in I} g_\iota^- \rho_\iota$ with $\rho_\iota \in \mathbb{D}$ for each $\iota \in I$. Therefore $f\xi \geq g_\iota^- \rho_\iota$ for each $\iota \in I$, that is, $g_\iota \circ f \in C(\xi, \rho_\iota) \subset C(\zeta, \rho_\iota)$, or equivalently $f\zeta \geq \bigvee_{\iota \in I} g_\iota^- \rho_\iota = \sigma$, which means that $f \in C(\zeta, \sigma)$. ∎

In Subsection 4.5 the *power convergence* (or the *continuous convergence*) $[\xi, \sigma]$ (of ξ with respect to σ) was defined as the coarsest among the convergences τ on $C(\xi, \sigma)$, for which the *evaluation* e is continuous from $\xi \times \tau$ to σ, that is, the coarsest among the convergences θ for which

$$\xi \times \theta \geq e^- \sigma, \tag{47}$$

where $e^-\sigma$ is the initial convergence of σ by e. The convergence σ in the definition above is called the *coupling convergence.*

The *exponential map* ^{t}

$$({}^t f)(y)(x) = f(x,y) \tag{48}$$

is a bijection of the set $Z^{X\times Y}$ (of all maps from $X \times Y$ to Z) onto the set $(Z^X)^Y$ (of all the maps from Y to the set of maps Z^X). Its inverse associates by (48) with each $g \in (Z^X)^Y$ an element $\hat{g}$ of $Z^{X\times Y}$. If now ξ, τ and σ are convergences on X, Y and Z respectively, then an immediate consequence of the definition of power convergence is that

$$C(\xi \times \tau, \sigma) \leftrightarrow C(\tau, [\xi, \sigma]), \tag{49}$$

where $\leftrightarrow$ denotes the restriction of that bijection. Moreover,

Theorem 78. *For convergences ξ, τ and σ, the exponential is a homeomorphism between the power convergences:*

$$[\tau, [\xi, \sigma]] \approx [\xi \times \tau, \sigma]. \tag{50}$$

Proof. A map $h \in Z^{X\times Y}$ belongs to $\lim_{[\xi\times\tau,\sigma]} \mathcal{H}$ if and only if for every $x \in \lim_\xi \mathcal{F}$ and for every $y \in \lim_\tau \mathcal{G}$, we have $h(x,y) \in \lim_\sigma \langle \mathcal{F} \times \mathcal{G}, \mathcal{H}\rangle$. As $\langle \mathcal{F} \times \mathcal{G}, \mathcal{H}\rangle = \langle \mathcal{F}, \langle \mathcal{G}, {}^t\mathcal{H}\rangle\rangle$, the preceding formula is equivalent to ${}^t h(y)(x) \in \lim_\sigma \langle \mathcal{F}, \langle \mathcal{G}, {}^t\mathcal{H}\rangle\rangle$, that is, to ${}^t h(y) \in \lim_{[\xi,\sigma]} \langle \mathcal{G}, {}^t\mathcal{H}\rangle$, hence equivalent to ${}^t h \in \lim_{[\tau,[\xi,\sigma]]} {}^t\mathcal{H}$. ∎

If $f : X \to Y$ and Z is a fixed set, then $f^* : Z^Y \to Z^X$ is defined by $f^*(h) = h \circ f$, that is,

$$\langle x, f^*(h)\rangle = \langle f(x), h\rangle$$

for every $x \in X$ and $h : Y \to Z$. If $f \in C(\xi, \tau)$ then $f^*(C(\tau, \sigma)) \subset C(\xi, \sigma)$.[72] Therefore for each $f \in C(\xi, \tau)$ we will see f^* as restricted to $C(\tau, \sigma)$, that is,

$$f^* : C(\tau, \sigma) \to C(\xi, \sigma).$$

Theorem 79. *If $f \in C(\xi, \tau)$, then f^* is continuous from $[\tau, \sigma]$ to $[\xi, \sigma]$.*

Proof. Let $h \in \lim_{[\tau,\sigma]} \mathcal{H}$. In order to prove that $h \in \lim_{(f^*)^-[\xi,\sigma]} \mathcal{H}$, or equivalently, $f^*(h) \in \lim_{[\xi,\sigma]} f^*(\mathcal{H})$, one must establish that $\langle x, f^*(h)\rangle \in \lim_\sigma \langle \mathcal{F}, f^*(\mathcal{H})\rangle$, that is,

$$\langle f(x), h\rangle \in \lim_\sigma \langle f(\mathcal{F}), \mathcal{H}\rangle \tag{51}$$

for every $x \in |\xi|$, and each filter $\mathcal{F}$ such that $x \in \lim_\sigma \mathcal{F}$. Because f is continuous $f(x) \in \lim_\tau f(\mathcal{F})$, and since by assumption, $h \in \lim_{[\tau,\sigma]} \mathcal{H}$, (51) holds. ∎

[72] In fact, if $h \in C(\tau, \sigma)$ then $f^*(h) = h \circ f \in C(\xi, \sigma)$.

In particular, if ξ and τ are convergences on a set X such that $\xi \geq \tau$, then the identity map i belongs to $C(\xi, \tau)$, hence i^* maps $C(\tau, \sigma)$ into $C(\xi, \sigma)$ for every convergence σ. Therefore we consider now

$$i^* : C(\tau, \sigma) \to C(\xi, \sigma).$$

Because i is the identity, i^* is the injection of $C(\tau, \sigma)$ into $C(\xi, \sigma)$. By Theorem 79, i^* is continuous from $[\tau, \sigma]$ to $[\xi, \sigma]$. Hence by Theorem 79, $i^{**} = (i^*)^*$ maps $C([\xi, \sigma], \sigma)$ into $C([\tau, \sigma], \sigma)$ and is continuous from $[[\xi, \sigma], \sigma]$ to $[[\tau, \sigma], \sigma]$. If $A \subset B$ and $i_{A,B} : A \to B$ is the injection and Z is a set, then $i^*_{A,B} : Z^B \to Z^A$ is the restriction. Indeed, if $h : B \to Z$, then by definition, $i^*_{A,B}(h) = h \circ i_{A,B}$. In particular, i^{**} is the restriction, wich associates with each map $h : C(\xi, \sigma) \to |\sigma|$ the map $h \circ i^* : C(\tau, \sigma) \to |\sigma|$.

Lemma 80. *If ξ is a convergence on X and σ is a convergence on Z, then*

$$j(x)(h) = \langle x, h \rangle$$

defines an embedding $j : X \to C([\xi, \sigma], \sigma)$, which is continuous from ξ to $[[\xi, \sigma], \sigma]$.

Proof. Indeed, if $h \in \lim_{[\xi,\sigma]} \mathcal{H}$ and $x \in \lim_\xi \mathcal{F}$, then by definition, $\langle x, h \rangle \in \lim_\sigma \langle \mathcal{F}, \mathcal{H} \rangle = \lim_\sigma j(\mathcal{F})(\mathcal{H})$. In the particular case $\mathcal{F} = (x)_\bullet$ this yields $j(x)(h) = \langle x, h \rangle \in \lim_\sigma \langle (x)_\bullet, \mathcal{H} \rangle = \lim_\sigma j(x)(\mathcal{H})$, which shows that $j(x) \in C([\xi, \sigma], \sigma)$ for every $x \in X$. ∎

Corollary 81. *For every ξ and σ, one has $\xi \geq j^-[[\xi, \sigma], \sigma]$.*

It is immediate that if $\rho \geq \sigma$, then $C(\xi, \rho) \subset C(\xi, \sigma)$ and the injection is continuous from $[\xi, \rho]$ to $[\xi, \sigma]$, in symbols, $[\xi, \rho] \triangleright [\xi, \sigma]$. Indeed, $h \in \lim_{[\xi,\rho]} \mathcal{H}$ whenever $x \in \lim_\xi \mathcal{F}$ implies $h(x) \in \lim_\rho \langle \mathcal{F}, \mathcal{H} \rangle$, hence $h(x) \in \lim_\sigma \langle \mathcal{F}, \mathcal{H} \rangle$, that is, $h \in \lim_{[\xi,\sigma]} \mathcal{H}$. In this way, we have defined an order $\triangleright$ on $C(\xi, o)$, where o stands for the *indiscrete topology*.[73]

Proposition 82. *For each ξ and a set Σ of convergences on a common underlying set,*

$$[\xi, \bigvee \Sigma] = \bigvee^{\triangleleft}_{\sigma \in \Sigma} [\xi, \sigma].$$

Proof. Let $h \in \lim_{[\xi, \bigvee \Sigma]} \mathcal{H}$. As we have seen $h \in C(\xi, \sigma)$ and $\mathcal{H}$ can be extended to a filter on $C(\xi, \sigma)$ for each $\sigma \in \Sigma$. By the definition of power convergence, $x \in \lim_\xi \mathcal{F}$ implies $h(x) \in \lim_{\bigvee \Sigma} \langle \mathcal{F}, \mathcal{H} \rangle = \bigcap_{\sigma \in \Sigma} \lim_\sigma \langle \mathcal{F}, \mathcal{H} \rangle$, hence $h \in \lim_{[\xi,\sigma]} \mathcal{H}$ for each $\sigma \in \Sigma$, that is, $h \in \lim_{\bigvee^{\triangleleft}_{\sigma \in \Sigma} [\xi,\sigma]} \mathcal{H}$. ∎

If $g : W \to Z$, then the map $g_* : W^X \to Z^X$ is defined by $g_*(h) = g \circ h$. If ρ is a convergence on W and σ is a convergence on Z, and $g \in C(\rho, \sigma)$, then (the restriction of) g_* maps $C(\xi, \rho)$ to $C(\xi, \sigma)$ and is continuous from $[\xi, \rho] \to [\xi, \sigma]$. In fact,

[73] The open sets of the *indiscrete topology* on Z are $\varnothing$ and Z.

Proposition 83.

$$[\xi, g^- \sigma] = g_*^- [\xi, \sigma].$$

Proof. Let $h \in \lim_{[\xi, g^- \sigma]} \mathcal{H}$, that is, $x \in \lim_\xi \mathcal{F}$ implies that $h(x) \in \lim_{g^- \sigma} \langle \mathcal{F}, \mathcal{H} \rangle$, that is, by the definition of initial convergence, $g(h(x)) \in \lim_\sigma g(\langle \mathcal{F}, \mathcal{H} \rangle)$. Because $g(\langle \mathcal{F}, \mathcal{H} \rangle) = \langle \mathcal{F}, g_*(\mathcal{H}) \rangle$, we conclude that this is equivalent to $g_*(h)(x) \in \lim_\sigma \langle \mathcal{F}, g_*(\mathcal{H}) \rangle$, which means that $g_*(h) \in \lim_{[\xi,\sigma]} g_*(\mathcal{H})$, that is, $h \in \lim_{g_*^- [\xi,\sigma]} \mathcal{H}$. ∎

Corollary 84. *If $g_i : Z \to Z_i$ is a surjection and σ_i is a convergence on Z_i for each $i \in I$, then*

$$[\xi, \bigvee_{i \in I} g_i^- \sigma_i] = \bigvee_{i \in I}^{\triangleleft} (g_i)_*^- [\xi, \sigma_i].$$

14.1. **Topologicity and other properties of power convergences.** As was said repeatedly, a power convergence of a topology with respect to another topology need not be a topology.[74] Now we will see a sufficient and necessary condition for a power convergence to belong to a given reflective class. If J is a reflector, then J-convergences are those convergences τ for which $J\tau \geq \tau$.

Proposition 85. *Let J be a reflector. Then $J[\xi, \sigma] \geq [\xi, \sigma]$ for each convergence $\sigma = J\sigma$ if and only if $\xi \times J\tau \geq J(\xi \times \tau)$ for each convergence τ.*[75]

Proof. Let J fulfill the condition and let $\sigma \leq J\sigma$. By definition, $[\xi, \sigma]$ is the coarsest convergence on $C(\xi, \sigma)$ for which

$$\xi \times [\xi, \sigma] \geq e^- \sigma.$$

On applying J to the inequality above, we get by (39),

$$\xi \times J[\xi, \sigma] \geq J(\xi \times [\xi, \sigma]) \geq J(e^- \sigma) \geq e^-(J\sigma) = e^- \sigma,$$

thus $J[\xi, \sigma] \geq [\xi, \sigma]$, and since J is contractive, $[\xi, \sigma]$ is a J-convergence. Conversely, suppose $J[\xi, \sigma] \geq [\xi, \sigma]$. By (76) it is enough to show that $C(\xi \times \tau, \sigma) \subset C(\xi \times J\tau, \sigma)$. If $f \in C(\xi \times \tau, \sigma)$ then by Theorem 78 ${}^t f \in C(\tau, [\tau, \sigma]) \subset C(J\tau, J[\tau, \sigma])$ by (46). By Proposition 79, $({}^t f)^* \in C([J[\xi, \sigma], \sigma], [J\tau, \sigma])$. On the other hand, $\xi \geq i^-([J[\xi, \sigma], \sigma])$, which means that the injection i from $|\xi|$ to $|C(C(\xi, \sigma), \sigma)|$ belongs to $C(\xi, [J[\xi, \sigma], \sigma])$. Therefore the composition $({}^t f)^* \circ i \in C(\xi, [J\tau, \sigma])$, which means that $f = \widehat{({}^t f)^* \circ i} \in C(\xi \times J\xi, \sigma)$. ∎

We know already a reflector that commutes with (arbitrary) products, thus *a fortiori* fulfills the assumption of Proposition 85.

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[74]A characterization of those underlying Hausdorff regular topologies for which the power convergence is topological will appear through Theorem 89 and Corollary 109.

[75]Notice that nothing was assumed about the underlying convergence ξ.

Corollary 86. *If a coupling convergence σ is a pseudotopology (a fortiori, a topology), then the power convergence $[\xi, \sigma]$ is a pseudotopology for each convergence ξ.*

A reflective class $\mathrm{fix}\, J$ of convergence spaces is said to be *exponential* if $J[\xi, \sigma] \geq [\xi, \sigma]$ for every $J\sigma \geq \sigma$. In other words, a class is exponential if the power does not lead out of this class. We conclude that the class of pseudotopologies is exponential, but that of topologies is not.

15. Hyperconvergences

15.1. **Sierpiński topology.** I have mentioned that if the coupling convegence is the *Sierpiński topology*, then the resulting power convergence is a hyperspace convergence. Let us investigate in detail this important special case of power convergences.

In spite of its great simplicity, the *Sierpiński topology* is a fundamental (non-Hausdorff) topology. It is often denoted by \$ and is defined on a two-element set, say, $\{0, 1\}$ by its open sets $\{\varnothing, \{1\}, \{0, 1\}\}$. This topology is not even T_1 (the singleton $\{1\}$ is not closed), but it is T_0. The Sierpiński topology is compact; even more: *every filter converges with respect to* \$.

Observe that a subset A of a topological space (X, τ) is closed if and only if the *indicator function* of A is continuous from τ to \$.[76] Therefore we can identify the set of all τ-closed subsets of X with $C(\tau, \$)$.

The Sierpiński topology plays an exceptional role among other topologies: it is *initially dense* in the category of all topologies. This means that every topology τ is the initial convergence with respect to maps valued in the Sierpiński topological space. Namely,

$$\tau = \bigvee_{f \in C(\tau, \$)} f^{-}\$, \tag{52}$$

(where $f^{-}\$$ denotes the initial convergence of \$ by f).

15.2. **Upper Kuratowski convergence.** If \$ is used as a coupling topology, then the coupling map goes from a subset of $X \times 2^X$ to $\{0, 1\}$ and is defined by

$$\langle x, A \rangle = \begin{cases} 0 \text{ if } x \in A \\ 1 \text{ if } x \notin A \end{cases}.$$

A filter $\mathcal{F}$ on the set $C(\tau, \$)$ (of closed subsets of a topological space (X, τ)) converges to $A_0 \in C(\tau, \$)$ in the *upper Kuratowski convergence* if for every $x_0 \notin A_0$ there exist a neighborhood V of x_0 and $F \in \mathcal{F}$ such that $V \cap A = \varnothing$ for each $A \in F$, in other words,

$$\bigcap_{F \in \mathcal{F}} \mathrm{cl}_\tau \bigcup_{A \in F} A \subset A_0. \tag{53}$$

[76] The indicator function $\langle \cdot, A \rangle$ of A is defined by $\langle x, A \rangle = 0$ if $x \in A$ and $\langle x, A \rangle = 1$ if $x \notin A$.

The concept can be naturally extended to the case of an arbitrary underlying convergence τ. To this end, I will use the notion of reduced filter. If $\mathcal{F}$ is a filter on (a subset of) 2^X then we denote by $|\mathcal{F}|$ the *reduced filter* of $\mathcal{F}$, that is, the filter on X generated by $\{\bigcup_{A\in F} A : F \in \mathcal{F}\}$. Let $\mathbb{F}$ be a set of filters on (a subset of) 2^X. Then

$$|\bigcap\nolimits_{\mathcal{F}\in\mathbb{F}} \mathcal{F}| = \bigcap\nolimits_{\mathcal{F}\in\mathbb{F}} |\mathcal{F}|. \tag{54}$$

Indeed, $\bigcap_{\mathcal{F}\in\mathbb{F}} \mathcal{F}$ consists of the sets of the form $\bigcup_{\mathcal{F}\in\mathbb{F}} F_{\mathcal{F}}$ (where $F_{\mathcal{F}} \in \mathcal{F}$ for each $\mathcal{F} \in \mathbb{F}$), hence its reduced filter is generated by the unions of the elements of $\bigcup_{\mathcal{F}\in\mathbb{F}} F_{\mathcal{F}}$. The filter $\bigcap_{\mathcal{F}\in\mathbb{F}} |\mathcal{F}|$ is generated by the sets of the form $\bigcup_{\mathcal{F}\in\mathbb{F}} \bigcup_{A\in F_{\mathcal{F}}} A$. Therefore the filters in (54) are equal.

We say that $\mathcal{F}$ *upper Kuratowski converges* to A_0 with respect to a convergence τ if

$$\mathrm{adh}_\tau |\mathcal{F}| \subset A_0. \tag{55}$$

The formula above means that $x_0 \notin A_0$ and $x_0 \in \lim_\tau \mathcal{G}$ imply that there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\bigcup_{A\in F} A \cap G = \varnothing$. In the particular case where τ is a topology the condition holds if and only if it holds for $\mathcal{G} = \mathcal{N}_\tau(x_0)$. Therefore, if τ is a topology then (55) is equivalent to (53).

Proposition 87. *A filter $\mathcal{F}$ upper Kuratowski converges to A_0 with respect to τ if and only if $A_0 \in \lim_{[\tau,\$]} \mathcal{F}$.*

Proof. Let τ be a convergence on X. By the definition of power convergence, $A_0 \in \lim_{[\tau,\$]} \mathcal{F}$ if and only if $\langle x_0, A_0\rangle \in \lim_\$ \langle \mathcal{G}, \mathcal{F}\rangle$ for every $x_0 \in X$ and every filter $\mathcal{G}$ on X such that $x_0 \in \lim_\tau \mathcal{G}$. The filter $\langle \mathcal{G}, \mathcal{F}\rangle$ on $\{0, 1\}$ is generated by

$$\{\{\langle x, A\rangle : x \in G, A \in F\} : G \in \mathcal{G}, F \in \mathcal{F}\}.$$

Because the only neighborhood of 0 in \$ is the whole space $\{0, 1\}$, the convergence condition is restrictive only at 1. Therefore $\langle x_0, A_0\rangle \in \lim_\$ \langle \mathcal{G}, \mathcal{F}\rangle$ if and only if $\langle x_0, A_0\rangle = 1$ (equivalently, $x_0 \notin A_0$) whenever there is $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $\langle x, A\rangle = 1$ (equivalently, $x \notin A$) for every $x \in G$ and $A \in F$, in other words, whenever $G \cap \bigcup_{A\in F} A = \varnothing$, which amounts to $\mathrm{adh}_\tau |\mathcal{F}| \subset A_0$. ∎

We see that the upper Kuratowski convergence is a pseudotopology, because it is a power convergence with respect to a topological (hence, *a fortiori* pseudotopological) coupling convergence \$. We know however no reason that it be a topology.

The *cocompact topology* on $C(\tau, \$)$ can be defined by a base of open sets consisting of

$$\{A \in C(\tau, \$) : A \cap K = \varnothing\},$$

where K is a τ-compact set.

Proposition 88. *If τ is a Hausdorff topology, then $[\tau, \$]$ is finer than the cocompact topology with respect to τ.*

Proof. If $A_0 \in \lim_{[\tau,\$]} \mathcal{F}$, and K is a compact set disjoint from A_0, then for each $x \in K$ there exist $F_x \in \mathcal{F}$ and a neighborhood V_x of x such that $V_x \cap \bigcup_{A \in F_x} A = \varnothing$. As K is compact there are finitely many x in K, say $x_1, x_2, \ldots, x_m$, such that $K \subset \bigcup_{1 \leq j \leq m} V_{x_j}$. Then $F = \bigcap_{1 \leq j \leq m} F_{x_j} \in \mathcal{F}$ and $K \cap \bigcup_{A \in F} A = \varnothing$. ∎

In fact, by a theorem of Choquet [8],

Theorem 89. *Let τ be a (Hausdorff) regular topology. Then the upper Kuratowski convergence $[\tau, \$]$ is a topology if and only if it coincides with the cocompact topology if and only if τ is locally compact.*

Proof. First let us show that if τ is a locally compact topology, then the cocompact topology with respect to τ coincides with $[\tau, \$]$. Let $\mathcal{F}$ converge to A_0 in the cocompact topology and let $x \notin A_0$. As A_0 is τ-closed and τ is locally compact there exists a compact neighborhood K of x such that $K \cap A_0 = \varnothing$, hence by assumption, there is $F \in \mathcal{F}$ such that $K \cap \bigcup_{A \in F} A = \varnothing$, thus $A_0 \in \lim_{[\tau,\$]} \mathcal{F}$.

If τ is not locally compact, then there is an element x_0, for every closed neighborhood W of which there exists an ultrafilter $\mathcal{U}_W$ such that $\lim_\tau \mathcal{U}_W = \varnothing$. On the other hand, each element of $\bigcap_{W \in \mathcal{N}_\tau(x)} \mathcal{U}_W = \bigcap_{\mathrm{cl}_\tau W = W \in \mathcal{N}_\tau(x)} \mathcal{U}_W$ is generated by the family consisting of $\bigcup_{W \in \mathcal{N}_\tau(x)} U_W$, where $U_W \in \mathcal{U}_W$ are all the possible selections. Therefore $(\bigcap_{W \in \mathcal{N}_\tau(x)} \mathcal{U}_W) \# \mathcal{N}_\tau(x)$, hence $x \in \mathrm{adh}_\tau(\bigcap_{W \in \mathcal{N}_\tau(x)} \mathcal{U}_W)$.

Because all the singletons $\{x\}$ are closed, the family

$$\{\{\{x\} : x \in U\} : U \in \mathcal{U}_W\}$$

generates a filter $\mathcal{Z}_W$ on $C(\tau, \$)$, the set of τ-closed sets. It is immediate that $\mathcal{U}_W$ is the reduced filter of $\mathcal{Z}_W$, therefore $\varnothing \in \lim_{[\tau,\$]} \mathcal{Z}_W$ by virtue of (55) and Proposition 87. We will see that $\varnothing \notin \lim_{[\tau,\$]}(\bigcap_{W \in \mathcal{N}_\tau(x)} \mathcal{Z}_W)$, and thus $[\tau, \$]$ is not even a pretopology.

Indeed, by (54) the reduced filter of $\bigcap_{W \in\, \mathcal{N}_\tau(x)} \mathcal{Z}_W$ is $\bigcap_{W \in \mathcal{N}_\tau(x)} \mathcal{U}_W$, and since $\mathrm{adh}_\tau(\bigcap_{W \in \mathcal{N}_\tau(x)} \mathcal{U}_W) \neq \varnothing$, we conclude that $\varnothing \notin \lim_{[\tau,\$]}(\bigcap_{W \in \mathcal{N}_\tau(x)} \mathcal{Z}_W)$. ∎

I shall characterize vicinities and open sets of $[\xi, \$]$ in terms of ξ.[77]

A family $\mathcal{A}$ of closed subsets of a topological space is *stable* if $B \in \mathcal{A}$ for every $A \in \mathcal{A}$ and each closed subset B of A. The vicinity filters of $[\xi, \$]$ admit bases of ξ-stable filters.

Theorem 90. *A family $\mathcal{A}$ is stable for ξ-closed subsets and a vicinity of A_0 with respect to $[\xi, \$]$ if and only if $\mathcal{A}_c = \mathcal{O}_\xi(\mathcal{A}_c)$ and is ξ-compact at A_0^c.*

Proof. A family $\mathcal{A}$ is a vicinity of A_0 for $[\xi, \$]$ if and only if $\mathcal{A} \in \mathcal{F}$ for every filter on $C(\xi, \$)$ with $A_0 \in \lim_{[\xi,\$]} \mathcal{F}$, equivalently if for each filter $\mathcal{H}$ on $|\xi|$ such that $\mathrm{adh}_\xi \mathcal{H} \subset A_0$ there exists $\mathrm{cl}_\xi H = H \in \mathcal{H}$ such that $H \in \mathcal{A}$.

[77] In order not to complicate the presentation, we shall present a special case of [20, Theorem 16.2, Corollary 16.3] where the underlying space is a topology.

Equivalently, $\mathcal{H}\#\mathcal{A}_c$ implies that $\operatorname{adh}_\xi \mathcal{H} \cap A_0^c \neq \varnothing$, which means that $\mathcal{A}_c$ is ξ-compact at A_0^c. ∎

Corollary 91. [17, Theorem 3.1] *A family $\mathcal{A}$ is open for $[\xi, \$]$ if and only if $\mathcal{A}_c = \mathcal{O}_\xi(\mathcal{A}_c)$ and is ξ-compact.*

Remark 92. *The upper Kuratowski convergence is* hypercompact*, that is, every filter converges.*

15.3. **Kuratowski convergence.** A filter $\mathcal{F}$ on $C(\xi, \$)$ *lower Kuratowski converges* to A_0 if for every filter $\mathcal{G}$ such that $\lim_\xi \mathcal{G} \cap A_0 \neq \varnothing$ and for each $G \in \mathcal{G}$, there exists $F \in \mathcal{F}$ such that $A \cap G \neq \varnothing$ for each $A \in F$ [18, p. 306] (equivalently, for each $G \in \mathcal{G}$ each $H \in \mathcal{F}^\#$ there exists $A \in H$ such that $A \cap G \neq \varnothing$)[78]. Therefore $\mathcal{F}$ on $C(\xi, \$)$ lower Kuratowski converges to A_0 whenever

$$A_0 \subset \operatorname{adh}_\xi |\mathcal{F}^\#|,$$

where $|\mathcal{F}^\#|$ is the reduced grill. If ξ is a topology, then it is enough, in the definition above, to take for every $x \in A_0$ the coarsest filter $\mathcal{G} = \mathcal{N}_\xi(x)$ that converges to x. Then the condition becomes: for every $x \in A_0$ and each neighborhood V of x there exists $F \in \mathcal{F}$ such that $V \cap A \neq \varnothing$ for each $A \in F$. Therefore if ξ is a topology, then the associated lower Kuratowski convergence is a topology.

A filter $\mathcal{F}$ on $C(\xi, \$)$ *Kuratowski converges* to A_0 if it lower and upper Kuratowski converges to A_0, that is whenever $\operatorname{adh}_\xi |\mathcal{F}| \subset A_0 \subset \operatorname{adh}_\xi |\mathcal{F}^\#|$.

If ξ is a topology, then the greatest element of $C(\xi, \$)$, to which a filter $\mathcal{F}$ converges in the lower Kuratowski topology, is[79]

$$\operatorname{Li}_\xi \mathcal{F} = \bigcap_{H \in \mathcal{F}^\#} \operatorname{cl}_\xi\Big(\bigcup_{A \in H} A\Big).$$

As the least element to which $\mathcal{F}$ upper Kuratowski converges is

$$\operatorname{Ls}_\xi \mathcal{F} = \bigcap_{F \in \mathcal{F}} \operatorname{cl}_\xi\Big(\bigcup_{A \in F} A\Big),$$

and $\operatorname{Li}_\xi \mathcal{F} \subset \operatorname{Ls}_\xi \mathcal{F}$ for every filter $\mathcal{F}$, a filter $\mathcal{F}$ Kuratowski converges to A_0 if and only if $\operatorname{Ls}_\xi \mathcal{F} \subset A_0 \subset \operatorname{Li}_\xi \mathcal{F}$. We infer that (without any assumption on the underlying topology)

If the underlying convergence is a topology, then the Kuratowski convergence is a Hausdorff pseudotopology.

Proposition 93. *If the underlying convergence is a topology, then the Kuratowski convergence is compact.*

Proof. For every ultrafilter $\mathcal{U}$ (on $C(\xi, \$)$), $\mathcal{U}^\# = \mathcal{U}$, hence $\operatorname{Ls}_\xi \mathcal{U} = \operatorname{Li}_\xi \mathcal{U}$.[80] ∎

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[78] This equivalence follows for [27].

[79] (see [17])

[80] If $\operatorname{Ls}_\xi \mathcal{U} = \operatorname{Li}_\xi \mathcal{U} = \varnothing$ then $\varnothing$ is the limit of the Kuratowski convergence with respect to ξ.

If the underlying convergence is a topology, then the Kuratowski convergence is a topology if and only if the upper Kuratowski convergence is a topology.[81] Consequently, if the underlying convergence ξ is a Hausdorff regular topology, then by Theorem 89 and Corollary 51, the Kuratowski is regular if and only ξ is locally compact.

16. Exponential hull of topologies

16.1. **Bidual convergences.** Recall that if ξ, σ are convergences, then $[\xi,\sigma]$ is a convergence on $C(\xi,\sigma)$ and $[[\xi,\sigma],\sigma]$ is a convergence on $C([\xi,\sigma],\sigma)$ such that the injection j of $|\xi|$ into $C([\xi,\sigma],\sigma)$ is continuous from ξ to $[[\xi,\sigma],\sigma]$. Let

$$\mathrm{Epi}^\sigma \xi = j^-[[\xi,\sigma],\sigma].$$

It follows from Corollary 56 that[82]

$$\xi \geq \mathrm{Epi}^\sigma \xi. \tag{56}$$

A convergence ξ is called *bidual* with respect to a convergence σ whenever $\xi = \mathrm{Epi}^\sigma \xi$.

Proposition 94. *Let $f : X \to Y$ and let ξ be a convergence on X. Then*

$$f(\mathrm{Epi}^\sigma \xi) \geq \mathrm{Epi}^\sigma f\xi.$$

Proof. Let $y \in \lim_{f(\mathrm{Epi}^\sigma \xi)} \mathcal{G}$, hence there exist x and $\mathcal{F}$ such that $x \in \lim_{\mathrm{Epi}^\sigma \xi} \mathcal{F}$, $f(x) = y$ and $f(\mathcal{F}) = \mathcal{G}$. The first equality means that $\langle x,k\rangle \in \lim_\sigma \langle \mathcal{F},\mathcal{K}\rangle$ for every $k \in \lim_{[\xi,\sigma]} \mathcal{K}$. To show that $y \in \lim_{\mathrm{Epi}^\sigma f\xi} \mathcal{G}$ consider $h \in C(f\xi,\sigma)$ and a filter $\mathcal{H}$ on $C(f\xi,\sigma)$ such that $h \in \lim_{[f\xi,\sigma]} \mathcal{H}$. It follows that $f^*(h) \in \lim_{[\xi,\sigma]} f^*(\mathcal{H})$, hence

$$\langle y,h\rangle = \langle f(x),h\rangle = \langle x,f^*(h)\rangle \in \lim_\sigma \langle \mathcal{F},f^*(\mathcal{H})\rangle = \lim_\sigma \langle f(\mathcal{F}),\mathcal{H}\rangle = \lim_\sigma \langle \mathcal{G},\mathcal{H}\rangle.$$

∎

Proposition 95. *If $\xi \geq \theta$ then* $\mathrm{Epi}^\sigma \xi \geq \mathrm{Epi}^\sigma \theta$.

Proof. Let $x \in \lim_{\mathrm{Epi}^\sigma \xi} \mathcal{F}$, that is, $h(x) \in \lim_\sigma \langle \mathcal{F},\mathcal{H}\rangle$ for every $h \in C(\xi,\sigma)$ and each filter $\mathcal{H}$ on $C(\xi,\sigma)$ such that $h \in \lim_{[\xi,\sigma]} \mathcal{H}$. Because $C(\theta,\sigma) \subset C(\xi,\sigma)$ and the injection is continuous from $[\theta,\sigma]$ to $[\xi,\sigma]$, if $h \in \lim_{[\theta,\sigma]} \mathcal{H}$ then $h \in \lim_{[\xi,\sigma]} \mathcal{H}$ and thus $h(x) \in \lim_\sigma \langle \mathcal{F},\mathcal{H}\rangle$, which proves that $x \in \lim_{\mathrm{Epi}^\sigma \theta} \mathcal{F}$. ∎

It follows from Propositions 94 and 95 that Epi^σ is a (concrete) functor.

Proposition 96.

$$[\xi,\sigma] = [j^-[[\xi,\sigma],\sigma],\sigma].$$

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[81] as the supremum (in a lattice of convergences) of two topologies.

[82] Here is a direct proof. Let $x \in \lim_\xi \mathcal{F}$ and let $h \in \lim_{[\xi,\sigma]} \mathcal{H}$. By definition of power convergence, this means that $h(x) \in \lim_\sigma \langle \mathcal{F},\mathcal{H}\rangle$. Hence $x \in \lim_{[[\xi,\sigma],\sigma]} \mathcal{F}$.

Proof. Let $h \in \lim_{[\xi,\sigma]} \mathcal{H}$. This means that $x \in \lim_\xi \mathcal{F}$ implies $h(x) \in \lim_\sigma \langle \mathcal{F}, \mathcal{H} \rangle$. In order to prove that $h \in \lim_{[j^-[[\xi,\sigma],\sigma],\sigma]} \mathcal{H}$, we need to show that $h(x) \in \lim_\sigma \langle \mathcal{F}, \mathcal{H} \rangle$ for filter $\mathcal{F}$ on X and every $x \in \lim_{[[\xi,\sigma],\sigma]} \mathcal{F}$. The latter means that $g(x) \in \lim_\sigma \langle \mathcal{F}, \mathcal{G} \rangle$ for every $g \in C(\xi, \sigma)$ and each filter $\mathcal{G}$ on $C(\xi, \sigma)$ such that $g \in \lim_{[\xi,\sigma]} \mathcal{G}$. In particular, this holds for $g = h$ and $\mathcal{G} = \mathcal{H}$. In particular, with $\mathcal{F} = (x)_\bullet$ we conclude that $C(\xi, \sigma) \subset C(j^-[[\xi, \sigma], \sigma], \sigma)$.

Conversely, by (56) the injection of $C(\operatorname{Epi}^\sigma \xi, \sigma)$ into $C(\xi, \sigma)$ is continuous from $[j^-[[\xi, \sigma], \sigma], \sigma]$ to $[\xi, \sigma]$. ∎

Therefore, $\operatorname{Epi}^\sigma$ is idempotent. We have already noticed that $\operatorname{Epi}^\sigma$ is a concrete functor. Therefore, on recalling (56), we conclude that

Theorem 97. *For every convergence σ, the map $\operatorname{Epi}^\sigma$ is a concrete reflector.*

For a given functor L, define

$$\operatorname{Epi}^L \xi = \bigvee\nolimits_{\sigma = L\sigma} \operatorname{Epi}^\sigma \xi.$$

A convergence ξ is *bidual* with respect to a functor L if $\xi = \operatorname{Epi}^L \xi$. It follows that

Corollary 98. *For every functor L, the map Epi^L is a concrete reflector.*

16.2. **Epitopologies.** If $L = T$ (the topologizer), then we abridge $\operatorname{Epi} = \operatorname{Epi}^T$. A convergence ξ is called an *epitopology* if $\operatorname{Epi} \xi \geq \xi$. By Corollary 98 the class of epitopologies is a concretely reflective subcategory of the category of convergence spaces. Of course, Epi is the corresponding reflector, that we call the *epitopologizer.* Because the Sierpiński topology $\$$ is initially dense in $\operatorname{fix} T$ (the category of topological spaces), by Corollary 84 implies that

$$\operatorname{Epi} \xi = \operatorname{Epi}^\$ \xi.$$

This fact is of great importance, because it reduces considerably the complexity of reasonings envolving the epitopologizer.

Proposition 99. *The epitopologizer commutes with finite products.*

Proof. By Proposition 96 $[\operatorname{Epi} \xi, \$] = [\xi, \$]$. This and (50) imply that

$$[\xi \times \operatorname{Epi} \tau, \$] \approx [\xi, [\operatorname{Epi} \tau, \$]] = [\xi, [\tau, \$]] \approx [\xi \times \tau, \$].$$

Hence $\operatorname{Epi}(\xi \times \operatorname{Epi} \tau) = j^-[[\xi \times \operatorname{Epi} \tau, \$], \$] = j^-[[\xi \times \tau, \$], \$] = \operatorname{Epi}(\xi \times \tau)$, and thus

$$\xi \times \operatorname{Epi} \tau \geq \operatorname{Epi}(\xi \times \operatorname{Epi} \tau) = \operatorname{Epi}(\xi \times \tau).$$

Therefore $\operatorname{Epi} \xi \times \operatorname{Epi} \tau \geq \operatorname{Epi}(\operatorname{Epi} \xi \times \tau) \geq \operatorname{Epi} \operatorname{Epi}(\xi \times \tau) = \operatorname{Epi}(\xi \times \tau)$. ∎

It follows from Proposition 85 that

Corollary 100. *The class of epitopologies is exponential.*

F. Mynard gave in [38] an explicit formula for the epitopologizer similar to that for adherence-determined convergences in (24), namely

$$\lim_{\mathrm{Epi}\,\xi}\mathcal{F}=\bigcap_{\mathbb{F}_\xi\ni\mathcal{H}\#\mathcal{F}}\mathrm{cl}_\xi(\mathrm{adh}_\xi\,\mathcal{H}),\tag{57}$$

where $\mathbb{F}_\xi$ stands for the class of the ξ-reduced filters, that is, the filters of the form $|\mathcal{G}|\approx\{\bigcup_{B\in G}B:G\in\mathcal{G}\}$ where $\mathcal{G}$ is a filter on $C(\xi,\$)$.

Proof. By definition, $x\in\lim_{\mathrm{Epi}\,\xi}\mathcal{F}$ if and only if $\langle x,A\rangle\in\lim_{\$}\langle\mathcal{F},\mathcal{G}\rangle$ for every $A\in C(\xi,\$)$ and each filter $\mathcal{G}$ on $C(\xi,\$)$ such that $A\in\lim_{[\xi,\$]}\mathcal{G}$. Because the only \$-neighborhood of 0 is the whole $\{0,1\}$, the condition $\langle x,A\rangle\in\lim_{\$}\langle\mathcal{F},\mathcal{G}\rangle$ need be considered only in case $\langle x,A\rangle=1$. Therefore, and on recalling (55), $x\in\lim_{\mathrm{Epi}\,\xi}\mathcal{F}$ if and only if for every filter $\mathcal{G}$ on the set of ξ-closed sets such that $\mathrm{adh}_\xi|\mathcal{G}|\subset A=\mathrm{cl}_\xi A$ if $x\notin A$ then there is $F\in\mathcal{F}$ and $G\in\mathcal{G}$ such that $F\cap\bigcup_{B\in G}B=\varnothing$, which means that $\mathcal{F}$ does not mesh with $|\mathcal{G}|$. By taking $A=\mathrm{cl}_\xi(\mathrm{adh}_\xi|\mathcal{G}|)$, our condition can be rephrased: $x\in\mathrm{cl}_\xi(\mathrm{adh}_\xi\,\mathcal{H})$ for every ξ-reduced filter $\mathcal{H}$ that meshes with $\mathcal{F}$. ∎

In order to avoid introducing several other concepts, the following characterization of epitopologies is given here only for T_1-convergences, that is, those for which all the singletons are closed.[83]

Proposition 101. *A T_1 convergence is an epitopology if and only if it is a pseudotopology with closed limits.*

Proof. If a convergence ξ is T_1, then every filter on $|\xi|$ is a ξ-reduced filter,[84] hence in this case (57) becomes

$$\lim_{\mathrm{Epi}\,\xi}\mathcal{F}=\bigcap_{\mathcal{H}\#\mathcal{F}}\mathrm{cl}_\xi(\mathrm{adh}_\xi\,\mathcal{H}).\tag{58}$$

If ξ is an epitopology, then by (58) it has closed limits, and is a pseudotopology, because $\lim_{S\xi}\mathcal{F}=\bigcap_{\mathcal{H}\#\mathcal{F}}\mathrm{adh}_\xi\,\mathcal{H}\subset\bigcap_{\mathcal{H}\#\mathcal{F}}\mathrm{cl}_\xi(\mathrm{adh}_\xi\,\mathcal{H})$. Conversely, if ξ is a pseudotopology with closed limits and $x\notin\lim_\xi\mathcal{F}$ then there is an ultrafilter $\mathcal{U}\#\mathcal{F}$ such that $x\notin\mathrm{adh}_\xi\,\mathcal{U}=\lim_\xi\mathcal{U}=\mathrm{cl}_\xi(\mathrm{adh}_\xi\,\mathcal{U})$ because the limits are closed. Hence $x\notin\lim_{\mathrm{Epi}\,\xi}\mathcal{F}$ showing that ξ is an epitopology. ∎

Proposition 102. *Each topology is an epitopology.*

Proof. Obviously, each principal filter of a ξ-closed set is ξ-reduced. Hence by (57) and (28) each topology is an epitopology. ∎

It was mentioned in a footnote that some authors (e.g., H. J. Kowalsky [36]) define a convergence as a relation fulfilling (8)(9) (as I do) and an additional axiom

$$\lim\mathcal{F}_0\cap\lim\mathcal{F}_1\subset\lim(\mathcal{F}_0\cap\mathcal{F}_1).\tag{59}$$

[83] One of the characterizations of epitopologies (due to Bourdaud) is: *A convergence is an epitopology if and only if it is a star-regular pseudotopology with closed limits* [20].

[84] Indeed, if $\mathcal{H}$ is a filter on $|\xi|$ then take the filter $\mathcal{G}$ on $C(\xi,\$)$ generated by $\{\{\{x\}:x\in H\}:H\in\mathcal{H}\}$. Then $|\mathcal{G}|=\mathcal{H}$.

I call a convergence a *prototopology* if it fulfills (59). It is straightforward that prototopologies constitute a concretely reflective subcategory of convergence spaces. An important fact is that

Proposition 103. *The class of topologies is finally dense in that of prototopologies.*

Proof. Let ξ be a prototopology on X. For every $x \in X$ and each filter $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F}$ let $\tau_{x,\mathcal{F}}$ be a topology on X such that $\mathcal{F} \wedge (x)_\bullet$ is the neighborhood filter of x and all other elements of X are isolated. Then $\xi = \bigwedge_{(\mathcal{F},x)\in\xi} \tau_{x,\mathcal{F}}$. ∎

Theorem 104. [6] *The category of epitopologies is the least exponential reflective subcategory of prototopologies that includes all topologies.*

Proof. Let $\mathbf{L}$ be an exponential reflective subcategory of prototopologies that contains all topologies. By Proposition 103 for every prototopology ξ, there exist a family $\{\tau_k : k \in K\}$ of topologies such that $\xi = \bigwedge_{k\in K} \tau_k$. By virtue of Corollary 84,

$$[\xi, \$] = \bigvee_{k\in K} (j^*)^-[\tau_k, \$].$$

Because $\mathbf{L}$ is exponential and contains all topologies, $[\tau_k, \$] \in \mathbf{L}$ for every $k \in K$, and since $\mathbf{L}$ is reflective, $[\xi, \$] \in \mathbf{L}$ as the initial object with respect to prototopologies in $\mathbf{L}$. Therefore $[[\xi, \$], \$] \in \mathbf{L}$ because $\mathbf{L}$ is exponential, and thus the initial prototopology $\mathrm{Epi}\,\xi = j^-[[\xi, \$], \$]$ belongs to $\mathbf{L}$, because $\mathbf{L}$ is reflective. It follows that every epitopology belongs to $\mathbf{L}$. ∎

17. Reflective properties of power convergences

We have seen that if M is an arbitrary functor, then the class of all the convergences τ such that $M\tau \geq \tau$ is reflective.[85] An important problem consists in characterizing convergences ξ such that

$$M[\xi, \sigma] \geq [\xi, \sigma] \tag{60}$$

for each $\sigma \in \mathbb{D} \subset \mathrm{fix}\, M$.[86]

Example 105. *If M is equal to the topologizer T, then (60) is equivalent to the problem, the solution of which was given in Theorem 89 in the special case of Hausdorff regular topologies ξ and the class $\mathbb{D}$ consisting of the Sierpiński topology \$.*

Let M be a functor, L a reflector. For a convergence σ, let

$$\mathrm{Epi}_M^\sigma\, \xi = j^-[M[\xi, \sigma], \sigma], \tag{61}$$

where j is the injection of $|\xi|$ into $C([\xi, \sigma], \sigma)$, and

$$\mathrm{Epi}_M^L\, \xi = \bigvee\nolimits_{\sigma = L\sigma} \mathrm{Epi}_M^\sigma\, \xi. \tag{62}$$

[85] This does not mean in general that M is the reflector of the reflective class it defines.

[86] This is a very special case of problems thoroughly studied by F. Mynard (e.g., [39]).

Theorem 106. [20, Theorem 15.2] *Let M be a functor, L a reflector, and $\xi \geq \theta$ be convergences on the same underlying set. The following are equivalent:*

$$\theta \times M\tau \geq L(\xi \times \tau) \text{ for each } \tau; \tag{63}$$

$$M[\xi, \sigma] \geq [\theta, \sigma] \text{ for every } \sigma = L\sigma; \tag{64}$$

$$\theta \geq \mathrm{Epi}_M^L \xi. \tag{65}$$

Proof. (63) $\Rightarrow$ (64). If τ is a topology on a singleton[87], then (63) implies $\theta \geq L\xi$, hence $\xi \geq \theta \geq L\xi$. Therefore $C(L\xi, \sigma) \subset C(\theta, \sigma) \subset C(\xi, \sigma)$ for every convergence σ. On the other hand, $C(\xi, \sigma) \subset C(L\xi, L\sigma)$ for every functor L, so that if $\sigma = L\sigma$ then $C(\xi, \sigma) \subset C(L\xi, \sigma)$. This implies that $[\xi, \sigma]$ and $[\theta, \sigma]$ have the same underlying set. For $\tau = [\xi, \sigma]$ the inequality (63) becomes

$$\theta \times M[\xi, \sigma] \geq L(\xi \times [\xi, \sigma]) \geq L(e^- \sigma) = e^- \sigma,$$

which means that $M[\xi, \sigma] \geq [\theta, \sigma]$.

(64) $\Rightarrow$ (65) On applying $j^-[\cdot, \sigma]$ to (64), we get

$$\mathrm{Epi}_M^\sigma \xi = j^-[M[\xi, \sigma], \sigma] \geq j^-[[\theta, \sigma], \sigma] \leq \theta$$

for every $\sigma = L\sigma$, thus (65).

(65) $\Rightarrow$ (63). By Lemma 76 it is enough to prove that $C(\xi \times \tau, \sigma) \subset C(\theta \times M\tau, \sigma)$ for each τ and every $\sigma = L\sigma$. If $f \in C(\xi \times \tau, \sigma)$ then by (49) ${}^t f \in C(\tau, [\xi, \sigma])$, hence by (46) ${}^t f \in C(M\tau, M[\xi, \sigma])$, and thus $({}^t f)^* \in C([M[\xi, \sigma], \sigma], [M\tau, \sigma])$ by Proposition 79. On the other hand, by (65), $\theta \geq j^-([M[\xi, \sigma], \sigma])$ for each $\sigma \in \mathrm{fix}\, L$, which means that the injection j belongs to $C(\theta, [M[\xi, \sigma], \sigma])$. Therefore the composition $({}^t f)^* \circ j \in C(\theta, [M\tau, \sigma])$, which means that $f = \widehat{({}^t f)^* \circ j} \in C(\theta \times M\xi, \sigma)$. ∎

In the particular case $M = JE$ where J is a reflector, E is a coreflector and $\theta = \xi$, Theorem 106 entails immediately[88] the following special case of a theorem of F. Mynard [39, Theorem 3.1].

Theorem 107. *Let J, L be reflectors, E a coreflector, and ξ a convergence. The following are equivalent:*

$$\xi \times J\tau \geq L(\xi \times \tau) \text{ for each } \tau \geq JE\tau \text{ (for each } \tau = E\tau\text{)}; \tag{66}$$

$$JE[\xi, \sigma] \geq [\xi, \sigma] \text{ for every } \sigma = L\sigma; \tag{67}$$

$$\xi \geq \mathrm{Epi}_{JE}^L \xi; \tag{68}$$

$$i \times f \text{ is } L\text{-quotient for every } J\text{-quotient } f \text{ with } JE\text{-domain.} \tag{69}$$

The last term of the equivalence above follows from Remark 74 and Theorem 71.

[87] There is a unique topology on a singleton.

[88] on using $JJ = J$.

Remark 108. *The list of equivalences in Theorem 106 can be extended. In particular, they hold if and only if*

$$M[\xi,\rho] \geq [\theta,\rho] \text{ for every } \rho \in \mathbb{D}, \tag{70}$$

where $\mathbb{D}$ is an initially dense subclass of $\operatorname{fix} J$. In fact, if $\sigma \in \operatorname{fix} J$, then there exists a class of maps $\{g_\iota : \iota \in I\}$ such that $\sigma = \bigvee_{\iota\in I} g_\iota^- \rho_\iota$ with $\rho_\iota \in \mathbb{D}$ for each $\iota \in I$. Therefore, by Corollary 84,

$$[\xi,\sigma] = [\xi, \bigvee\nolimits_{\iota\in I} g_\iota^- \rho_\iota] = \bigvee\nolimits_{\iota\in I}^{\lhd} (g_\iota^*)^- [\xi,\rho_\iota], \tag{71}$$

hence, by (39),

$$M[\xi,\sigma] \geq \bigvee\nolimits_{\iota\in I}^{\lhd} M(g_\iota^*)^-[\xi,\rho_\iota] \geq \bigvee\nolimits_{\iota\in I}^{\lhd} (g_\iota^*)^- M[\theta,\rho_\iota],$$

so that $M[\xi,\sigma] \geq [\theta,\sigma]$ by virtue of (70) and (71). On the other hand, (65) implies

$$\theta \geq \bigvee\nolimits_{\rho\in\mathbb{D}} j^-[M[\xi,\rho],\rho], \tag{72}$$

and (72) entails (63), because on one hand, our proof of the implication (63) by (65) hinges entirely on Lemma 76, and on the other, the conclusion of Lemma 76 depends on initially dense subclasses by virtue of Proposition 77.

This fact has considerable importance in deciding whether a power convergence with respect to a topology is in the reflective class determined by M.

Corollary 109. *If $M \geq T$ is a functor, and ξ is a convergence, then $M[\xi,\sigma] \geq [\xi,\sigma]$ for every topology σ if and only if $M[\xi,\$] \geq [\xi,\$]$ if and only if $\xi \geq j^-[M[\xi,\$],\$]$.*[89]

By Theorem 89 the upper Kuratowski convergence with respect to a Hausdorff regular topology is topological (equivalently, pretopological) if and only if the underlying topology is locally compact. Therefore

Corollary 110. *The topologizer T and the pretopologizer P do not commute with finite products.*

Example 111. *Let us come back to the case where M is equal to the topologizer T and $\mathbb{D}$ consists of the Sierpiński topology. Then the power convergences are upper Kuratowski convergences. Now Theorem 106 furnishes a sufficient and necessary condition for the topologicity of $[\xi,\$]$ for an arbitrary convergence (without any separation assumptions). We just need to interpret the condition $\xi \geq \operatorname{Epi}_T \xi$, which amounts to $j^-[T[\xi,\$],\$] \leq \xi$.*

A topology ξ is called *core-compact* [9] if for every element x and each $O \in \mathcal{N}_\xi(x)$ there exist $V \in \mathcal{N}_\xi(x)$ such that V is ξ-compact at O. The following theorem was established in different terms[90] by Hofmann and Lawson in [32]; a more general result (for arbitrary convergences) was proved in [20].

[89] In fact, it can be shown [39] that $\operatorname{Epi}_M^T \xi = j^-[M[\xi,\$],\$]$.

[90] of the Scott topology

Theorem 112. *The upper Kuratowski convergence with respect to a topology ξ is topological if and only if ξ is core-compact.*

Proof. By Corollary 109 with $M = T$, we need show that ξ is core-compact if and only if $\xi \geq j^{-}[T[\xi, \$], \$]$. Because ξ is a topology, it is enough to show that for every element x the neighborhood filter $\mathcal{N}_{\xi}(x)$ converges to x in $j^{-}[T[\xi, \$], \$]$. In other words, we need prove that for every ξ-closed set A

$$\langle x, A\rangle \in \lim_{\xi}\langle \mathcal{N}_{\xi}(x), \mathcal{N}_{[\xi,\$]}(A)\rangle. \tag{73}$$

In the Sierpiński topology the only case of the formula above which is not always fulfilled is when $\langle x, A\rangle = 1$ (equivalently, if $x \notin A$). In this case, there exist $V \in \mathcal{N}_{\xi}(x)$ and $\mathcal{A} \in \mathcal{N}_{[\xi,\$]}(A)$ such that $V \cap D = \varnothing$ for each $D \in \mathcal{A}$. By Corollary 91 this is equivalent to the existence of a ξ-compact family $\mathcal{B} = \ \mathcal{A}_c$ such that $O \in \mathcal{B}$ and $\bigcap_{B\in\mathcal{B}} B \supset V$ and thus $\bigcap_{B\in\mathcal{B}} B \in \mathcal{N}_{\xi}(x)$. Therefore if $\mathcal{H}$ is a filter such that $V \in \mathcal{H}^{\#}$ then $\mathcal{H}\#\mathcal{B}$ and by compactness $\mathrm{adh}_{\xi}\,\mathcal{H} \in \mathcal{B}^{\#}$ hence in particular, $\mathrm{adh}_{\xi}\,\mathcal{H}\cap O \neq \varnothing$, that is, ξ is core-compact. Conversely if V is compact at O, then the family $\mathcal{O}_{\xi}(V)$ of all ξ-open sets which contain V, is ξ-compact at O. ∎

Similarly, if $M = L = T$ Theorems 106 and 112, and Proposition 75 yield a generalization of Theorem 72:

Theorem 113. *A topological space is core-compact if and only if the product of its identity map with every quotient map is quotient.*

These results are instances of a general scheme, which enables one, for example, to characterize those ξ for which $[\xi, \$]$ is a TE-convergence.

If in the definition of core-compactness we replace the topology ξ by $T\,\mathrm{Seq}\,\xi$ (equivalently, by $T\,\mathrm{First}\,\xi$) then we get countable core-compactness: a topology ξ is called *countably core-compact* if for every element x and each $O \in \mathcal{N}_{\xi}(x)$ and each countably based filter $\mathcal{F}$ which converges to x, there exists $F \in \mathcal{F}$ which is ξ-compact at O. Also Corollary 91 is a special case of a more abstract result, which in particular gives a characterization of sequentially open subsets of $[\xi, \$]$ in terms of countably compact families of open sets (see [2]). Because a topology τ is sequential whenever $\tau = T\,\mathrm{First}\,\tau$, on replacing T by $T\,\mathrm{First}$ in Theorem 112 and in its proof, we conclude that

Theorem 114. [19] *The upper Kuratowski convergence with respect to a topology ξ is a sequential topology if and only if ξ is countably core-compact.*

On setting in Theorem 107 $J = L = T$ and $E = \mathrm{First}$ we recover this generalization of Theorem 73

Theorem 115. [19] *A topological space is countably core-compact if and only if the product of its identity map with every quotient map from a sequential topological space is quotient.*

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