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Hyperconvergences

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Abstract. The hyperconvergence (upper Kuratowski convergence) is the coarsest convergence on the set of closed subsets of a convergence space that make the canonical evaluation continuous. Sundry reflective and coreflective properties of hyperconvergences are characterized in terms of the underlying convergence.

1. Introduction

Convergences on hyperspaces (spaces of closed sets) have been attracting attention due to their numerous applications, but also for theoretic reasons. Closed sets (of a convergence) can be identified with continuous maps (from that convergence) valued in the *Sierpiński topology* $\$ = \{\varnothing, \{1\}, \{0,1\}\}$. Therefore, a hyperspace is an instance of a function space.

A convergence on a hyperspace that is of particular interest, is the coarsest among those convergences for which the natural evaluation map (from the product of a convergence space with its hyperspace) valued in the Sierpiński space is continuous. We shall call it the *hyperconvergence* (or the *upper Kuratowski convergence*). It follows from the definition that hyperconvergences constitute a particular instance of *continuous convergences*; recall that the continuous convergence (of a convergence) is the coarsest convergence on the set of continuous maps from that convergence (valued in another convergence space) for which the natural evaluation map is continuous.

On the other hand, hyperconvergences play a universal role among all continuous convergences (valued in topologies). This is because the Sierpiński topology is initially dense in the category of all topologies, which means that every topology is the initial object with respect to maps valued in the Sierpiński space [1]. Therefore, by duality, every continuous convergence valued in a topology is the initial object with respect to maps valued in hyperconvergences. This fact has important consequences for continuous convergences as well as for quotient maps.

Hyperconvergence can be characterized as the (lower) order convergence on the complete lattice of closed sets; transposed onto the lattice of open sets, it is the *Scott convergence* ([13] in the case of open sets of a topological space).

We study hyperspaces with respect to convergences that are not necessarily T_1. This is not because of a desire of utmost generality, but because of the very nature of the considered topic. Indeed the hyperconvergence (for a nonempty convergence) is never T_1, but is always T_0. Another peculiar property of hyperconvergences is *hypercompactness*: every filter converges.

To express properties of hyperconvergences in terms of the corresponding properties of the underlying convergences, is an essential element of our quest. Because hyperconvergences are defined on the spaces of closed sets, hyperconvergence properties typically correspond to hereditary properties (with respect to open sets) of the underlying convergences. For example, the hyperconvergence of a topology is topological if and only if the underlying topology is core-compact [13]. The very fact that the hyperconvergence of a topology need not be topological has been at the origin of convergence theory [5].

In this survey paper we not only present a collection of our recent results related to hyperconvergence [11], [17], [16], [15], [14], but also refine several of them and clarify certain links between them. Therefore we have tried to make this paper self-contained providing full proofs in most cases.

2. Notation

Following Engelking $A \subset X$ means that A is a (not necessarily proper) subset of X, and $A^c = X \setminus A$ when X has been fixed. We consider now families of a fixed set. A family $\mathcal{B}$ is *finer* than a family $\mathcal{A}$ (in symbols, $\mathcal{A} \leq \mathcal{B}$) if for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $B \subset A$. Families of sets $\mathcal{E}$ and $\mathcal{H}$ *mesh* (in symbols, $\mathcal{E}\#\mathcal{H}$) provided $E \cap H \neq \varnothing$ for each $E \in \mathcal{E}$ and every $H \in \mathcal{H}$; we denote by $\mathcal{E}^{\#}$ the *grill* of $\mathcal{E}$, that is, the family of all sets that intersect every element of $\mathcal{E}$. The symbol $\mathcal{E}_c$ stands for the family $\{E^c : E \in \mathcal{E}\}$. If $\mathcal{F}$ is a filter, then we denote by $\beta\mathcal{F}$ the set of all the ultrafilters which are finer than $\mathcal{F}$.

If $\mathrm{a} : 2^Y \to 2^Y$ and if $\mathcal{A}$ is a family of subsets of Y, then we denote by

$$\mathrm{a}^{\natural}\mathcal{A} = \{\mathrm{a}A : A \in \mathcal{A}\}$$

the a-*regularization* of $\mathcal{A}$. If now $\mathbb{H}$ is a class of families of sets, then $\mathrm{a}^{\natural}\mathbb{H} = \{\mathrm{a}^{\natural}\mathcal{H} : \mathcal{H} \in \mathbb{H}\}$.

On the other hand, if $\mathcal{V}(y)$ is a family of subsets of X for every $y \in Y$, then for a family $\mathcal{A}$ of subsets of Y,

$$\mathcal{V}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \bigcap_{y \in A} \mathcal{V}(y)$$

is the *contour* of $\mathcal{V}(\cdot)$ along $\mathcal{A}$. If $\mathcal{A} = \{A\}$ then we abridge $\mathcal{V}(A) = \mathcal{V}(\{A\})$. If now $X = Y$ and $\{\mathcal{V}(x) : x \in X\}$ is given and we define $x \in \mathrm{a}_{\mathcal{V}}A$ if and only if $A \in \mathcal{V}^{\#}(x)$, then we have

$$\mathcal{V}(\mathcal{A})\#\mathcal{B} \iff \mathcal{A}\#\mathrm{a}_{\mathcal{V}}^{\natural}\mathcal{B}. \tag{2.1}$$

By $\mathcal{O}_\xi(x)$ we denote the family of all ξ-open sets which contain x. Therefore, $\mathcal{O}_\xi(A)$ stands for the family of all ξ-open sets which include A and $\mathcal{O}_\xi(\mathcal{A}) = \bigcup_{A\in\mathcal{A}} \mathcal{O}_\xi(A)$.

Finally, we denote respectively by $f^-(y), f^-(B)$ and $f^-(\mathcal{B})$ the preimage by f of y, B and $\mathcal{B}$. If $H \subset X \times Y$ and $A \subset X$ then the image of A by H is denoted by HA while H^-B stands for the preimage of $B \subset Y$ by H.

3. Hyperspace

The *hyperspace* associated with a topology (more generally, with a convergence) τ is the set of all τ-closed subsets. It is well known that there exists a one-to-one correspondence between closed subsets and continuous maps which take values in a two-element set endowed with the *Sierpiński* topology $\$$. Indeed, if τ is a convergence (see Section 5) on a set X and $A \subset X$, then the indicator map

$$\langle x, A\rangle = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if } x \notin A \end{cases} \tag{3.1}$$

is continuous (in the first variable) from τ to $\$ = \{\varnothing, \{1\}, \{0,1\}\}$ if and only if A is τ-closed. Therefore $C(\tau, \$)$ stands for the set of all τ-closed sets, equivalently for the set of all maps continuous from τ to $\$$. In particular, the constant maps $\langle x, \varnothing\rangle = 1$ and $\langle x, X\rangle = 0$ correspond to the empty set and the whole of X respectively.

We shall sometimes call *hyperpoints* the elements of a hyperspace, *hypersets* the subsets of a hyperspace (that is, families of closed sets), and *hyperfilters* the filters on a hyperspace.

The Sierpiński topology plays an exceptional role among other topologies: it is *initially dense* in the category of all topologies. This means that every topology τ is the initial object with respect to maps valued in the Sierpiński topological space:

$$\tau = \bigvee_{f\in C(\tau,\$)} f^-\$, \tag{3.2}$$

where $f^-\theta$ denotes the initial convergence of θ by f. Indeed, the statement above is an immediate consequence of the representation of closed sets with the aid of the coupling map (3.1) continuous in the first variable for every fixed A.

4. Polarities, Galois connections

We will see later that minimality of hyperconvergence can be rephrased in terms of polarity. A map $f : X \to Y$ (between complete lattices X, Y) is called an *inverse polarity* whenever

$$f(\bigwedge A) = \bigvee f(A)$$

for every $A \subset X$. Each inverse polarity admits its *adjoint map* $f^*(y) = \bigwedge\{x : y \geq f(x)\}$ which is also a polarity. A couple of maps

$$f : X \rightleftarrows Y : g \tag{4.1}$$

is called a $(-,-)$-*connection* if f and g are order-inverting and if gf and fg are *contractive*, that is, $gf(x) \leq x$ and $fg(y) \leq y$. It is well known [3] that (4.1) is a $(-,-)$-connection if and only if f is an inverse polarity and $g = f^*$. It is often handy to invert order on one or both the lattices X, Y. If we invert the order on X only, then we obtain a $(+,-)$-*connection*, with f and g order-preserving, with gf *expansive* (that is, $gf(x) \geq x$) and fg contractive. Similarly we define $(-,+)$-connections and $(+,+)$-connections, and refer to all of them as *Galois connections.*

A *projector* is an order-preserving, contractive and idempotent map, while an order-preserving, expansive, idempotent map is called a *coprojector*. An important consequence of the definitions above is that if (4.1) is a $(-,-)$-connection, then $f = fgf$ and $gfg = g$, hence gf and fg are projectors. Moreover, the restrictions

$$f : \operatorname{fix}(gf) \rightleftarrows \operatorname{fix}(fg) : g$$

constitute a couple of reciprocal lattice inverse isomorphisms. Of course, if (4.1) is a $(+,-)$-connection, then gf is a coprojector while fg is a projector. Finally, let us mention that (4.1) is a $(-,-)$-connection if and only if $f(x) \geq y$ is equivalent to $x \leq g(y)$ for every x and y.

5. Convergence

We refer to [11], [17] for details of convergence theory. A *convergence* ξ on a set X is a relation between filters $\mathcal{F}$ on X and elements x of X

$$x \in \lim_\xi \mathcal{F}$$

such that $\lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G}$ if $\mathcal{F} \leq \mathcal{G}$ and $x \in \lim_\xi \{x\}^\uparrow$ for each $x \in X$, where $\{x\}^\uparrow$ stands for the principal ultrafilter of x; the latter condition is that of *strictness*. Please, have in mind that various authors give various (slightly different) definitions of convergence; a most frequent additional axiom is the one that we use to define prototopologies (see Section 11).

The *adherence* of a family $\mathcal{H}$ of sets is defined by

$$\operatorname{adh}_\xi \mathcal{H} = \bigcup_{\mathcal{E} \# \mathcal{H}} \lim_\xi \mathcal{E}.$$

If ξ is a convergence then $|\xi|$ stands for the set on which ξ is defined. A convergence ζ is *finer* then a convergence ξ (ξ is *coarser* than ζ) whenever $\lim_\zeta \mathcal{F} \subset \lim_\xi \mathcal{F}$ for every filter $\mathcal{F}$. If ξ is a convergence on X and τ is a convergence on Y, then a map $f : X \to Y$ is *continuous* if $f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F})$. The *final* convergence (the finest convergence on Y for which f is continuous from ξ) is denoted by $f\xi$ and the *initial* convergence (the coarsest convergence on X for which f is continuous to τ) is denoted by $f^-\tau$.

A set V is a *ξ-vicinity* of x if $V \in \mathcal{F}$ for every filter $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F}$; the family $\mathcal{V}_\xi(x)$ of vicinities of x is a filter. A convergence ξ is a pretopology if and only if $x \in \lim_\xi \mathcal{V}_\xi(x)$ for every x. A set O is *ξ-open* if $O \in \mathcal{F}$ for every filter $\mathcal{F}$ such that $\lim_\xi \mathcal{F} \cap O \neq \varnothing$ (*ξ-closed* if its complement is ξ-open). In other words, a set is ξ-open if and only if it is a ξ-vicinity of each its elements. A set W is a *ξ-neighborhood* of x if it includes a ξ-open set that contains x; the set of all ξ-neighborhoods of x is denoted by $\mathcal{N}_\xi(x)$. A convergence ξ is a topology if and only if $x \in \lim_\xi \mathcal{N}_\xi(x)$ for every x. Of course, a topological closure $a_{\mathcal{N}_\xi} = \mathrm{cl}_\xi$ is defined via (2.1). It is the closure for a topology $T\xi$ called *topologization* of ξ.

A family $\mathfrak{G}$ of filters is a *base* for a convergence ξ whenever $x \in \lim_\xi \mathcal{F}$ implies the existence of $\mathcal{G} \in \mathfrak{G}$ such that $\mathcal{G} \leq \mathcal{F}$ and $x \in \lim_\xi \mathcal{G}$.

6. Hyperconvergence

The *hyperconvergence* (also called the *upper Kuratowski convergence*) is the power convergence with respect to the Sierpiński topology. In other words, if ξ is a convergence on X, then the *hyperconvergence* $[\xi, \$]$ associated with ξ is the coarsest convergence on $C(\xi, \$)$ such that $\langle \cdot, \cdot \rangle : \xi \times C(\xi, \$) \to \$$ is continuous. We say that ξ is a *primal* convergence of $[\xi, \$]$ and that $[\xi, \$]$ is the *dual* convergence of ξ. It follows directly from the definition of power convergence that $A_0 \in \lim_{[\xi,\$]} \mathfrak{G}$ if and only if for every $x \in |\xi|$ and each filter $\mathcal{F}$ on $|\xi|$ such that $x \in \lim_\xi \mathcal{F}$,

$$\langle x, A_0 \rangle \in \lim_{\$} \langle \mathcal{F}, \mathfrak{G} \rangle. \tag{6.1}$$

Here $\mathfrak{G}$ is a hyperfilter and A_0 is a hyperpoint. The filter $\langle \mathcal{F}, \mathfrak{G} \rangle$ is generated by $\langle F, \mathcal{G} \rangle = \{\langle x, G \rangle : x \in F, G \in \mathcal{G}\}$.

It is possible to characterize the hyperconvergence in terms of the primal convergence. To this end we will need a notion of reduced filter. Let us have in mind that if $\mathfrak{G}$ is a ξ-hyperfilter (that is, a filter on $C(\xi, \$)$) then every $\mathcal{G} \in \mathfrak{G}$ is a ξ-hyperset (that is, a family of ξ-closed sets). The *reduced filter* of a hyperfilter $\mathfrak{G}$ is defined in [10] by

$$\mathrm{r}\mathfrak{G} \approx \{ \bigcup_{A \in \mathcal{G}} A : \mathcal{G} \in \mathfrak{G} \},$$

where $\mathcal{F} \approx \mathcal{B}$ means that $\mathcal{F}$ is generated by $\mathcal{B}$. Notice that the non degeneracy of a hyperfilter $\mathfrak{G}$ does not imply the non degeneracy of the reduced filter $\mathrm{r}\mathfrak{G}$; indeed, $\mathrm{r}\mathfrak{G}$ is degenerate if and only if there exists $\varnothing \neq \mathcal{G} \in \mathfrak{G}$ such that $\bigcup_{A \in \mathcal{G}} A = \varnothing$, that is, whenever $\mathcal{G} = \{\varnothing\}$ which means that $\mathfrak{G} = \{\varnothing\}^{\uparrow}$ (the principal ultrafilter of the empty set $\varnothing$). Of course, $\lim_{[\xi,\$]} \{\varnothing\}^{\uparrow}$ contains the hyperpoint $\varnothing$ and thus every ξ-closed subset A of $|\xi|$. On the other hand, $\mathrm{adh}_\xi \mathrm{r}(\{\varnothing\}^{\uparrow}) = \mathrm{adh}_\xi \varnothing^{\uparrow} = \varnothing$.

Proposition 6.1. [10] *Let $\mathfrak{G}$ be a hyperfilter on $C(\xi, \$)$. Then*

$$A_0 \in \lim_{[\xi,\$]} \mathfrak{G} \Longleftrightarrow \mathrm{adh}_\xi \mathrm{r}\mathfrak{G} \subset A_0. \tag{6.2}$$

Proof. We have already noted that $A_0 \in \lim_{[\xi,\$]}\mathfrak{G}$ is equivalent to (6.1) for every $x \in |\xi|$ and each filter $\mathcal{F}$ on $|\xi|$ such that $x \in \lim_\xi \mathcal{F}$. Because the only \$-neighborhood of 0 is the whole of $\{0,1\}$, the formula above is significant only in the case of $\langle x, A_0\rangle = 1$, that is, $x \notin A_0$. Therefore (6.1) means that if $x \notin A_0$ then there exist $F \in \mathcal{F}$ and $\mathcal{G} \in \mathfrak{G}$ such that $F \cap \bigcup_{A\in\mathcal{G}} A = \varnothing$, that is, $\mathcal{F}$ does not mesh with $\mathrm{r}\mathfrak{G}$. We conclude that if $x \notin A_0$, then $x \notin \mathrm{adh}_\xi \mathrm{r}\mathfrak{G}$, and thus the proof is complete. □

Corollary 6.2. *A hyperfilter $\mathfrak{G}$ converges to A_0 in $[\xi,\$]$ if and only if for every filter $\mathcal{F}$ such that $\lim_\xi \mathcal{F} \setminus A_0 \neq \varnothing$, there is $\mathcal{G} \in \mathfrak{G}$ for which $\bigcap_{A\in\mathcal{G}} A^c \in \mathcal{F}$.*

Proof. By (6.2) $A_0 \in \lim_{[\xi,\$]}\mathfrak{G}$ if and only if every filter $\mathcal{F}$ which meshes with $\mathrm{r}\mathfrak{G}$ does not ξ-converge to an element of A_0^c. Therefore $\lim_\xi \mathcal{F} \cap A_0^c \neq \varnothing$ implies that $\mathcal{F}$ does not mesh with $\mathrm{r}\mathfrak{G}$, that is, there exist $\mathcal{G} \in \mathfrak{G}$ and $F \in \mathcal{F}$ such that $F \cap \bigcup_{A\in\mathcal{G}} A = \varnothing$, in other words, $\bigcap_{A\in\mathcal{G}} A^c \in \mathcal{F}$. □

A convergence is *hypercompact* provided that every filter converges. It follows from (6.2) that

Proposition 6.3. *Each hyperconvergence is hypercompact.*

Proof. Indeed $|\xi| \in \lim_{[\xi,\$]}\mathfrak{G}$ for every filter $\mathfrak{G}$, because $\mathrm{adh}_\xi \mathrm{r}\mathfrak{G} \subset |\xi|$. □

A convergence is said to be T_1 if all its singletons are closed. Except for the trivial case (of the discrete topology on a singleton), no hypercompact convergence is T_1; in fact, if τ is a hypercompact convergence on Y and $y_0 \neq y_1$, then $\varnothing \neq \lim_\tau \{y_0, y_1\}^\uparrow \subset \lim_\tau \{y_0\}^\uparrow \cap \lim_\tau \{y_1\}^\uparrow$. Hence if τ were T_1, then $\lim_\tau \{y_0\}^\uparrow \cap \lim_\tau \{y_1\}^\uparrow = \varnothing$.

Proposition 6.4. *If $|\xi|$ is non empty, then the hyperconvergence $[\xi,\$]$ is not T_1.*

More precisely, if $A \in C(\xi,\$)$, then

$$\mathrm{cl}_{[\xi,\$]}\{A\} = \{B \in C(\xi,\$) : A \subset B\}, \tag{6.3}$$

so that $|\xi| \in \mathrm{cl}_{[\xi,\$]}\{\varnothing\}$. Therefore, if $X \neq \varnothing$ and ξ is any convergence on X, then the ξ-closed sets $\varnothing$ and X are distinct and $X \in \mathrm{cl}_{[\xi,\$]}\{\varnothing\}$.

Proposition 6.5. *Each hyperconvergence is T_0.*

Proof. Indeed, if $A_0, A_1 \in C(\xi,\$)$ and $x \in A_1 \setminus A_0$, then by (6.3) $x \in A$ for every $A \in \mathrm{cl}_{[\xi,\$]}\{A_1\}$, hence $A_0 \notin \mathrm{cl}_{[\xi,\$]}\{A_1\}$. □

The hyperconvergence of the (unique) convergence on a singleton (this is necessarily the discrete topology ι) is homeomorphic to the Sierpiński topology. In fact the (ι-closed) subsets of $\{x\}$ are $\varnothing$ and $\{x\}$; for $[\iota,\$]$, the principal hyperfilter of $\varnothing$ converges to $\varnothing$ and to $\{x\}$, while the principal hyperfilter of $\{x\}$ converges to $\{x\}$; hence the map which associates $\{x\}$ to 0 and $\varnothing$ to 1 is a homeomorphism.

7. Point and star topologies

With a view to characterizing reduced filters, we associate with each convergence two special topologies [4], [11]. If ξ is a convergence on X, then $\mathrm{cl}_{\xi^\bullet} x = \mathrm{cl}_\xi\{x\}$ defines a (binary) relation on X. For our purposes, it is handy to use a special notation for the inverse relation, namely,

$$\mathrm{cl}_{\xi^*} y = \mathrm{cl}^-_{\xi^\bullet} y = \{x : y \in \mathrm{cl}_\xi\{x\}\}.$$

As usual, the image and the preimage of sets by a relation are

$$\mathrm{cl}_{\xi^\bullet} A = \bigcup_{x\in A} \mathrm{cl}_\xi\{x\} \text{ and } \mathrm{cl}_{\xi^*} B = \{x : \mathrm{cl}_\xi\{x\} \cap B \neq \varnothing\} = \bigcup_{y\in B} \mathrm{cl}_{\xi^*} y.$$

Both the operations $\mathrm{cl}_{\xi^\bullet}$ and cl_{ξ^*} turn out to be topological closures. In fact, it is straightforward that they are expansive and that $\mathrm{cl}_{\xi^\bullet}\varnothing = \mathrm{cl}_{\xi^*}\varnothing = \varnothing$. The idempotency of $\mathrm{cl}_{\xi^\bullet}$ and of cl_{ξ^*} follows immediately from that of cl_ξ. As for finite additivity, we get more than needed: as the operations are relations, they are (fully) additive:

$$\mathrm{cl}_{\xi^\bullet}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \mathrm{cl}_{\xi^\bullet} A_i; \quad \mathrm{cl}_{\xi^*}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \mathrm{cl}_{\xi^*} A_i.$$

The topology $\xi^\bullet$ is called the *point topology* of ξ and the topology ξ^* is called the *star topology* of ξ. If ξ is T_1, then $\xi^\bullet$ and ξ^* are equal to the discrete topology. Of course, $\xi^\bullet \geq \xi$, but this is not necessarily the case with ξ^*.

Example 7.1. Consider the discrete topology ι on $\{0,1\}$. Then the hyperconvergence $[\iota, \$]$ is a topology and for instance $\mathrm{cl}_{[\iota,\$]^\bullet}\{\{0\}\} = \mathrm{cl}_{[\iota,\$]}\{\{0\}\} = \{\{0\},\{0,1\}\}$ while $\mathrm{cl}_{[\iota,\$]^*}\{\{0\}\} = \{\varnothing, \{0\}\}$.

As both the point- and the star- closures are additive, the collection of all $\xi^\bullet$-closed (respectively, ξ^*-closed) sets is also that of open sets of some topology.

Proposition 7.2. *For every convergence ξ, a set is $\xi^\bullet$-closed if and only if it is ξ^*-open.*

Proof. If $y \notin \mathrm{cl}_{\xi^\bullet} A = A$ then $x \notin \mathrm{cl}_{\xi^*} y$ for every $x \in A$, that is, $A \cap \mathrm{cl}_{\xi^*} y = \varnothing$. In other words, $\mathrm{cl}_{\xi^*} A^c \subset A^c$. □

The topology $\xi^\bullet$ has been defined with the aid of the ξ-closures of singletons. Because there exists a unique filter that contains the singleton $\{x\}$, namely the principal ultrafilter $\{x\}^\uparrow$, $\mathrm{cl}_{\xi^\bullet} x = \mathrm{cl}_\xi\{x\} = \lim_{\mathrm{T}\xi}\{x\}^\uparrow$. Therefore $y \in \lim_{\xi^\bullet} \bigwedge_{x\in \mathrm{cl}_{\xi^*} y}\{x\}^\uparrow = \lim_{\xi^\bullet}\{\mathrm{cl}_{\xi^*} y\}^\uparrow$ and thus $\mathrm{cl}_{\xi^*} y$ is a neighborhood of y with respect to $\xi^\bullet$. Actually, another way of proving that a set is $\xi^\bullet$-open if and only if it is ξ^*-closed is to notice that

Lemma 7.3. *$\mathcal{N}_{\xi^\bullet}(x)$ is the principal filter generated by $\mathrm{cl}_{\xi^*} x$.*

Proof. Observe that $W \in \mathcal{N}_{\xi^\bullet}(x)$ whenever $x \in \lim_{\xi^\bullet}\{y\}^\uparrow = \mathrm{cl}_{\xi^\bullet} y$ implies that $y \in W$, equivalently $\mathrm{cl}_{\xi^*} x \subset W$. This means that $\mathrm{cl}_{\xi^*} x$ is the least $\xi^\bullet$-neighborhood of x. □

Therefore, by (2.1), for two families $\mathcal{A}$ and $\mathcal{B}$,

$$\mathrm{cl}^{\natural}_{\xi^*}\mathcal{A}\#\mathcal{B} \Longleftrightarrow \mathcal{A}\#\mathrm{cl}^{\natural}_{\xi^\bullet}\mathcal{B}. \tag{7.1}$$

8. Reduced and erected filters

If ξ is a convergence on X, then for each subset H of X, we define the *erected set* $\mathrm{e}_\xi H = \{A = \mathrm{cl}_\xi A : A \subset H\}$, and for a filter $\mathcal{H}$ on X, the *erected filter* of $\mathcal{H}$

$$\mathrm{e}^{\natural}_{\xi}\mathcal{H} = \{\mathrm{e}_\xi H : H \in \mathcal{H}\}. \tag{8.1}$$

The erected filter of a non degenerate filter is never degenerate. Indeed, $\{\varnothing\}$ belongs to every element of an erected filter.

The (possibly degenerate) filter $\mathrm{re}^{\natural}_{\xi}\mathcal{H}$ is generated by the sets of the form $\mathrm{int}_{\xi^*} H = \bigcup_{\mathrm{cl}_\xi A = A \subset H} A$, where $H \in \mathcal{H}$, that is, $\mathrm{int}^{\natural}_{\xi^*}\mathcal{H} = \mathrm{re}^{\natural}_{\xi}\mathcal{H}$, and thus

$$\mathrm{re}^{\natural}_{\xi}\mathcal{H} \geq \mathcal{H}.$$

A hyperset $\mathcal{G}$ is *ξ-stable* if $B = \mathrm{cl}_\xi B \subset G \in \mathcal{G}$ implies $B \in \mathcal{G}$. Notice that $\mathcal{G}$ is ξ-stable if and only if $\mathcal{G}_c = \mathcal{O}_\xi(\mathcal{G}_c)$. A hyperset $\mathcal{G}$ is *ξ-saturated* if $B = \mathrm{cl}_\xi B \subset \bigcup_{A \in \mathcal{G}} A$ implies that $B \in \mathcal{G}$. A hyperfilter $\mathfrak{G}$ on $C(\xi, \$)$ is *ξ-stable* (respectively, *ξ-saturated*) if it admits a filter base consisting of ξ-stable (respectively, ξ-saturated) hypersets. It is straightforward that

Proposition 8.1. *A hyperfilter is an erected filter if and only if it is saturated.*

Proposition 8.2. *Let $\mathcal{F}$ be a ξ-reduced filter and let $\mathfrak{G}$ be a stable ξ-hyperfilter. Then $\mathcal{F}$ meshes with $\mathrm{r}\mathfrak{G}$ if and only if the filters $\mathfrak{G}$ and $\mathrm{e}^{\natural}_{\xi}\mathcal{F}$ restricted to $C(\xi, \$) \setminus \{\varnothing\}$ mesh.*

Proof. By definition, a ξ-reduced filter $\mathcal{F}$ meshes with $\mathrm{r}\mathfrak{G}$ if and only if $\bigcup_{\mathrm{cl}_\xi A = A \subset F} A \cap \bigcup_{B \in \mathcal{G}} B \neq \varnothing$ for every $F \in \mathcal{F}$ and $\mathcal{G} \in \mathfrak{G}$. Equivalently, there is a non empty ξ-closed set $A \in \mathrm{e}_\xi F \cap \mathcal{G}$ for every $F \in \mathcal{F}$ and $\mathcal{G} \in \mathfrak{G}$, because $\mathfrak{G}$ is stable. □

If now $\mathfrak{G}$ is an arbitrary hyperfilter on $C(\xi, \$)$, then $\mathrm{e}^{\natural}_{\xi}\mathrm{r}\mathfrak{G}$ is coarser than $\mathfrak{G}$. Indeed, a base of $\mathrm{r}\mathfrak{G}$ consists of $\bigcup_{A \in \mathcal{G}} A$ where $\mathcal{G} \in \mathfrak{G}$; therefore, $\mathrm{e}^{\natural}_{\xi}\mathrm{r}\mathfrak{G}$ admits a base of the form $\{B = \mathrm{cl}_\xi B \subset \bigcup_{A \in \mathcal{G}} A\}$ with $\mathcal{G} \in \mathfrak{G}$, and since $\mathcal{G} \subset \{B = \mathrm{cl}_\xi B \subset \bigcup_{A \in \mathcal{G}} A\}$,

$$\mathrm{e}^{\natural}_{\xi}\mathrm{r}\mathfrak{G} \leq \mathfrak{G}.$$

It follows that

Proposition 8.3. *The hyperconvergence $[\xi, \$]$ has a convergence base consisting of saturated hyperfilters.*

Proof. If $A \in \lim_{[\xi,\$]}\mathfrak{G}$, then $\mathrm{adh}_\xi \mathrm{r}(\mathrm{e}^{\natural}_{\xi}\mathrm{r}\mathfrak{G}) = \mathrm{adh}_\xi \mathrm{r}\mathfrak{G} \subset A$, hence $A \in \lim_{[\xi,\$]}\mathrm{e}^{\natural}_{\xi}\mathrm{r}\mathfrak{G}$; as $\mathrm{e}^{\natural}_{\xi}\mathrm{r}\mathfrak{G}$ is erected, it is saturated. □

Proposition 8.4. [16] *A filter is a reduced filter if and only if it is regular for the point topology if and only if it is open for the star topology.*

Proof. If $\mathcal{G} \in \mathfrak{G}$, and if $x \in \bigcup_{A \in \mathcal{G}} A$, then there is $A \in \mathcal{G}$ and thus $\mathrm{cl}_\xi\{x\} \subset \mathrm{cl}_\xi A = A \subset \bigcup_{A \in \mathcal{G}} A$, so that $\mathrm{r}\mathfrak{G}$ is regular for the point topology of ξ. Conversely, if $\mathrm{cl}_{\xi^\bullet} H \subset H$, then $\bigcup_{\mathrm{cl}_\xi A = A \subset H} A \subset \bigcup_{x \in H} \mathrm{cl}_\xi\{x\} = \mathrm{cl}_{\xi^\bullet} H \subset H$, that is, if $\mathcal{H}$ is regular for $\xi^\bullet$, then $\mathrm{re}_\xi^\natural \mathcal{H} = \mathcal{H}$. By definition, a filter is $\xi^\bullet$-regular if and only if it admits a base of $\xi^\bullet$-closed sets, that is, by Proposition 7.2, of ξ^*-open sets; equivalently the filter is ξ^*-open. □

In general

$$\mathrm{int}_{\xi^*}^\natural \mathcal{H} = \mathrm{re}_\xi^\natural \mathcal{H} \geq \mathcal{H} \geq \mathrm{cl}_{\xi^\bullet}^\natural \mathcal{H}.$$

By the proposition above, $\mathcal{H} = \mathrm{r}\mathfrak{G}$ for some hyperfilter $\mathfrak{G}$ on $C(\xi, \$)$ if and only if $\mathrm{int}_{\xi^*}^\natural \mathcal{H} = \mathcal{H} = \mathrm{cl}_{\xi^\bullet}^\natural \mathcal{H}$. Consequently, we shall use the term ξ-*reduced* filter for *regular for the point topology of* ξ. Let us denote by $\mathrm{cl}_{\xi^\bullet}^\natural \mathbb{S}$ the set of filters that are regular for the point topology of ξ.

If we observe that $\mathfrak{G}_0 \leq \mathfrak{G}_1$ implies $\mathrm{r}\,\mathfrak{G}_0 \leq \mathrm{r}\mathfrak{G}_1$ and $\mathcal{H}_0 \leq \mathcal{H}_1$ implies $\mathrm{e}_\xi^\natural \mathcal{H}_0 \leq \mathrm{e}_\xi^\natural \mathcal{H}_1$, then we have proved that

Theorem 8.5. *The operations of erection* $\mathrm{e}_\xi^\natural$ *and of reduction* r *constitute a Galois* $(+,-)$*-connection. Hence they form a lattice isomorphism between reduced filters and filters based in saturated families.*

It follows that for every family $\{\mathcal{H}_i : i \in I\}$ of filters on $|\xi|$,

$$\mathrm{e}_\xi^\natural \left(\bigvee_{i \in I} \mathcal{H}_i \right) = \bigvee_{i \in I} \mathrm{e}_\xi^\natural\, \mathcal{H}_i, \tag{8.2}$$

and for every family $\{\mathfrak{G}_i : i \in I\}$ of filters on $C(\xi, \$)$,

$$\mathrm{r}(\bigwedge_{i \in I} \mathfrak{G}_i) = \bigwedge_{i \in I} \mathrm{r}\mathfrak{G}_i. \tag{8.3}$$

Another immediate consequence of general properties of Galois connections, is that

$$\mathcal{H} \leq \mathrm{r}\mathfrak{G} \Longleftrightarrow \mathrm{e}_\xi^\natural \mathcal{H} \leq \mathfrak{G}. \tag{8.4}$$

Because $[\xi, \$]$ has a convergence base consisting of ξ-saturated filters,

$$\mathrm{adh}_{[\xi,\$]} \mathfrak{G} = \mathrm{adh}_{[\xi,\$]} \mathrm{e}_\xi^\natural \mathrm{r}\, \mathfrak{G}. \tag{8.5}$$

9. Reflective and coreflective properties of convergences

The category of convergences with continuous maps as morphisms is concrete over the category of sets. This means that every morphism from ξ to τ is uniquely determined by a map from $|\xi|$ to $|\tau|$. A (covariant) *functor* F is a map on morphisms which preserves the composition; as every object ξ can be identified with the identity morphism $\iota_\xi : \xi \to \xi$, each functor F maps also objects following the rule $\iota_{F(\xi)} = F(\iota_\xi)$. A functor F between two concrete categories (over the category of sets) is *concrete* whenever each morphism and its image by F have the same underlying map.

A concrete subcategory of convergences is *reflective* (respectively, *coreflective*) if it is stable for initial (respectively, final) convergences; in other words, $\mathbf{M}$ is reflective if $\tau \in \mathbf{M}$ implies $f^{-}\tau \in \mathbf{M}$, and $\bigvee_{i\in I} \tau_i \in \mathbf{M}$ whenever $\tau_i \in \mathbf{M}$ for each $i \in I$ ($\mathbf{M}$ is coreflective if $\xi \in \mathbf{M}$ implies $f\xi \in \mathbf{M}$, and $\bigwedge_{i\in I} \xi_i \in \mathbf{M}$ whenever $\xi_i \in \mathbf{M}$ for each $i \in I$).

Every concrete functor in the category of convergences is determined by its action on objects [1, Remark 5.3]. An order-preserving map M on convergences such that $|M\xi| = |\xi|$, is a concrete functor if and only if $M(f^{-}\tau) \geq f^{-}(M\tau)$ (equivalently, $f(M\xi) \geq M(f\xi)$) for every f and every τ on the domain (respectively, each ξ on the range) of f. A concrete functor is called a *reflector* if it is a projector, a *coreflector* if it is a coprojector.

From our point of view, a property of convergences is tantamount to a subcategory of convergences. We shall consider reflective properties, like topologicity, pretopologicity (corresponding to reflective subcategories) and coreflective properties, like countable character, Fréchetness, sequentiality (corresponding to coreflective subcategories). A reflective concrete subcategory can be described as the class of those convergences τ for which

$$M\tau \geq \tau, \tag{9.1}$$

where M is a concrete functor; a coreflective concrete subcategory is the class of those convergences τ for which

$$\tau \geq M\tau, \tag{9.2}$$

where M is a concrete functor.

If $\mathbb{J}$ is a class of filters possibly depending on convergence, then a convergence ξ is called *adherence-determined* by $\mathbb{J}$ if $\lim_\xi \mathcal{F} = \bigcap_{\mathbb{J}(\xi)\ni\mathcal{H}\#\mathcal{F}} \operatorname{adh}_\xi \mathcal{H}$ for every filter $\mathcal{F}$ [7]. Let

$$\lim_{\mathrm{A}_{\mathbb{J}}\xi} \mathcal{F} = \bigcap_{\mathbb{J}(\xi)\ni\mathcal{H}\#\mathcal{F}} \operatorname{adh}_\xi \mathcal{H}. \tag{9.3}$$

We assume that $\zeta \leq \xi$ implies $\mathbb{J}(\zeta) \subset \mathbb{J}(\xi)$, $\mathbb{J}(\mathrm{A}_{\mathbb{J}}\xi) = \mathbb{J}(\xi)$ and that $\mathcal{G} \in \mathbb{J}(\xi)$ implies that (the filter generated by) $f^{-}(\mathcal{G})$ belongs to $\mathbb{J}(\xi)$. Then $\mathrm{A}_{\mathbb{J}}$ is a reflector. If $\mathbb{J}$ does not depend on convergence, then the first two conditions are automatically fulfilled. A convergence is a *pseudotopology, paratopology, pretopology* if it is adherence-determined by the class $\mathbb{S}$ (of all filters), $\mathbb{P}_\omega$ (of countably based filters), $\mathbb{P}$ (of principal filters), respectively. These classes of filters are independent of convergence; moreover the preimage by a map of a filter from any of these classes, remains in the class. The corresponding reflectors S, P_ω, P are called the *pseudotopologizer*, the *paratopologizer* and the *pretopologizer*. A convergence is a *topology* if it is adherence-determined by the class $\mathbb{T}$ of principal filters of closed sets. The corresponding reflector T is the *topologizer*.

A convergence ξ is called $\mathbb{E}$-*based* (or, *based in* $\mathbb{E}$) if $x \in \lim_\xi \mathcal{F}$ implies the existence of a filter $\mathcal{G} \in \mathbb{E}$ such that $\mathcal{G} \leq \mathcal{F}$ and $x \in \lim_\xi \mathcal{G}$ [7]. Let

$$\lim_{\mathrm{B}_\mathbb{E}\xi} \mathcal{F} = \bigcup_{\mathbb{E}(\xi) \ni \mathcal{G} \leq \mathcal{F}} \lim_\xi \mathcal{G}. \tag{9.4}$$

We assume that $\zeta \leq \xi$ implies $\mathbb{E}(\zeta) \subset \mathbb{E}(\xi)$ and $\mathbb{E}(\mathrm{B}_\mathbb{E}\xi) = \mathbb{E}(\xi)$ and that $\mathcal{G} \in \mathbb{E}$ implies that (the filter generated by) $f(\mathcal{G})$ belongs to $\mathbb{E}$. Then $\mathrm{B}_\mathbb{E}$ is a coreflector. If $\mathbb{E}$ does not depend on convergence, then the first two conditions are automatically fulfilled. For instance, a convergence is *first-countable* (or of *countable character*) if and only if it is based in the class $\mathbb{P}_\omega$ (of countably based filters); we denote by $\mathrm{First} = \mathrm{B}_{\mathbb{P}_\omega}$ the corresponding coreflector.

We shall also consider the categories of convergences τ characterized by the inequality

$$\tau \geq JE\tau,$$

where J is a concrete reflector and E is a concrete coreflector. Such categories are coreflective, by (9.2). In particular, τ is *bisequential* if $\tau \geq S\mathrm{First}\tau$, *strongly Fréchet* if $\tau \geq P_\omega \mathrm{First}\tau$, *Fréchet* if $\tau \geq P\mathrm{First}\tau$ and *sequential* if $\tau \geq T\mathrm{First}\tau$.

10. Power convergence

The *power convergence* (or the *continuous convergence*) $[\xi, \sigma]$ of ξ with respect to σ is the coarsest convergence on the set $C(\xi, \sigma)$ of continuous maps from ξ to σ for which the natural evaluation map is continuous. This is the only convergence structure satisfying the *exponential law* for arbitrary convergences ξ, τ, σ:

$$[\xi \times \tau, \sigma] = [\tau, [\xi, \sigma]], \tag{10.1}$$

where $=$ stands for a homeomorphism via the transposition map ${}^t f(y)(x) = f(x, y)$. Of course, hyperconvergences constitute a special case of power convergences, namely with respect to $\sigma = \$$.

Recall that for a map $f : X \to Y$, the map $f^* : Z^Y \to Z^X$ is defined by $f^*(h) = h \circ f$. It turns out that if $f : \xi \to \tau$ is a continuous map, then (the restriction of) $f^* : [\tau, \sigma] \to [\xi, \sigma]$ is also continuous. In particular, if $\xi \geq \tau$ then $C(\tau, \sigma) \subset C(\xi, \sigma)$ and the injection from $[\tau, \sigma]$ to $[\xi, \sigma]$ is continuous. Moreover,

$$[\bigwedge_{i \in I} f_i \tau_i, \sigma] = \bigvee_{i \in I} (f_i^*)^- [\tau_i, \sigma] \tag{10.2}$$

for every family of surjective maps $\{f_i : i \in I\}$ with a common range.

If $g : W \to Z$, then the map $g_* : W^X \to Z^X$ is defined by $g_*(h) = g \circ h$. Let us observe that if $g : \sigma \to \theta$ is a continuous map, then for every ξ the map $g_* : [\xi, \sigma] \to [\xi, \theta]$ is also continuous. By (3.2),

$$[\xi, \sigma] = [\xi, \bigvee_{g \in C(\sigma, \$)} g^- \$] = \bigvee_{g \in C(\sigma, \$)} g_*^- [\xi, \$], \tag{10.3}$$

so that every continuous convergence with respect to a topology σ is the initial convergence with respect to hyperconvergences. This fact implies that

Theorem 10.1. *If for a given convergence ξ, the hyperconvergence $[\xi,\$]$ belongs to a concrete reflective subcategory* **J** *of convergences, then $[\xi,\sigma]$ also belongs to* **J** *for every topology σ.*

A subcategory **L** of convergences is called *Cartesian-closed* if $[\xi,\sigma]\in\mathbf{L}$ whenever $\xi,\sigma\in\mathbf{L}$. The categories of convergences and of pseudotopologies are Cartesian-closed, but that of topologies is not.

Theorem 10.2. *If* **L** *is a concrete reflective subcategory of convergences and L is the corresponding reflector, then* **L** *is Cartesian-closed if and only if L commutes with finite products.*

Proof. Assume that L commutes with finite products and that $\xi,\sigma\in\mathbf{L}$. By definition, $[\xi,\sigma]$ is the least convergence (on $C(\xi,\sigma)$) for which $\xi\times[\xi,\sigma]\geq e^{-}\sigma$ (where $e^{-}\sigma$ stands for the initial convergence of σ by the natural evaluation). Because $\sigma\in\mathbf{L}$ and thus $e^{-}\sigma\in\mathbf{L}$,

$$\xi\times L[\xi,\sigma]\geq L(\xi\times[\xi,\sigma])\geq L(e^{-}\sigma)=e^{-}\sigma,$$

hence $L[\xi,\sigma]\geq[\xi,\sigma]$ proving that $[\xi,\sigma]\in\mathbf{L}$.

Conversely, suppose that $L[\xi,\sigma]\geq[\xi,\sigma]$ for every $\xi,\sigma\in\mathbf{L}$. In order to prove that

$$\xi\times L\tau\geq L(\xi\times\tau)$$

for all convergences ξ and τ, it is enough to show that $C(\xi\times\tau,\sigma)\subset C(\xi\times L\tau,\sigma)$ for each $\sigma=L\sigma$. If $f\in C(\xi\times\tau,\sigma)$, then ${}^{t}f\in C(\tau,[\xi,\sigma])$ hence ${}^{t}f\in C(L\tau,L[\xi,\sigma])$ because L is a functor. It follows that

$$({}^{t}f)^{*}\in C([L[\xi,\sigma],\sigma],[L\tau,\sigma])\subset C([[\xi,\sigma],\sigma],[L\tau,\sigma]),$$

because $[L[\xi,\sigma],\sigma]\geq[[\xi,\sigma],\sigma]$ by virtue of the assumption. As the injection $i_{|\xi|}$ is continuous from ξ to $[[\xi,\sigma],\sigma]$, and $({}^{t}f)^{*}\circ i_{|\xi|}=f$, we conclude that $f\in C(\xi\times L\tau,\sigma)$. □

Let $X\subset Y,\theta$ be a convergence on X and τ be a convergence on Y. If the injection $i_X:X\to Y$ is continuous, then we write $\theta\rhd\tau$. It follows that $i_X^{*}:C(\tau,\sigma)\to C(\theta,\sigma)$ is continuous from $[\tau,\sigma]$ to $[\theta,\sigma]$; on the other hand, $i_X^{*}(f)=f\circ i_X$, that is, i_X^{*} is the restriction to X of maps on Y.

In the special case of $\sigma=\$$, the restriction $i_X^{*}(A)$ is equivalent to the intersection of a τ-closed set $A\in C(\tau,\$)$ with X. Indeed, by definition, $i_X^{*}(\psi_A)(x)=\psi_A(i_X(x))$, where ψ_A is the indicator function of a subset A of Y, hence $i_X^{*}(\psi_A)(x)=0$ if and only if an element x of X belongs to A. Accordingly, for a family $\mathcal{A}$ of θ-closed sets,

$$i_X^{**}(\mathcal{A})=\{A\in C(\tau,\$):A\in\mathcal{A}\}.$$

If $\zeta\geq\xi$, then $C(\xi,\sigma)\subset C(\zeta,\sigma)$ for every convergence σ, and the injection is continuous from $[\xi,\sigma]$ to $[\zeta,\sigma]$. Let

$$\mathrm{Epi}^{\sigma}\xi=i_{|\xi|}^{-}[[\xi,\sigma],\sigma], \tag{10.4}$$

where $i_{|\xi|}$ is the injection of $|\xi|$ in $C(C(\xi,\sigma),\sigma)$ (it is straightforward that $i_{|\xi|}$ is continuous from ξ to $[[\xi,\sigma],\sigma]$).

Consider now $[\cdot,\sigma]$ that associates with every convergence ξ on X, the power convergence $[\xi,\sigma]$. On the other hand, $[\theta,\sigma]$ is a convergence on $C(\theta,\sigma)$ for every convergence θ on a subset of $|\sigma|^X$ and each $x\in X$ can be identified with a map continuous from $[\theta,\sigma]$ to σ. Therefore, the couple of maps $[\cdot,\sigma], i_X^-[\cdot,\sigma]$ (where $i_X^-[\theta,\sigma]$ is the restriction of $[\theta,\sigma]$ to X) is a $(-,-)$-connection with respect to the usual order on convergences at one end and to $\lhd$ on the other.

It follows from the general properties of Galois connections that $[\mathrm{Epi}^\sigma\xi,\sigma]=[\xi,\sigma]$ and that Epi^σ is a projector. It is clear from the definition that Epi^σ is a concrete functor, hence it is a reflector.

11. Epitopologies

By taking the supremum on the right-hand side of (10.4) over all topologies σ, we define

$$\mathrm{Epi}\xi=\bigvee_{\sigma=\mathrm{T}\sigma}\mathrm{Epi}^\sigma\xi=\bigvee_{\sigma=\mathrm{T}\sigma} i_{|\xi|}^-[[\xi,\sigma],\sigma],$$

which in view of the preliminaries above is a concrete reflector. It is called the *epitopologizer* and each convergence τ such that $\mathrm{Epi}\tau=\tau$ is called an *epitopology* (or *Antoine* convergence).

It can be deduced from (10.3) that the epitopologizer can be represented by

$$\mathrm{Epi}\xi=i_{|\xi|}^-[[\xi,\$],\$],$$

thus $\mathrm{Epi}\xi=\mathrm{Epi}^\$\xi$. Consequently, the epitopologizer is the composition of two branches of the $(-,-)$-connection $[\cdot,\$], i_X^-[\cdot,\$]$ (which is a special case of the connection considered in the previous section). It follows, in particular, that

$$[\mathrm{Epi}\xi,\$]=[\xi,\$] \tag{11.1}$$

and that $\mathrm{Epi}\xi$ is the coarsest convergence ζ for which $[\zeta,\$]=[\xi,\$]$.

Proposition 11.1. *The epitopologizer commutes with finite products:*

$$\xi\times\mathrm{Epi}\tau\geq\mathrm{Epi}(\xi\times\tau). \tag{11.2}$$

Notice that (11.2) applied twice yields $\mathrm{Epi}\xi\times\mathrm{Epi}\tau\geq\mathrm{Epi}(\xi\times\tau)$.

Proof. As Epi is idempotent, it is enough to prove that $\xi\times\mathrm{Epi}\tau\geq\mathrm{Epi}(\xi\times\tau)$. By virtue of the exponential law (10.1) and (11.1),

$$[\xi\times\mathrm{Epi}\tau,\$]=[\xi,[\mathrm{Epi}\tau,\$]]=[\xi,[\tau,\$]]=[\xi\times\tau,\$].$$

Therefore, $\xi\times\mathrm{Epi}\tau\geq\mathrm{Epi}(\xi\times\mathrm{Epi}\tau)=\mathrm{Epi}(\xi\times\tau)$. □

Hence, in view of Theorem 10.2, the category of epitopologies is Cartesian-closed. On the other hand, every topology is an epitopology. Indeed, denote by ι_Y the discrete topology on Y. Because $\iota_{C(\xi,\$)}\geq[\xi,\$]$, the injection from $C([\xi,\$],\$)$ to $C(\iota_{C(\xi,\$)},\$)$ is continuous from $[[\xi,\$],\$]$ to $[\iota_{C(\xi,\$)},\$]$, and since the topologizer T can be characterized via

$$T\xi=\bigvee_{f\in C(\xi,\$)} f^-\$=i_{|\xi|}^-[\iota_{C(\xi,\$)},\$],$$

it is clear that $\mathrm{Epi}\xi \geq T\xi$ for every convergence ξ, which proves the claim.

We say that a convergence is a *prototopology* if $\lim_\xi \mathcal{F} \cap \lim_\xi \mathcal{G} \subset \lim_\xi(\mathcal{F} \wedge \mathcal{G})$ for every couple of filters $\mathcal{F}, \mathcal{G}$. It is immediate that prototopologies constitute a Cartesian-closed reflective subcategory of convergences.

Theorem 11.2. [4] *The category of epitopologies is the least Cartesian-closed reflective subcategory of prototopologies that includes all topologies.*

Proof. Let $\mathbf{L}$ be a Cartesian-closed reflective subcategory of prototopologies that contains all topologies. For every prototopology ξ, there exist a family $\{\tau_i : i \in I\}$ of topologies and maps $f_i : |\tau_i| \to |\xi|$ such that $\xi = \bigwedge_{i\in I} f_i\tau_i$. By virtue of (10.2),

$$[\xi, \$] = \bigvee_{i\in I} (f_i^*)^- [\tau_i, \$].$$

Because $\mathbf{L}$ is Cartesian-closed and contains all topologies, $[\tau_i, \$] \in \mathbf{L}$ for every $i \in I$, and since $\mathbf{L}$ is reflective $[\xi, \$] \in \mathbf{L}$ as the initial object with respect to prototopologies in $\mathbf{L}$. Therefore $[[\xi, \$], \$] \in \mathbf{L}$ by the Cartesian-closedness of $\mathbf{L}$, and thus the initial prototopology $\mathrm{Epi}\xi = i^-_{|\xi|}[[\xi, \$], \$]$ belongs to $\mathbf{L}$, because $\mathbf{L}$ is reflective. It follows that every epitopology belongs to $\mathbf{L}$. □

We shall now characterize epitopologies internally. We say that ξ is *point-diagonal* if

$$\lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{N}_{\xi^\bullet}(\mathcal{F}) \tag{11.3}$$

for every filter $\mathcal{F}$. If ξ is a pseudotopology, then (11.3) is equivalent to

$$\mathrm{adh}_\xi \mathrm{cl}^\natural_{\xi^\bullet} \mathcal{H} \subset \mathrm{adh}_\xi \mathcal{H} \tag{11.4}$$

for every filter $\mathcal{H}$. Indeed, if $x \in \mathrm{adh}_\xi \mathrm{cl}^\natural_{\xi^\bullet}\mathcal{H}$ and ξ is point-diagonal, then there is a filter $\mathcal{F}\#\mathrm{cl}^\natural_{\xi^\bullet}\mathcal{H}$ (equivalently, $\mathcal{H}\#\mathcal{N}_{\xi^\bullet}(\mathcal{F})$) and such that $x \in \lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{N}_{\xi^\bullet}(\mathcal{F})$, and thus $x \in \mathrm{adh}_\xi\mathcal{H}$. Conversely, if (11.4) holds, ξ is a pseudotopology, and $x \notin \lim_\xi \mathcal{N}_{\xi^\bullet}(\mathcal{F})$, then there is a filter $\mathcal{H}\#\mathcal{N}_{\xi^\bullet}(\mathcal{F})$ (equivalently, $\mathcal{F}\#\mathrm{cl}^\natural_{\xi^\bullet}\mathcal{H}$) and such that $x \notin \mathrm{adh}_\xi\mathcal{H}$, hence by (11.4) $x \notin \mathrm{adh}_\xi \mathrm{cl}^\natural_{\xi^\bullet}\mathcal{H}$ thus $x \notin \lim_\xi \mathcal{F}$.

It follows from (12.2) (to be proved later) that

$$\lim_{\mathrm{Epi}\xi} \mathcal{F} = \bigcap_{\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{S} \ni \mathcal{H}\#\mathcal{F}} \mathrm{cl}_\xi(\mathrm{adh}_\xi \mathcal{H}). \tag{11.5}$$

Therefore,

Theorem 11.3. *A convergence ξ is an epitopology if and only if it is a point-diagonal pseudotopology with closed limits.*

Proof. Let ξ be an epitopology. Then it is a pseudotopology, because $\lim_{S\xi}\mathcal{F} = \bigcap_{\mathcal{H}\#\mathcal{F}} \mathrm{adh}_\xi \mathcal{H} \subset \lim_{\mathrm{Epi}\xi}\mathcal{F}$. The closedness of limits follows from (11.5). If $x \notin \lim_\xi \mathcal{N}_{\xi^\bullet}(\mathcal{F})$, then by (11.5) there exists a ξ-reduced filter $\mathcal{H}\#\mathcal{N}_{\xi^\bullet}(\mathcal{F})$ such that $x \notin \mathrm{cl}_\xi \mathrm{adh}_\xi \mathcal{H}$. Because $\mathcal{H} \# \mathcal{N}_{\xi^\bullet}(\mathcal{F})$ is equivalent to $\mathrm{cl}^\natural_{\xi^\bullet} \mathcal{H} \# \mathcal{F}$ and since $\mathcal{H} = \mathrm{cl}^\natural_{\xi^\bullet} \mathcal{H}$, also $\mathcal{H} \# \mathcal{F}$, hence $x \notin \lim_\xi \mathcal{F}$.

Conversely, let ξ be a point-diagonal pseudotopology with closed limits, and let $x \notin \lim_\xi \mathcal{F} = \lim_\xi \mathcal{N}_{\xi^\bullet}(\mathcal{F})$. As ξ is a pseudotopology, there is an ultrafilter $\mathcal{U} \# \mathcal{N}_{\xi^\bullet}(\mathcal{F})$ (equivalently, $\mathcal{F} \# \mathrm{cl}^\natural_{\xi^\bullet} \mathcal{U}$) and such that $x \notin \mathrm{adh}_\xi \mathcal{U} = \mathrm{adh}_\xi \mathrm{cl}^\natural_{\xi^\bullet} \mathcal{U}$. Because $\mathcal{U}$ is an ultrafilter and ξ has closed limits, $\mathrm{adh}_\xi \mathcal{U} = \lim_\xi \mathcal{U} = \mathrm{cl}_\xi \mathrm{adh}_\xi \mathcal{H}$ where $\mathcal{H} = \mathrm{cl}^\natural_{\xi^\bullet} \mathcal{U}$ is a reduced filter which meshes with $\mathcal{F}$. In view of (11.5), $x \notin \lim_{\mathrm{Epi}\xi} \mathcal{F}$. □

It follows from (11.5) and (11.4) that

$$\mathrm{cl}_\xi(\mathrm{adh}_\xi(\mathrm{cl}^\natural_{\xi^\bullet} \mathcal{H})) = \mathrm{cl}_\xi(\mathrm{adh}_{\mathrm{Epi}\xi} \mathcal{H}). \tag{11.6}$$

This formula which will be instrumental in several arguments, is a special case of (12.3) that we will prove later.

Since $\xi^\bullet$ is a *principally based* (called also *finitely generated*) topology, and $\mathcal{N}_{\xi^\bullet}(x) = \mathrm{cl}_{\xi^*} x$, the filter $\mathcal{N}_{\xi^\bullet}(\mathcal{F})$ is generated by $\{\mathrm{cl}_{\xi^*} F : F \in \mathcal{F}\}$, that is, $\mathcal{N}_{\xi^\bullet}(\mathcal{F}) = \mathrm{cl}^\natural_{\xi^*} \mathcal{F}$. Therefore the condition (11.3) is equivalent to

$$\lim_\xi \mathcal{F} \subset \lim_\xi \mathrm{cl}^\natural_{\xi^*} \mathcal{F},$$

and thus

Corollary 11.4. [4] *A convergence ξ is an epitopology if and only if it is a star-regular pseudotopology with closed limits.*

Proposition 11.5. *Every hyperconvergence is an epitopology.*

Proof. ¿From $i^-_{|\xi|}[[\xi,\$],\$] \geq [[\xi,\$],\$]$ it follows that $[i^-_{|\xi|}[[\xi,\$],\$],\$] \lhd [[[\xi,\$],\$],\$]$, hence $[\xi,\$] = [i^-_{|\xi|}[[\xi,\$],\$],\$] \leq i^-_{C(\xi,\$)}[[[\xi,\$],\$],\$] = \mathrm{Epi}[\xi,\$]$. □

12. Duality

Our goal is to characterize various properties of hyperconvergences in terms of primal convergences, or rather in terms of equivalence classes of primal convergences which correspond to the same hyperconvergence. As for every convergence ξ the epitopologization $\mathrm{Epi}\xi$ of ξ is the coarsest convergence such that $[\mathrm{Epi}\xi,\$] = [\xi,\$]$, primal characterizations of sundry properties of hyperconvergences are naturally formulated as properties of epitopologies.

If M is a concrete functor such that $M \geq \mathrm{T}$, then $C(M\tau,\sigma) \subset C(\tau,\sigma)$ for every convergence τ and each topology σ. Thus $i_{|\xi|}$ is an injection into $C(M[\xi,\$],\$)$ for every convergence ξ. Therefore one can define [17][14] the *M-epitopologizer* by

$$\mathrm{Epi}_M \xi = i^-_{|\xi|}([M[\xi,\$],\$]).$$

We say that ξ is an *M-epitopology* if $\xi = \mathrm{Epi}_M \xi$. Because \$ is an initially dense object of the category of topologies,

$$\mathrm{Epi}_M \xi \geq i^-([M[\xi,\sigma],\sigma]) \tag{12.1}$$

for every topology σ (here i stands for the natural injection of $|\xi|$ to $C(C(\xi,\sigma),\sigma)$).

As Epi_M is a composition of two branches of a $(-,-)$-connection and of M (inserted between them), we refer to this situation as to *modified duality.* It follows that if M is a coreflector, then Epi_M is a reflector; however, if M is a reflector, then Epi_M need not be a coreflector (in the category of convergences).

We can describe explicitly the convergence of filters for $\mathrm{Epi}_M\xi$ (a filter $\mathcal{F}$ converges to x in $\mathrm{Epi}_M\xi$ if and only if for every ξ-closed set A with $x \notin A$ and each ξ-hyperfilter $\mathfrak{G}$ such that $\mathrm{r}\mathfrak{G}$ meshes with $\mathcal{F}$ implies that $A \notin \lim_{M[\xi,\$]}\mathfrak{G}$). However significant interpretations of this general description will be given for special cases, where $\mathbb{J}$ and $\mathbb{E}$ are \$-compatible classes of filters, and M is either the reflector $\mathrm{A}_\mathbb{J}$, or the coreflector $\mathrm{B}_\mathbb{E}$, or the composition $\mathrm{A}_\mathbb{J}\mathrm{B}_\mathbb{E}$ (which is in general neither reflector nor coreflector). In this section we shall merely consider the reflector $\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}$ which will be instrumental in the study of $\mathrm{Epi}_{\mathrm{A}_\mathbb{J}\mathrm{B}_\mathbb{E}}$ that we will undertake in subsequent sections.

A class $\mathbb{E}$ of filters is said to be *\$-compatible* whenever $\mathfrak{G} \in \mathbb{E}$ implies that $\mathrm{r}\mathfrak{G} \in \mathbb{E}$ and if $\mathcal{H} \in \mathbb{E}$ implies that $\mathrm{e}^\natural_\xi\mathcal{H} \in \mathbb{E}$. The classes of all filters, of countably based filters, of countably deep filters (a filter $\mathcal{F}$ is *countably deep* if $\bigcap_{A\in\mathcal{A}} A \in \mathcal{F}$ for every countable $\mathcal{A} \subset \mathcal{F}$) and of principal filters are all \$-compatible.

According to our notation $\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{E}$ stands for the class of ξ-reduced $\mathbb{E}$-filters. In particular, $\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{S}$ is the class of ξ-reduced filters, $\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{P}_\omega$ of countably based ξ-reduced filters, and $\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{P}$ of principal ξ-reduced filters. We notice that $\mathrm{B}_\mathbb{E}[\xi,\$]$ is based in ξ-saturated filters for every convergence ξ, but $\mathrm{A}_\mathbb{E}[\xi,\$]$ is not in general.

Proposition 12.1. [16] *If $\mathbb{E}$ is \$-compatible, then a filter $\mathcal{F}$ converges to x in $\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi$ if and only if $x \in \mathrm{cl}_\xi(\mathrm{adh}_\xi\mathcal{H})$ for every filter $\mathcal{H} \in \mathrm{cl}^\natural_{\xi^\bullet}\mathbb{E}$ that meshes with $\mathcal{F}$.*

In other words,

$$\lim_{\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi}\mathcal{F} = \bigcap_{\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{E}\ni\mathcal{H}\#\mathcal{F}} \mathrm{cl}_\xi(\mathrm{adh}_\xi\mathcal{H}), \tag{12.2}$$

Proof. Because $\mathrm{B}_\mathbb{E}[\xi,\$]$ is based in ξ-saturated $\mathbb{E}$-filters, a filter $\mathcal{F}$ converges to x in $\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi$ if and only if $\mathrm{adh}_\xi\mathrm{r}\mathfrak{G} \setminus A \neq \varnothing$ for every ξ-closed set A with $x \notin A$ and each ξ-saturated $\mathbb{E}$-filter $\mathfrak{G}$ such that $\mathrm{r}\mathfrak{G}$ meshes with $\mathcal{F}$, equivalently, if $x \in \mathrm{cl}_\xi(\mathrm{adh}_\xi\mathrm{r}\mathfrak{G})$ for every ξ-saturated $\mathbb{E}$-filter $\mathfrak{G}$ such that $\mathrm{r}\mathfrak{G}$ meshes with $\mathcal{F}$. Because of the duality between ξ-saturated and ξ-reduced $\mathbb{E}$-filters for a \$-compatible class $\mathbb{E}$, this means that $x \in \mathrm{cl}_\xi(\mathrm{adh}_\xi\mathcal{H})$ for every ξ-reduced $\mathbb{E}$-filter $\mathcal{H}$ that meshes with $\mathcal{F}$. □

Since $\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{E} \subset \mathbb{E}$ and, of course, $\mathrm{adh}_\xi\mathcal{H} \subset \mathrm{cl}_\xi(\mathrm{adh}_\xi\mathcal{H})$, we infer that $\mathrm{A}_\mathbb{E}\xi \geq \mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi$ for every convergence ξ. Therefore every $\mathrm{B}_\mathbb{E}$-epitopology is adherence-determined by $\mathbb{E}$-filters. In particular we recover the already established fact that each epitopology is a pseudotopology. Another already established fact that each $\mathrm{B}_{\mathbb{E}}$-epitopology is an epitopology, is also an immediate consequence of (12.2).

Let us observe that $\mathrm{Epi}_{\mathrm{B}_{\mathbb{P}}}\xi = T\xi$, that is, $\mathrm{Epi}_{\mathrm{B}_{\mathbb{P}}}$ is the topologizer. Indeed, in this case (12.2) becomes $\bigcap_{\mathrm{cl}_{\xi^\bullet} H \in \mathcal{F}^\#} \mathrm{cl}_\xi(\mathrm{adh}_\xi H)$ which is equal to $\bigcap_{H \in \mathcal{F}^\#} \mathrm{cl}_\xi H$. Therefore, every topology is a $\mathrm{B}_{\mathbb{E}}$-epitopology under the provision that $\mathbb{E}$ includes $\mathbb{P}$, the class of principal filters.

We notice that

$$\mathrm{cl}_\xi(\mathrm{adh}_\xi(\mathrm{cl}^\natural_{\xi^\bullet}\mathcal{H})) = \mathrm{cl}_\xi(\mathrm{adh}_{\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi}\mathcal{H}), \tag{12.3}$$

for every $\mathbb{E}$-filter $\mathcal{H}$. Indeed, if $x \in \mathrm{adh}_{\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi}\mathcal{H}$ then there exists a filter $\mathcal{F}$ that meshes with $\mathcal{H}$ and converges to x for $\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi$, hence $\mathrm{cl}^\natural_{\xi^\bullet}\mathcal{H}$ meshes with $\mathcal{F}$, thus by (12.2), $x \in \mathrm{cl}_\xi(\mathrm{adh}_\xi(\mathrm{cl}^\natural_{\xi^\bullet}\mathcal{H}))$. On the other hand, by (11.4) $\mathrm{adh}_\xi(\mathrm{cl}^\natural_{\xi^\bullet}\mathcal{H}) \subset \mathrm{adh}_{\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi}\mathcal{H}$ because $\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi$ is an epitopology. Therefore, if $\mathcal{B} = \mathcal{O}_\xi(\mathcal{B})$, then

$$\mathrm{adh}_{\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi}\mathcal{H}\#\mathcal{B} \Longleftrightarrow \mathrm{adh}_\xi(\mathrm{cl}^\natural_{\xi^\bullet}\mathcal{H})\#\mathcal{B} \tag{12.4}$$

for every $\mathbb{E}$-filter $\mathcal{H}$.

13. Compactoidness

It turns out that some properties of hyperconvergences can be rephrased in terms of some compactness-like properties of their underlying convergences.

The concept of a set relatively compact with respect to another set has been extended by various authors (see [8] for a historical account) to a filter (and, more generally, family of sets) relatively compact with respect to a set, and in [9], with respect to another family.

Let $\mathbb{F}$ be a class of filters. If $\mathcal{A}, \mathcal{B}$ are families of subsets of $|\xi|$, then $\mathcal{A}$ is said to be $\mathbb{F}$-*compactoid in* $\mathcal{B}$ (for ξ) if $\mathrm{adh}_\xi\mathcal{F} \in \mathcal{B}^\#$ for every $\mathcal{F} \in \mathbb{F}$ such that $\mathcal{F}\#\mathcal{A}$. A family $\mathcal{A}$ is $\mathbb{F}$-*compact* if it is $\mathbb{F}$-compactoid in itself. In particular, if $A \subset X$, then $\{A\}$ (equivalently, $\mathcal{A} = \{A\}^\uparrow$) is compactoid in X (for a convergence on X) if and only if A is relatively compact; $\{A\}$ (equivalently, $\mathcal{A} = \{A\}^\uparrow$) is compact if and only if A is compact. Compactoidness is a common generalization of compactness (of sets) and of convergence (of filters). Indeed, a filter $\mathcal{F}$ is compactoid in $\{x\}$ for ξ if and only if $x \in \lim_{S\xi}\mathcal{F}$, where S is the pseudotopologizer.

This notion of compactoidness needs to be ultimately extended. Let $\mathbb{F}$ and $\mathbb{G}$ be classes of filters, and let ξ and τ be convergences on a common set. A family $\mathcal{A}$ is said to be $\frac{\mathbb{F}}{\mathbb{G}}$-*compactoid in* $\mathcal{B}$ (for $\frac{\xi}{\tau}$) [8, Section 8] if

$$\forall_{\mathcal{F}\in\mathbb{F}}\forall_{\mathcal{G}\in\mathbb{G}}\ \ \mathcal{F} \geq \mathcal{G}, \mathrm{adh}_\tau\mathcal{G} \in \mathcal{A}^\# \Longrightarrow \mathrm{adh}_\xi\mathcal{F} \in \mathcal{B}^\#. \tag{13.1}$$

If $\mathbb{G} = \mathbb{P}$ is the class of principal filters and $\tau = \iota$ is the discrete topology, then $\forall_{\mathcal{G}\in\mathbb{P}}\ \mathcal{F} \geq \mathcal{G}, \mathrm{adh}_\iota\mathcal{G} \in \mathcal{A}^\#$ is equivalent to $\mathcal{F}\#\mathcal{A}$, and the whole condition (13.1) becomes

$$\forall_{\mathcal{F}\in\mathbb{F}}\mathcal{F}\#\mathcal{A} \Longrightarrow \mathrm{adh}_\xi\mathcal{F} \in \mathcal{B}^\#,$$

that is, to the $\mathbb{F}$-compactoidness of $\mathcal{A}$ in $\mathcal{B}$ for ξ.

The use of two classes of filters enables one to recover such compactness-like notions like Lindelöf property, which is in these terms $\frac{\mathbb{S}}{\mathbb{P}_\omega}$-compactoidness.

Another important special case of (13.1) is that of cover compactoidness. A family $\mathcal{P}$ of subsets of $|\xi|$ is a *ξ-cover* of a set A if every filter such that $\lim_\xi \mathcal{F} \cap A \neq \varnothing$ contains an element of $\mathcal{P}$; a family $\mathcal{P}$ is an *ideal* if $Q \subset P \in \mathcal{P}$ implies that $Q \in \mathcal{P}$ and if $\bigcup \mathcal{R} \in \mathcal{P}$ for every finite subfamily $\mathcal{R}$ of $\mathcal{P}$. It was observed in [8] that a family $\mathcal{P}$ is a ξ-cover of A if and only if $\mathrm{adh}_\xi \mathcal{P}_c \cap A = \varnothing$ (Recall that $\mathcal{P}_c = \{P^c : P \in \mathcal{P}\}$). It follows that

Theorem 13.1. [8] *A family $\mathcal{A}$ is* $\frac{\mathbb{F}}{\mathbb{G}}$-compactoid in $\mathcal{B}$ *(for $\frac{\xi}{\tau}$) if and only if for every $B \in \mathcal{B}$ and each ξ-cover $\mathcal{P}$ of B such that $\mathcal{P}_c \in \mathbb{F}$ there exist $A \in \mathcal{A}$ and a refinement $\mathcal{R}$ of $\mathcal{P}$ which is a τ-cover of A such that $\mathcal{R}_c \in \mathbb{G}$.*

If $\xi = \tau$ in the theorem above, then we say that $\mathcal{A}$ is $\frac{\mathbb{F}}{\mathbb{G}}$*-cover-compactoid in* $\mathcal{B}$ for ξ. For example, $\mathcal{A}$ is $\frac{\mathbb{S}}{\mathbb{P}_\omega}$-cover-compactoid for ξ in $\mathcal{B}$ if for every (ideal) ξ-cover $\mathcal{P}$ of $B \in \mathcal{B}$ there exist $A \in \mathcal{A}$ and a countable subfamily $\mathcal{R}$ of $\mathcal{P}$ which is a ξ-cover of A. If ξ is a topology, then *ideal* can be dropped without altering the meaning [8], hence this means that A is Lindelöf in B for ξ.

Let us recall that cover-compactoidness and compactoidness coincide for topologies, but do not for arbitrary convergences.

At first we notice that convergence with respect to the epitopologization is a compactoidness property. More generally,

Proposition 13.2. *If $\mathbb{E}$ is \$-compatible, then a filter converges to x in* $\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi$ *if and only if it is* $\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{E}$*-compactoid in $\mathcal{N}_\xi(x)$ for ξ.*

Proof. On rephrasing Proposition 12.1, a filter $\mathcal{F}$ converges to x in $\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi$ if and only if $\mathrm{adh}_\xi \mathcal{H} \cap O \neq \varnothing$ for every ξ-open set O that contains x and each ξ-reduced filter $\mathcal{H} \in \mathbb{E}$ that meshes with $\mathcal{F}$. □

In particular,

Proposition 13.3. *A filter converges to x in* $\mathrm{Epi}\xi$ *if and only if it is* $\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{S}$*-compactoid in $\mathcal{N}_\xi(x)$ for ξ.*

In the sequel we shall make repeated use of the following immediate consequence of (12.4):

Lemma 13.4. [11, Lemma 6.3] *Let $\mathbb{E}$ be a \$-compatible class of filters. If $\mathcal{B}$ is a family of ξ-open sets, then a family is $\mathbb{E}$-compactoid in $\mathcal{B}$ for* $\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi$ *if and only if it is* $\mathrm{cl}^\natural_{\xi^\bullet}(\mathbb{E})$*-compactoid in $\mathcal{B}$ for ξ.*

It follows from Proposition 13.2 and from Lemma 13.4 that

Corollary 13.5. *A filter converges to x in* $\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi$ *if and only if it is* $\mathbb{E}$*-compactoid in $\mathcal{N}_\xi(x)$ for* $\mathrm{Epi}\xi$.

14. Topological core compactness and commutativity with finite products

Given a subcategory $\mathbf{L}$ of convergences, a convergence ξ is called $\mathbf{L}$-*exponential* if $[\xi,\sigma]\in\mathbf{L}$ for every $\sigma\in\mathbf{L}$. It follows immediately from the definitions that a subcategory is Cartesian-closed if and only each of its objects is exponential. By Theorem 10.2, a concrete reflective subcategory of convergences is Cartesian-closed if and only if the corresponding reflector commutes with finite products. In fact more is true, namely, a convergence ξ is $\mathbf{L}$-exponential if and only if

$$\xi\times L\tau\geq L(\xi\times\tau)$$

for every convergence τ (where L is the reflector corresponding to $\mathbf{L}$).

It is known [18] that a topology is T-exponential if and only if it is *core compact*, that is, for every x and each neighborhood O of x there is a neighborhood V of x which is compactoid in O [13]. We shall refine this classical result.

Let $\mathbb{J}$ and $\mathbb{E}$ be classes of filters. The following functor $\mathrm{Q}_{\mathbb{E}}^{\mathbb{J}}$ in the category of convergences was defined in [17]:

$$x\in\lim\nolimits_{\mathrm{Q}_{\mathbb{E}}^{\mathbb{J}}\xi}\mathcal{F}$$

whenever for every $O\in\mathcal{O}_{\xi}(x)$ there exists a filter $\mathcal{J}\in\mathbb{J}$ such that $\mathcal{J}\leq\mathcal{F}$ and $\mathcal{J}$ is $\mathbb{E}$-compactoid in O for ξ. We say that a convergence ξ *is topologically* $\mathbb{J}$-*core* $\mathbb{E}$-*compact at* x if $x\in\lim_{\xi}\mathcal{F}$ implies that $x\in\lim_{S\mathrm{Q}_{\mathbb{E}}^{\mathbb{J}}\xi}\mathcal{F}$. If $\mathbb{J}=\mathbb{P}$, the class of principal filters, then we abridge $\mathbb{P}$-*core* to *core*. Let us observe that

$$S\mathrm{Q}_{\mathbb{E}}^{\mathbb{P}}=\mathrm{Q}_{\mathbb{E}}^{\mathbb{P}}. \tag{14.1}$$

Indeed, if $x\in\lim_{S\mathrm{Q}_{\mathbb{E}}^{\mathbb{P}}}\mathcal{F}$ then for every ultrafilter $\mathcal{U}$ finer than $\mathcal{F}$ and each $O\in\mathcal{O}_{\xi}(x)$ there exists $F_{\mathcal{U}}\in\mathcal{U}$ which is $\mathbb{E}$-compactoid in O for ξ. Therefore there exists $n<\omega$ and $\mathcal{U}_1,\mathcal{U}_2,\ldots,\mathcal{U}_n$ such that $F=F_{\mathcal{U}_1}\cup F_{\mathcal{U}_2}\cup\ldots\cup F_{\mathcal{U}_n}\in\mathcal{F}$ and obviously F is $\mathbb{E}$-compactoid in O for ξ.

A class $\mathbb{G}$ of filters is *composable* if $\mathcal{H}\mathcal{A}\in\mathbb{G}(Y)$ for every $\mathcal{H}\in\mathbb{G}(X\times Y)$ and $\mathcal{A}\in\mathbb{G}(X)$ and if it includes the class of principal filters. Because the star and the point closures can be defined as relations, we conclude that

Proposition 14.1. *Every composable class is* \$-*compatible.*

Theorem 14.2. [17, Theorem 6.2] *If* $\mathbb{E}$ *and* $\mathbb{J}$ *are composable classes of filters, then*

$$\theta\times\mathrm{A}_{\mathbb{J}}\tau\geq T(\xi\times\tau) \tag{14.2}$$

for every $\tau=\mathrm{B}_{\mathbb{E}}\tau$ *if and only if*

$$\theta\geq S\mathrm{Q}_{\mathbb{E}}^{\mathbb{J}}\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi. \tag{14.3}$$

Proof. Assume (14.3) and $\tau=\mathrm{B}_{\mathbb{E}}\tau$. In particular, $\theta\geq\mathrm{Q}_{\mathbb{E}}^{\mathbb{J}}\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi$. Consider filters $\mathcal{F}$ and $\mathcal{G}$ such that $x\in\lim_{\theta}\mathcal{F}$ and $y\in\lim_{\mathrm{A}_{\mathbb{J}}\tau}\mathcal{G}$ and a $(\xi\times\tau)$-closed set H that meshes with $\mathcal{F}\times\mathcal{G}$. We will show that $(x,y)\in H$ and thus (14.2) holds. By (14.3) for every $W\in\mathcal{O}_{\xi}(x)$ there exists a $\mathbb{J}$-filter $\mathcal{D}_W\leq\mathcal{F}$ which is $\mathbb{E}$-compactoid in W for $\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi$. It follows that $H\mathcal{D}_W\#\mathcal{G}$

and $H\mathcal{D}_W \in \mathbb{J}$ because $\mathbb{J}$ is composable. Hence $y \in \mathrm{adh}_\tau H\mathcal{D}_W$ and thus there exists an $\mathbb{E}$-filter $\mathcal{L}_W \# H\mathcal{D}_W$ such that $y \in \lim_\tau \mathcal{L}_W$. Therefore $\mathcal{D}_W$ meshes with the $\mathbb{E}$-filter $H^-\mathcal{L}_W$ (because $\mathbb{E}$ is composable). We infer that there is $x_W \in \mathrm{adh}_{\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi} H^-\mathcal{L}_W \cap W$ as $\mathcal{D}_W$ is $\mathbb{E}$-compactoid in W for $\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi$, and thus $(x_W, y) \in H$, because $\mathrm{T}(\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi \times \tau) = \mathrm{T}(\xi \times \tau)$. Accordingly, $x \in \mathrm{cl}_\xi H^- y = H^- y$, that is, $(x, y) \in H$.

Conversely, if (14.3) does not hold, then there exist x_0 and a free ultrafilter $\mathcal{U}$ such that $x_0 \in \lim_\theta \mathcal{U}$ and there is $W \in \mathcal{O}_\xi(x_0)$ such that for every $\mathbb{J}$-filter $\mathcal{J} \leq \mathcal{U}$, there exists an $\mathbb{E}$-filter $\mathcal{E}_\mathcal{J}$ that meshes with $\mathcal{J}$ and such that $\mathrm{adh}_{\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi} \mathcal{E}_\mathcal{J} \cap W = \varnothing$. Denote by $\mathbb{J}(\mathcal{U})$ the set of $\mathbb{J}$-filters which are coarser than $\mathcal{U}$. Let τ be the convergence on X for which every point is isolated except for x_0, and a free filter $\mathcal{G}$ converges to x_0 if there is $\mathcal{J}$ in $\mathbb{J}(\mathcal{U})$ for which $\mathcal{G} \geq \mathcal{E}_\mathcal{J}$. This is an $\mathbb{E}$-based convergence and $x_0 \in \lim_{\mathrm{A}_\mathbb{J}\tau} \mathcal{U}$ thus $(x_0, x_0) \in \lim_{\theta \times \mathrm{A}_\mathbb{J}\tau}(\mathcal{U} \times \mathcal{U})$. The filter $\mathcal{U} \times \mathcal{U}$ meshes with the set $A = \{(x, y) : x \in \lim_{\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi}\{y\}^\uparrow, y \neq x_0\}$. Let $\mathcal{C}$ be a filter on A such that $(x, y) \in \lim_{\xi \times \tau} \mathcal{C}$. If $y \neq x_0$ then $(x, y) \in A$, because y is isolated in τ and $\lim_{\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi}\{y\}^\uparrow$ is ξ-closed for every y. If $y = x_0$ then there exist filters $\mathcal{J} \in \mathbb{J}(\mathcal{U})$ and $\mathcal{F}$ such that $x \in \lim_\xi \mathcal{F}$ and $\mathcal{F} \times \mathcal{E}_\mathcal{J} \leq \mathcal{C}$. Obviously A meshes with $\mathcal{F} \times \mathcal{E}_\mathcal{J}$, that is, for each $F \in \mathcal{F}$ and every $E \in \mathcal{E}_\mathcal{J}$ there exist $x \in F$ and $y \in E$ with $x \in \lim_{\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi}\{y\}^\uparrow = \mathrm{cl}_{(\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi)^*} y$. Therefore $\mathrm{cl}^\natural_{(\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi)^*} \mathcal{F}$ meshes with $\mathcal{E}_\mathcal{J}$. Because $\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi$ is star-regular, $x \in \lim_{\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi} \mathrm{cl}^\natural_{(\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi)^*} \mathcal{F}$, hence $x \in \mathrm{adh}_{\mathrm{Epi}_{\mathrm{B}_\mathbb{E}}\xi} \mathcal{E}_\mathcal{J}$ and hence (x, x_0) belongs to the $(\xi \times \tau)$-closed set $W^c \times \{x_0\}$. It follows that $(x_0, x_0) \notin A \cup W^c \times \{x_0\} \supset \mathrm{cl}_{\xi \times \tau} A$ so that $(x_0, x_0) \notin \lim_{T(\xi \times \tau)}(\mathcal{U} \times \mathcal{U})$. □

It is obvious that

Proposition 14.3. *(14.2) holds for every $\tau = \mathrm{B}_\mathbb{E}\tau$ if and only if*

$$\theta \times \mathrm{A}_\mathbb{J}\mathrm{B}_\mathbb{E}\tau \geq T(\xi \times \tau)$$

for every convergence τ.

15. Reflective properties of hyperconvergence

It has already been mentioned that a hyperconvergence associated with a topology need not be a pretopology (*a fortiori* need not be a topology). Topologicity ($T\tau \geq \tau$) and pretopologicity ($P\tau \geq \tau$) are examples of reflective properties. We shall provide a general characterization of a reflective property of hyperconvergences in terms of a coreflective property of primal convergences. If $\mathbf{J}$ is a property of convergences, then by definition $\xi \in \mathbf{J}_*$ whenever $[\xi, \$] \in \mathbf{J}$. If $\mathbf{J}$ is a concrete reflective subcategory of convergences, then by Theorem 10.1, $\xi \in \mathbf{J}_*$ if and only if $[\xi, \sigma] \in \mathbf{J}$ for every topology σ.

Proposition 15.1. *If $\mathbf{J}$ is a concrete reflective subcategory, then $\mathbf{J}_*$ is a (concrete) coreflective subcategory.*

Proof. Let $f_i : |\xi_i| \to Y$ and $\xi_i \in \mathbf{J}_*$ for every $i \in I$, that is, $[\xi_i, \$] \in \mathbf{J}$ for each $i \in I$. Then by (10.2), $[\bigwedge_{i\in I} f_i\xi_i, \$] = \bigvee_{i\in I}(f_i^*)^-[\xi_i, \$]$, and since by assumption $\mathbf{J}$ is a concrete reflective subcategory of convergences, $\bigvee_{i\in I}(f_i^*)^-[\xi_i, \$] \in \mathbf{J}$, thus $\bigwedge_{i\in I} f_i\xi_i \in \mathbf{J}_*$. □

Theorem 15.2. [17] *If $M \geq T$ is a concrete functor, then the following statements are equivalent:*

$$M[\xi, \$] \geq [\xi, \$]; \tag{15.1}$$

$$\xi \geq \mathrm{Epi}_M \xi; \tag{15.2}$$

$$\xi \times M\tau \geq T(\xi \times \tau) \text{ for every } \tau. \tag{15.3}$$

Proof. If $M[\xi, \$] \geq [\xi, \$]$, then $[[\xi, \$], \$] \geq [M[\xi, \$], \$]$ so that $\xi \geq \mathrm{Epi}\xi \geq \mathrm{Epi}_M\xi$. Conversely if $\xi \geq \mathrm{Epi}_M\xi$ then by (12.1) the injection $i : \xi \to [M[\xi, \sigma], \sigma]$ is continuous. By (10.1), the evaluation

$${}^t i : \xi \times M[\xi, \sigma] \to \sigma$$

is also continuous. But $[\xi, \$]$ is the coarsest convergence on $C(\xi, \$)$ that makes the evaluation continuous. Therefore, $M[\xi, \$] \geq [\xi, \$]$.

Assume (15.1) and let $f \in C(\xi \times \tau, \sigma)$. Then by (10.1) and because M is a functor, ${}^t f \in C(\tau, [\xi, \sigma]) = C(M\tau, M[\xi, \sigma]) \subset C(M\tau, [\xi, \sigma])$ by (15.1). Again by (10.1), $f \in C(M\tau, [\xi, \sigma]) = C(\xi \times M\tau, \sigma)$. Therefore (15.3) holds.

The hyperconvergence $[\xi, \$]$ is the coarsest convergence τ for which the natural evaluation is continuous from $\xi \times \tau$ to \$. Therefore if (15.3) holds, then the evaluation is also continuous from $\xi \times M[\xi, \$]$ to \$ and thus $M[\xi, \$] \geq [\xi, \$]$. □

It follows from Theorems 15.2 and 14.2 and from Proposition 14.3 that

$$\xi \geq \mathrm{Epi}_{\mathrm{A}_{\mathbb{J}}\mathrm{B}_{\mathbb{E}}}\xi \Longleftrightarrow \xi \geq S\mathrm{Q}_{\mathbb{E}}^{\mathbb{J}}\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi$$

(but this does not mean that $\mathrm{Epi}_{\mathrm{A}_{\mathbb{J}}\mathrm{B}_{\mathbb{E}}} = S\mathrm{Q}_{\mathbb{E}}^{\mathbb{J}}\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}$). This property of ξ is equivalent to a (reflective) property of $[\xi, \$] = [\mathrm{Epi}\xi, \$]$, hence it is a property of epitopologies. We conclude that

Corollary 15.3. *If $\mathbb{E}$ and $\mathbb{J}$ are composable classes of filters, then* $\mathrm{A}_{\mathbb{J}}\mathrm{B}_{\mathbb{E}}[\xi, \$] \geq [\xi, \$]$ *if and only if* $\mathrm{Epi}\xi$ *is topologically $\mathbb{J}$-core $\mathbb{E}$-compact for* $\mathrm{Epi}_{\mathrm{B}_{\mathbb{E}}}\xi$ *(equivalently ξ is $\mathbb{J}$-core* $\mathrm{cl}_{\xi^\bullet}^{\natural}\mathbb{E}$*-compact).*

Consider now several special cases of the condition above. As $\mathrm{B}_{\mathbb{S}}$ is the identity and $\mathrm{A}_{\mathbb{S}}$ is the pseudotopologizer, and since every hyperconvergence is a pseudotopology, the condition $\mathrm{A}_{\mathbb{S}}[\xi, \$] \geq [\xi, \$]$ is always fulfilled. On the other hand, by Corollary 15.3, if $x \in \lim_\xi \mathcal{F}$ then $\mathrm{adh}_{\mathrm{Epi}\xi}\, \mathcal{H} \cap W \neq \varnothing$ for every $W \in \mathcal{O}_\xi(x)$ and for each ultrafilter $\mathcal{U}$ finer than $\mathcal{F}$ and for every filter $\mathcal{H}$ that meshes with $\mathcal{U}$; in other words, $x \in \mathrm{cl}_\xi(\mathrm{adh}_{\mathrm{Epi}\xi}\mathcal{H})$, which is of course always the case.

We apply now Corollary 15.3 in the case $\mathbb{E} = \mathbb{S}$ (the class of all filters) and $\mathbb{J} = \mathbb{P}_\omega$ (the class of countably based filters).

Theorem 15.4. *The hyperconvergence* $[\xi, \$]$ *is a paratopology if and only if* $\mathrm{Epi}\xi$ *is topologically* $\mathbb{P}_\omega$*-core compact.*

If $\mathbb{E} = \mathbb{S}$ is the class of all filters and $\mathbb{J} = \mathbb{P}$ is the class of principal filters, then Corollary 15.3 yields

Theorem 15.5. *The hyperconvergence* $[\xi, \$]$ *is a pretopology if and only if* $\mathrm{Epi}\xi$ *is topologically core compact.*

Thanks to Theorem 10.1 we infer that

Corollary 15.6. *If an epitopology* ξ *is topologically core (respectively,* $\mathbb{P}_\omega$*-core) compact, then* $[\xi, \sigma]$ *is pretopological (respectively, paratopological) for each topology* σ*.*

Corollary 15.3 cannot be used to characterize the topologicity of hyperconvergences, because $T = \mathrm{A}_{\mathbb{T}}$, but the class $\mathbb{T}$ of principal filters of closed sets is not composable; instead one can apply directly Theorem 15.2 to another representation of the topologizer [11]. However, in the next section we will use yet another approach.

16. Vicinities and neighborhoods

Certain objects related to hyperconvergences, like vicinity or neighborhood, can be characterized in terms of primal convergences. This enables one to characterize, in terms of primal convergences, various reflective and coreflective properties of hyperconvergences that can be formulated with the aid of such objects.

Some of the reflective properties of hyperconvergences have been characterized already by virtue of Theorem 15.2. The situation is different for the coreflective properties of hyperconvergences, because for them there exists no equivalence scheme similar to Theorem 15.2.

In order to characterize hypervicinities and open hypersets, we introduce a notion of rigid compactoidness. A family $\mathcal{H}$ ξ*-rigidly meshes* with a family $\mathcal{A}$ if for every $H \in \mathcal{H}$, there is $B = \mathrm{cl}_\xi B \subset H$ such that $B \in \mathcal{A}^\#$. Of course, if $\mathcal{H}$ ξ-rigidly meshes with $\mathcal{A}$, then it meshes with $\mathcal{A}$. On the other hand, if $\mathcal{H}$ is ξ*-closed* (that is, $\mathcal{H} = \mathrm{cl}_\xi^\natural \mathcal{H}$) and meshes with $\mathcal{A}$, then it ξ-rigidly meshes with $\mathcal{A}$.

A family $\mathcal{A}$ is *rigidly* $\mathbb{E}$*-compactoid* in $\mathcal{B}$ for ξ if $\mathrm{adh}_\xi \mathcal{E} \in \mathcal{B}^\#$ for each $\mathbb{E}$-filter $\mathcal{E}$ which ξ-rigidly meshes with $\mathcal{A}$; a family is *rigidly* $\mathbb{E}$*-compact* if it is rigidly $\mathbb{E}$-compactoid in itself. Every $\mathbb{E}$-compactoid ($\mathbb{E}$-compact) family is rigidly $\mathbb{E}$-compactoid (rigidly $\mathbb{E}$-compact). If ξ is a topology, then the converse also holds, because then for every filter $\mathcal{H}$, one has $\mathrm{adh}_\xi \mathcal{H} = \bigcap_{H \in \mathcal{H}} \mathrm{cl}_\xi H$ and thus $\mathrm{adh}_\xi \mathcal{H} = \mathrm{adh}_\xi \mathrm{cl}_\xi^\natural \mathcal{H}$.

Lemma 16.1. *A set* F *is* $\mathrm{cl}_{\xi_\bullet}^\natural \mathbb{S}$*-compactoid in* B *for* ξ *if and only if* $\mathcal{O}_\xi(F)$ *is rigidly compactoid in* B *for* ξ*.*

Proof. We notice that $\mathcal{H}$ rigidly meshes with $\mathcal{O}_\xi(F)$ if and only if $\mathrm{int}^\natural_{\xi^*}\mathcal{H}$ meshes with F. Indeed, the former means that for each $H \in \mathcal{H}$ there is a subset $D = \mathrm{cl}_\xi D$ of H such that $D \in \mathcal{O}_\xi(F)^\#$, equivalently $\mathrm{cl}_\xi D \cap F \neq \varnothing$, or else $\mathrm{int}^\natural_{\xi^*}\mathcal{H}$ meshes with F.

Therefore if $\mathcal{O}_\xi(F)$ is rigidly compactoid in B for ξ and $\mathrm{int}^\natural_{\xi^*}\mathcal{H} = \mathcal{H} = \mathrm{cl}^\natural_{\xi^\bullet}\mathcal{H}$ meshes with F, then $\mathcal{H}$ rigidly meshes with $\mathcal{O}_\xi(F)$ and thus $\mathrm{adh}_\xi\mathcal{H} \cap B \neq \varnothing$. Conversely if F is $\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{S}$-compactoid in B for ξ and $\mathcal{H}$ rigidly meshes with $\mathcal{O}_\xi(F)$, then the ξ-reduced filter $\mathrm{int}^\natural_{\xi^*}\mathcal{H}$ meshes with F and thus $\mathrm{adh}_\xi(\mathrm{int}^\natural_{\xi^*}\mathcal{H}) \cap B \neq \varnothing$. □

It is straightforward that each intersection of stable hyperfilters is stable; in particular, each vicinity filter for a hyperconvergence is stable. On the other hand, an open hyperset (that is, open for a hyperconvergence) is always stable. Observe that $\mathcal{A}$ is ξ-stable if and only if $\mathcal{A}_c = \mathcal{O}_\xi(\mathcal{A}_c)$. Let us characterize stable vicinities.

Theorem 16.2. *Let $\mathbb{E}$ be \$-compatible. A hyperset $\mathcal{A}$ is a stable vicinity of A_0 for $\mathrm{B}_\mathbb{E}[\xi, \$]$ if and only if $\mathcal{A}_c = \mathcal{O}_\xi(\mathcal{A}_c)$ is rigidly $\mathbb{E}$-compactoid in A_0^c for ξ.*

Proof. A hyperset $\mathcal{A}$ is a vicinity of A_0 for $\mathrm{B}_\mathbb{E}[\xi, \$]$ if and only if $\mathcal{A} \in \mathfrak{E}$ for every $\mathbb{E}$-hyperfilter $\mathfrak{E}$ that $[\xi, \$]$-converges to A_0. Equivalently, if for each ξ-reduced $\mathbb{E}$-filter $\mathcal{E}$ with $\mathrm{adh}_\xi\mathcal{E} \subset A_0$ there exists $E \in \mathcal{E}$ such that if $B = \mathrm{cl}_\xi B \subset E$, then $B \in \mathcal{A}$. In other words, for every $E \in \mathcal{E}$ there exists $B = \mathrm{cl}_\xi B \subset E$ such that $B \notin \mathcal{A}_c^\#$, then $\mathrm{adh}_\xi\mathcal{E} \cap A_0^c \neq \varnothing$; equivalently, $\mathcal{A}_c$ is rigidly $\mathrm{cl}^\natural_{\xi^\bullet}\mathbb{E}$-compactoid in A_0^c for ξ. □

Corollary 16.3. *Let $\mathbb{E}$ be \$-compatible. A hyperset $\mathcal{A}$ is open for $\mathrm{B}_\mathbb{E}[\xi, \$]$ if and only if $\mathcal{A}_c = \mathcal{O}_\xi(\mathcal{A}_c)$ and is rigidly $\mathbb{E}$-compact for ξ.*

This refines a result of [11, Corollary 5.3]; if ξ is a topology, in the case where $\mathbb{E}$ is the class of all filters this recovers [10, Theorem 3.1] of Dolecki, Greco and Lechicki, and for the class $\mathbb{E}$ of countably based filters [2, Theorem 2.1] of Alleche and Calbrix.

We will use the characterizations above to describe the pretopologicity and the topologicity of hyperconvergences in terms of primal convergences. The theorem below is identical with Theorem 15.5; what differs is the proof.

First observe that, as every hyperconvergence has a convergence base consisting of saturated filters, every vicinity filter has a filter base consisting of saturated, hence stable, hypersets.

Theorem 16.4. *The hyperconvergence $[\xi, \$]$ is a pretopology if and only if* $\mathrm{Epi}\xi$ *is topologically core compact.*

Proof. By definition, $[\xi, \$]$ is a pretopology if and only if $A_0 \in \lim_{[\xi,\$]}\mathfrak{V}_{[\xi,\$]}(A_0)$, that is, $\mathrm{adh}_\xi \mathrm{r}\mathfrak{V}_{[\xi,\$]}(A_0) \subset A_0$. Equivalently, if $\lim_\xi\mathcal{F} \cap A_0^c \neq \varnothing$, then $\mathcal{F}$ does not mesh with $\mathrm{r}\mathfrak{V}_{[\xi,\$]}(A_0)$, in other words, there is $F \in \mathcal{F}$ and a stable $\mathcal{A} \in \mathfrak{V}_{[\xi,\$]}(A_0)$ such that $F \cap \bigcup_{A \in \mathcal{A}} A = \varnothing$, that is, $F \subset \bigcap_{A \in \mathcal{A}} A^c$. By Theorem 16.2, this means that for every ξ-open set O and each filter $\mathcal{F}$ such that $\lim_\xi\mathcal{F} \cap O \neq \varnothing$

there exists a family $\mathcal{B} = \mathcal{O}_\xi(\mathcal{B})$ which is rigidly compactoid in O for ξ and such that $\bigcap_{B\in\mathcal{B}} B \in \mathcal{F}$. It follows that ξ is topologically core $\mathrm{cl}^{\natural}_{\xi^\bullet}\mathbb{S}$-compact. Indeed, as $\mathcal{B}$ is rigidly compactoid in O for ξ, and $\mathcal{B} \subset \mathcal{O}_\xi(F)$, the latter is also rigidly compactoid in O for ξ, thus by Lemma 16.1, F is $\mathrm{cl}^{\natural}_{\xi^\bullet}\mathbb{S}$-compactoid in O for ξ. Conversely, if ξ is topologically core $\mathrm{cl}^{\natural}_{\xi^\bullet}\mathbb{S}$-compact and $\lim_\xi \mathcal{F} \cap O \neq \varnothing$, then there exists F which is $\mathrm{cl}^{\natural}_{\xi^\bullet}\mathbb{S}$-compactoid in O for ξ, hence by Lemma 16.1 $\mathcal{B} = \mathcal{O}_\xi(F)$ is rigidly compactoid in O for ξ and of course $\bigcap_{B\in\mathcal{B}} B \in \mathcal{F}$. Finally, since O is ξ-open, by Lemma 13.4, it follows that F is compactoid in O for $\mathrm{Epi}\xi$. □

A convergence is a topology if and only if the filter of neighborhoods of every point converges to that point. Once again, as $[\xi, \$] = [\mathrm{Epi}\xi, \$]$, we can formulate any characterization of the pretopologicity of hyperconvergences in terms of the underlying epitopologies.

Theorem 16.5. *The hyperconvergence* $[\xi, \$]$ *is a topology if and only if for every* ξ*-open set* O *and each filter* $\mathcal{F}$ *such that* $\lim_\xi \mathcal{F} \cap O \neq \varnothing$ *there exists a family* $\mathcal{B} = \mathcal{O}_\xi(\mathcal{B})$ *which is rigidly compact for* $\mathrm{Epi}\xi$ *and such that* $O \in \mathcal{B}$ *and* $\bigcap_{B\in\mathcal{B}} B \in \mathcal{F}$.

As every compact family is rigidly compact, the existence of a compact family $\mathcal{B}$ in the statement above is a sufficient condition for the topologicity of $[\xi, \$]$ (and by Theorem 10.1, of $[\xi, \sigma]$ for every topology σ).

In the special case where ξ is a topology, this is a condition of Day and Kelly [6] (who did not use the name *compact family*, but gave an equivalent definition) and the characterization of core compactness [13, Proposition 4.2] by Hofmann and Lawson, if we remember that a family $\mathcal{B} = \mathcal{O}_\xi(\mathcal{B})$ is compact if and only if it is an open set for the Scott topology associated with a topology ξ. Therefore it must follow that if the primal convergence is a topology, then the topologicity and the pretopologicity of the hyperconvergence coincide. This is actually the case. We do not provide here the proof, because it uses the technique of multifilters, and would require quite a few preliminaries.

Theorem 16.6. [11, Theorem 5.6] *If* ξ *is a topology, then* $[\xi, \$]$ *is a topology if and only if* $[\xi, \$]$ *is a pretopology.*

Example 16.7. Consider the bisequence $\{x_\infty\} \cup \{x_n : n < \omega\} \cup \{x_{n,k} : n, k < \omega\}$ endowed with the canonical pretopology π: the filter associated with $(x_n)_n$ is the coarsest free filter that converges to x_∞ and for every $n < \omega$, the filter associated with $(x_{n,k})_k$ is the coarsest free filter that converges to x_n. This is a Hausdorff locally compact pretopology (hence a point-diagonal pseudotopology with closed limits), so that its hyperconvergence is pretopological in view of Theorem 16.4. On the other hand, the topologization of π is Hausdorff regular, but not locally compact, thus $[T\pi, \$]$ is not a pretopology.

17. Coreflective properties of hyperconvergence

Let us start by considering the case of coreflectors of the form $M = \mathrm{B}_{\mathbb{E}}$, where $\mathrm{B}_{\mathbb{E}}$ is defined in (9.4).

Theorem 17.1. *Let $\mathbb{E}$ be a \$-compatible class of filters. The hyperconvergence $[\xi, \$]$ is $\mathbb{E}$-based if and only if every ξ-open set is $\frac{\mathbb{S}}{\mathbb{E}}$-compactoid for ξ (equivalently, for $\mathrm{Epi}\xi$).*

Proof. Let $[\xi, \$]$ be $\mathbb{E}$-based and let $\mathcal{G}$ be a filter such that $\mathrm{adh}_\xi \mathcal{G} \subset A = \mathrm{cl}_\xi A$, that is, $A \in \lim_{[\xi,\$]} \mathrm{cl}^\natural_{\xi^\bullet} \mathcal{G}$ and by the assumption there exists an $\mathbb{E}$-filter $\mathfrak{E}$ coarser than $\mathrm{cl}^\natural_{\xi^\bullet} \mathcal{G}$ for which $A \in \lim_{[\xi,\$]} \mathfrak{E}$, hence $\mathrm{adh}_\xi \mathrm{r}\mathfrak{E} \subset A$. The filter $\mathcal{E} = \mathrm{r}\mathfrak{E}$ is a ξ-reduced $\mathbb{E}$-filter, because $\mathbb{E}$ is \$-compatible, and $\mathcal{E} \leq \mathrm{re}^\natural_\xi \mathcal{G}$. For every $E \in \mathcal{E}$ let $G_E \in \mathcal{G}$ be such that $\mathrm{int}_{\xi^*} G_E \subset E$. Then the filter $\mathcal{H}$ generated by $\{G_E : E \in \mathcal{E}\}$ is an $\mathbb{E}$-filter coarser than $\mathcal{G}$ and than $\mathcal{E}$ and hence $\mathrm{adh}_\xi \mathcal{H} \subset A$.

Conversely if the condition holds and $A \in \lim_{[\xi,\$]} \mathfrak{G}$, then $\mathrm{adh}_\xi \mathrm{r}\mathfrak{G} \subset A$, and thus there exists an $\mathbb{E}$-filter $\mathcal{E}$ coarser than $\mathrm{r}\mathfrak{G}$ such that $\mathrm{adh}_\xi \mathcal{E} \subset A$, hence $A \in \lim_{[\xi,\$]} \mathrm{e}^\natural_\xi \mathcal{E}$ and $\mathrm{e}^\natural_\xi \mathcal{E}$ belongs to $\mathbb{E}$, because $\mathbb{E}$ is \$-compatible, and $\mathrm{e}^\natural_\xi \mathcal{E} \leq \mathrm{e}^\natural_\xi \mathrm{r}\mathfrak{G} \leq \mathfrak{G}$. The condition follows, because for every ξ-open set O the complement is obviously ξ-closed. □

As a corollary we get (for general convergences, *open hereditarily cover Lindelöf* need not entail *hereditarily cover Lindelöf.*)

Theorem 17.2. [15] *The hyperconvergence $[\xi, \$]$ is of countable character if and only if ξ (equivalently, $\mathrm{Epi}\xi$) is open hereditarily cover Lindelöf.*

Observe that a convergence τ is respectively bisequential, strongly Fréchet, Fréchet and sequential if and only if $S\tau \geq S\mathrm{First}\tau, P_\omega\tau \geq P_\omega \mathrm{First}\tau$, $P\tau \geq P\mathrm{First}\tau$ and $T\tau \geq T\mathrm{First}\tau$.

In other words, a convergence is Fréchet whenever every sequential vicinity of a point is a vicinity. Therefore,

Theorem 17.3. *The hyperconvergence $[\xi, \$]$ is Fréchet if and only if every family $\mathcal{B} = \mathcal{O}_\xi(\mathcal{B})$ which is rigidly countably compactoid in a ξ-open set O, is rigidly ξ-compactoid in O.*

A convergence is sequential whenever every sequentially open set is open. Hence

Theorem 17.4. *The hyperconvergence $[\xi, \$]$ is sequential if and only if every family $\mathcal{B} = \mathcal{O}_\xi(\mathcal{B})$ which is rigidly countably ξ-compact is rigidly ξ-compactoid.*

Recall that a topology τ is *tight* if $x \in \mathrm{cl}_\tau A$ implies the existence of a countable subset E of A such that $x \in \mathrm{cl}_\tau E$. As a special case of [15, Theorem 3.7], we have

Theorem 17.5. *If ξ is a topology and if $T[\xi, \$]$ is tight, then ξ is hereditarily Lindelöf.*

Proof. Let $\mathcal{G}$ be a filter such that $\mathrm{adh}_\xi \mathcal{G} \subset A = \mathrm{cl}_\xi A$, that is, $A \in \lim_{[\xi,\$]} \mathrm{e}_{\xi}^{\natural} \bullet \mathcal{G}$. In particular, $A \in \mathrm{cl}_{[\xi,\$]}\{\mathrm{cl}_\xi G : G \in \mathcal{G}\}$, and since $T[\xi, \$]$ is tight, there exists a sequence (G_n) such that $A \in \mathrm{cl}_{[\xi,\$]}\{\mathrm{cl}_\xi G_n : n < \omega\}$, that is, for every ξ-compact family $\mathcal{B} = \mathcal{O}_\xi(\mathcal{B})$ such that $A^c \in \mathcal{B}$ there exists $n < \omega$ such that $(\mathrm{cl}_\xi G_n)^c \in \mathcal{B}$. It follows that $\mathrm{adh}_\xi(G_n) = \bigcap_{n<\omega} \mathrm{cl}_\xi G_n \subset A$, for if $x \in \bigcap_{n<\omega} \mathrm{cl}_\xi G_n \setminus A$, then $(\mathrm{cl}_\xi G_n)^c \notin \mathcal{O}_\xi(x)$ for every $n < \omega$ and $A^c \in \mathcal{O}_\xi(x)$. Of course, $\mathcal{O}_\xi(x)$ is $T\xi$-compact, which leads to a contradiction. □

Corollary 17.6. *If ξ is a topology and if $T[\xi, \$]$ is tight, then $[\xi, \$]$ is of countable character.*

Recall that every perfectly normal Lindelöf topology is hereditarily Lindelöf [12, Exercise 3.8.A]. Notice that, unlike in the cited book of Engelking, our definition of Lindelöf convergence does not involve any separation axiom. On the other hand, the atomic cofinite topology on an uncountable set is (Hausdorff) compact, hence Lindelöf, but not hereditarily Lindelöf (for the *atomic cofinite topology* on X all elements but one are isolated, and a subset of X is a neighborhood of the non isolated point whenever its complement is finite). Indeed, the restriction of this topology to the (open) complement of the non isolated element is the discrete topology on an uncountable set.

18. Lattice-theoretic approach to hyperconvergence

If L is a complete lattice with respect to an order $\leq$, then three convergences arise naturally: lower, upper, and their infimum. A filter $\mathcal{F}$ converges to x in the *upper convergence* ($x \in \lim_+ \mathcal{F}$) if

$$\limsup \mathcal{F} = \bigwedge_{F \in \mathcal{F}} \bigvee F \leq x.$$

This formula obviously defines a convergence, because $\mathcal{F} \leq \mathcal{G}$ implies $\limsup \mathcal{F} \geq \limsup \mathcal{G}$, and $x = \bigvee_{F \ni x} \bigwedge F$. Clearly, $\limsup \mathcal{F} \leq x \leq y$ implies that $\limsup \mathcal{F} \leq y$, that is, every limit is upper stable. It follows that every $+$-closed set is upper stable, hence each $+$-open set is lower stable. By definition, a subset O of L is $+$-open if $\bigwedge_{F \in \mathcal{F}} \bigvee F \leq x$ for a filter $\mathcal{F}$ and $x \in O$ then $O \in \mathcal{F}$; in other words, if $\bigwedge A \in O$ then there is a finite subset B of A such that $\bigwedge B \in O$.

Open sets for the *lower convergence* (that is, the upper convergence with respect to the inverse order) were characterized by D. Scott [19] with the aid of the relation of being way below: an element x (of a complete lattice) is *way below* an element y (in symbols, $x \ll y$) whenever $\bigvee A \leq y$ implies $\bigvee F \leq x$ for some finite subset F of A. Accordingly, a set O is open for the lower convergence if $x \in O$ and $x \leq y$ imply $y \in O$, and if $\bigvee A \in O$, then there is a finite subset B of A such that $\bigvee B \in O$. In the special case of the lattice of open sets for a topology ξ, this means precisely that $\mathcal{B}$ is open for the lower topology if and only if $\mathcal{O}_\xi(\mathcal{B}) = \mathcal{B}$ is ξ-compact.

A complete lattice is called *continuous* if its lower convergence is topological, that is, if each neighborhood filter converges to its defining point; in other words, if $y = \bigvee\{x : x \ll y\}$ for every $y \in L$. In [13] Hofmann and Lawson proved that the lattice of open sets of a topology ξ is continuous if and only if ξ is core compact. This is dual to the result from preceding paragraphs that a topology ξ is core compact if and only if the hyperconvergence $[\xi, \$]$ is topological. Moreover, Hofmann and Lawson showed that in the class of so-called *sober* topologies, core compactness is equivalent to hereditary local compactness. We shall relate these facts in terms of hyperspaces.

The hyperspace $C(\xi, \$)$ ordered by inclusion is a complete lattice in which the supremum and the infimum are given by

$$\bigvee \mathcal{A} = \mathrm{cl}_\xi(\bigcup_{A \in \mathcal{A}} A); \ \bigwedge \mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

If ξ is a topology, then $\mathrm{adh}_\xi \mathcal{H} = \bigcap_{H \in \mathcal{H}} \mathrm{cl}_\xi H$, hence $A \in \lim_{[\xi,\$]} \mathfrak{F}$ if and only if $\bigwedge_{\mathcal{F} \in \mathfrak{F}} \bigvee \mathcal{F} \subset A$, that is, the hyperconvergence (of a topology) is the upper convergence in the lattice of closed sets.

An element p of a lattice is *prime* if $a \vee b \geq p$ implies that either $a \geq p$ or $b \geq p$. A ξ-closed set is called *irreducible* if it is prime in the lattice $C(\xi, \$)$, that is, if it cannot be written as the union of two proper ξ-closed subsets. Notice that $\mathrm{cl}_\xi\{x\}$ is irreducible for every $x \in |\xi|$ and for each convergence ξ. The whole space of the cofinite topology on an infinite set is an example of irreducible closed set which is not the closure of a point. A T_0 convergence is called *sober* if every irreducible closed set contains a dense point. Each Hausdorff topology is sober; the cofinite topology on an infinite set is a T_1 non sober topology. Being sober is of course a topological property.

The *lower Vietoris* (or *lower Kuratowski*) topology $V_-(\xi)$ on $C(\xi, \$)$ has the following subbase of open sets: $O^- = \{A \in C(\xi, \$) : A \cap O \neq \varnothing\}$ where $O \in \mathcal{O}_\xi$. Notice that for a hyperset $\mathcal{A}$, a hyperpoint $A_0 \in \mathrm{cl}_{V_-(\xi)} \mathcal{A}$ if and only if $A_0 \subset \mathrm{cl}_\xi(\bigcup_{A \in \mathcal{A}} A)$, that is, whenever $A_0 \leq \bigvee \mathcal{A}$ in the lattice $C(\xi, \$)$.

The *spectrum* $\Sigma(\xi)$ of $C(\xi, \$)$ is the set of non empty irreducible ξ-closed sets, endowed with the restriction of the lower Vietoris topology. In other words, if we let $\mathrm{cl}_\xi^\natural U = \{\mathrm{cl}_\xi\{x\} : x \in U\}$, then the topology of $\Sigma(\xi)$ is generated by the hypersets $\Sigma(\xi) \setminus \mathrm{cl}_\xi^\natural A$ with $A \in C(\xi, \$)$.

In our terms, [13, Proposition 2.7] implies that

Proposition 18.1. *The map* $\mathrm{cl}_\xi : |\xi| \to \Sigma(\xi)$ *is continuous and open on its image; it is injective if and only if it is an embedding if and only if* ξ *is* T_0*; it is bijective if and only if it is a homeomorphism if and only if* ξ *is sober.*

Proof. The first assertion follows from the fact that if O is ξ-open, then $\mathrm{cl}_\xi\{x\} \in O^-$ whenever $\mathrm{cl}_\xi\{x\} \cap O \neq \varnothing$, that is, whenever $x \in O$. To prove the second statement, notice that ξ is T_0 if and only if $x_0 \neq x_1$ implies $\mathrm{cl}_\xi\{x_0\} \neq \mathrm{cl}_\xi\{x_1\}$. The third statement is a consequence of the fact that ξ is sober if and only if the only non empty irreducible ξ-closed sets are the closures of singletons. □

We prove here directly the following conclusion of [13, Proposition 2.7].

Proposition 18.2. *The spectrum is sober.*

Proof. Let $\mathcal{A} \subset \Sigma(\xi)$ be a $V_-(\xi)$-closed hyperset (that is, $\{A \in \Sigma(\xi) : A \subset \mathrm{cl}_\xi(\bigcup_{B\in\mathcal{A}} B)\}$) which is not the closure of a hyperpoint of $\Sigma(\xi)$; in other words, the set $\mathrm{cl}_\xi(\bigcup_{B\in\mathcal{A}} B)$ is reducible: there exist ξ-closed sets A_0, A_1 such that $\mathrm{cl}_\xi(\bigcup_{B\in\mathcal{A}} B) = A_0 \cup A_1$ and $A_1 \setminus A_0 \neq \varnothing$ and $A_0 \setminus A_1 \neq \varnothing$. The hypersets $\mathcal{A}_0 = \{A \in \Sigma(\xi) : A \subset A_0\}$ and $\mathcal{A}_1 = \{A \in \Sigma(\xi) : A \subset A_1\}$ are $V_-(\xi)$-closed. Let $A \subset A_0 \cup A_1$, where A is an element of $\Sigma(\xi)$. As A is irreducible, either $A \cap A_0$ is a subset of $A \cap A_1$ and thus $A \subset A_1$ or vice versa and thus $A \subset A_0$. This shows that $\mathcal{A}$ is reducible and hence $\Sigma(\xi)$ endowed with $V_-(\xi)$ is sober. □

It follows that every T_0 topology ξ can be embedded in a sober topology (the *sobrification* of ξ).

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