GROUP TOPOLOGIES COARSER THAN THE ISBELL TOPOLOGY

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Abstract. The Isbell, compact-open and point-open topologies on the set $C(X,\mathbb{R})$ of continuous real-valued maps can be represented as the dual topologies with respect to some collections $\alpha(X)$ of compact families of open subsets of a topological space X. Those $\alpha(X)$ for which addition is jointly continuous at the zero function in $C_\alpha(X,\mathbb{R})$ are characterized, and sufficient conditions for translations to be continuous are found. As a result, collections $\alpha(X)$ for which $C_\alpha(X,\mathbb{R})$ is a topological vector space are defined canonically. The Isbell topology coincides with this vector space topology if and only if X is infraconsonant. Examples based on measure theoretic methods, that $C_\alpha(X,\mathbb{R})$ can be strictly finer than the compact-open topology, are given.

1. Introduction

The *Isbell*, *compact-open* and *point-open* topologies on the set $C(X,Y)$ of continuous real-valued maps from X to Y, can be represented as the *dual topologies* with regard to some collections $\alpha = \alpha(X)$ of compact openly isotone families of a topological space X, that is, the topology $\alpha(X,Y)$ is determined by a subbase of open sets of the form

$$[\mathcal{A},U] := \left\{ f \in C(X,\mathbb{R}) : f^-(U) \in \mathcal{A} \right\}, \tag{1.1}$$

where $\varnothing \notin \mathcal{A} \in \alpha$ and U are open subsets of Y (and $f^-(U) := \{x \in X : f(x) \in U\}$[1]). They are dual with regard to the collections, respectively, $\kappa(X)$ of all *compact families*, $k(X)$ of *compactly generated families* and $p(X)$ *finitely generated families* on X. Although $p(X,\mathbb{R})$ and $k(X,\mathbb{R})$ are topological vector spaces for each X, the Isbell topology $\kappa(X,\mathbb{R})$ need not be even translation-invariant. If X is *consonant* (that is, if $k(X,\$^*) = \kappa(X,\$^*)$, where $\$^*$ designs the *Sierpiński topology*) then $k(X,\mathbb{R})$ and $\kappa(X,\mathbb{R})$ coincide, and in particular $\kappa(X,\mathbb{R})$ is a group topology. In [5] we characterized those topologies X, for which addition is jointly continuous at the zero function for the Isbell topology $\kappa(X,\mathbb{R})$; the class of such topologies, called *infraconsonant*, is larger than that of consonant topologies, but we do not know if the two classes coincide in case of completely regular topologies X. In this paper we prove that the Isbell topology $\kappa(X,\mathbb{R})$ is a group topology if and only if X is infraconsonant. More generally, for each X there exists a largest hereditary ([2]) collection $\Lambda^{\downarrow}(X) \subseteq \kappa(X)$, for which the addition is jointly continuous at the zero function in $\Lambda^{\downarrow}(X,\mathbb{R})$. It turns out that $\Lambda^{\downarrow}(X,\mathbb{R})$ is a vector space topology and that a completely regular space X is infraconsonant if and only if $\Lambda^{\downarrow}(X,\mathbb{R}) = \kappa(X,\mathbb{R})$.

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[1] $f^-(U)$ is a shorthand for $f^{-1}(U)$

[2] A collection α is *hereditary* if $\mathcal{A} \downarrow A \in \alpha$ whenever $A \in \mathcal{A} \in \alpha$, where $\mathcal{A} \downarrow A$ is defined by (2.2).

Using measure theoretic methods, we show in particular that if a completely regular X is not pre-Radon, then $k(X,\mathbb{R})$ is strictly included in $\Lambda^{\downarrow}(X,\mathbb{R})$.

2. Generalities

If $\mathcal{A}$ is a family of subsets of a topological space X then $\mathcal{O}_X(\mathcal{A})$ denotes the family of open subsets of X containing an element of $\mathcal{A}$. In particular, if $A\subset X$ then $\mathcal{O}_X(A)$ denotes the family of open subsets of X containing A. We denote by $\mathcal{O}_X$ the set of open subsets of X.

If X and Y are topological spaces, $C(X,Y)$ denotes the set of continuous functions from X to Y. If $A\subset X$, $U\subset Y$, then $[A,U]:=\{f\in C(X,Y):f(A)\subset U\}$. A family $\mathcal{A}$ of subsets of X is *openly isotone* if $\mathcal{O}_X(\mathcal{A})=\mathcal{A}$. If $\mathcal{A}$ is openly isotone and U is open, then $[\mathcal{A},U]=\bigcup_{A\in\mathcal{A}}[A,U]$.

If α is a collection of openly isotone families $\mathcal{A}$ of open subsets of X, such that each open subset of X belongs to an element of α, then

$$\{[\mathcal{A},U]:\mathcal{A}\in\alpha,U\in\mathcal{O}_Y\}$$

forms a subbase for a topology $\alpha(X,Y)$ on $C(X,Y)$. We denote the set $C(X,Y)$ endowed with this topology by $C_\alpha(X,Y)$. Note that because

$$[\mathcal{A},U]\cap[\mathcal{B},U]=[\mathcal{A}\cap\mathcal{B},U],$$

$\alpha(X,Y)$ and $\alpha^{\cap}(X,Y)$ coincide, where $\alpha^{\cap}$ consists of finite intersections of the elements of α. Therefore, we can always assume that α is stable under finite intersections.

In the sequel, we will focus on the case where α consists of compact families. A family $\mathcal{A}=\mathcal{O}_X(\mathcal{A})$ is *compact* if whenever $\mathcal{P}\subset\mathcal{O}_X$ and $\bigcup\mathcal{P}\in\mathcal{A}$ then there is a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup\mathcal{P}_0\in\mathcal{A}$. Of course, for each compact subset K of X, the family $\mathcal{O}_X(K)$ is compact.

We denote by $\kappa(X)$ the collection of compact families on X. Seen as a family of subsets of $\mathcal{O}_X$ (the set of open subsets of X), $\kappa(X)$ is the set of open sets for the *Scott topology*; hence every union of compact families is compact, in particular $\bigcup_{K\in\mathcal{K}}\mathcal{O}_X(K)$ is compact if $\mathcal{K}$ is a family of compact subsets of X. A topological space is called *consonant* if every compact family $\mathcal{A}$ is *compactly generated*, that is, there is a family $\mathcal{K}$ of compact sets such that $\mathcal{A}=\bigcup_{K\in\mathcal{K}}\mathcal{O}_X(K)$. Similarly, $p(X):=\{\mathcal{O}_X(F):F\in[X]^{<\omega}\}$ and $k(X):=\{\mathcal{O}(K):K\subseteq X \text{ compact }\}$ are basis for topologies on $\mathcal{O}_X$. Accordingly, $p(X,Y)$ is the *topology of pointwise convergence*, $k(X,Y)$ is the *compact-open topology* and $\kappa(X,Y)$ is the *Isbell topology* on $C(X,Y)$.

If $\$^*:=\{\varnothing,\{0\},\{0,1\}\}$ the function spaces $C(X,\$^*)$ can be identified with the set of open subsets of X ([3]). In this notation, X is consonant if and only if $C_k(X,\$^*)=C_\kappa(X,\$^*)$.

More generally, a space X is called Z-*consonant* if $C_\kappa(X,Z)=C_k(X,Z)$ [11, chapter 3]. [11, Problem 62] asks for what spaces Z (other than $\$^*$) Z-consonance impies consonance. A still more general problem is, given a collection α of compact families defined for each space X, to determine for what spaces Z,

$$C_k(X,Z)=C_\alpha(X,Z)\Longleftrightarrow C_k(X,\$^*)=C_\alpha(X,\$^*). \tag{2.1}$$

[3] In [3], [5] and [6], we distinguish two homeomorphic copies $\$:=\{\varnothing,\{1\},\{0,1\}\}$ and $\$^*:=\{\varnothing,\{0\},\{0,1\}\}$ of the *Sierpiński topology* on $\{0,1\}$ and identify the function spaces $C(X,\$)$ and $C(X,\$^*)$ with the set of closed subsets of X and open subsets of X respectively. This is why we use $\$^*$here.

The latter equality always implies the former. More generally, in view of the definition of $\alpha(X,Z)$, if α and γ are collections (of compact families on X), then

$$C_\alpha(X,\$^*) \leq C_\gamma(X,\$^*) \Longrightarrow C_\alpha(X,Z) \leq C_\gamma(X,Z)$$

for every topological space Z. To show the converse implication under some additional assumptions, recall that the *restriction of* $\mathcal{A}$ *to* $A \in \mathcal{A}$ is defined by

$$\mathcal{A} \downarrow A := \{U \in \mathcal{O}_X : \exists B \subseteq A \cap U, B \in \mathcal{A}\}. \tag{2.2}$$

[5, Lemma 2.8] shows that if $\mathcal{A}$ is a compact family and $A \in \mathcal{A}$, then $\mathcal{A} \downarrow A$ is compact too. A collection α of families of open subsets of a given set is *hereditary* if $\mathcal{A} \downarrow A \in \alpha$ whenever $\mathcal{A} \in \alpha$ and $A \in \mathcal{A}$.

It was shown in [5, Proposition 2.4] that if X is completely regular and $\mathbb{R}$-consonant, then it is consonant. More generally:

Proposition 2.1. *If $\alpha, \gamma \subseteq \kappa(X)$ are two topologies, α is hereditary, X is completely regular, and $C_\alpha(X,\mathbb{R}) \leq C_\gamma(X,\mathbb{R})$, then $C_\alpha(X,\$^*) \leq C_\gamma(X,\$^*)$.*

Proof. The neighborhood filter of an open set A with respect to $\alpha(X,\$^*)$ is generated by a base of the form $\{\mathcal{A} \in \alpha : A \in \mathcal{A}\}$. Therefore we need show that for each $\mathcal{A} \in \alpha$ and each $A \in \mathcal{A}$, there exists $\mathcal{G} \in \gamma$ such that $\mathcal{G} \subseteq \mathcal{A} \downarrow A$. By assumption, $\mathcal{N}_\gamma(\overline{0}) \geq \mathcal{N}_\alpha(\overline{0})$ so that for each $\mathcal{A} \in \alpha$ and each $A \in \mathcal{A}$, there exists $\mathcal{G} \in \gamma$ and $r > 0$ such that $[\mathcal{G}, (-r,r)] \subset \left[\mathcal{A} \downarrow A, (-\frac{1}{2}, \frac{1}{2})\right]$. Suppose that there exists $G \in \mathcal{G} \setminus (\mathcal{A} \downarrow A)$, hence $X \setminus G \in (\mathcal{A} \downarrow A)^{\#}$. Because X is completely regular and $\mathcal{G}$ is compact there is $G_0 \in \mathcal{G}$ and a continuous function f such that $f(G_0) = \{0\}$ and $f(X \setminus G) = \{1\}$, by [5, Lemma 2.5]. Then $f \in [\mathcal{G}, (-r,r)]$ but $f \notin \left[\mathcal{A} \downarrow A, (-\frac{1}{2}, \frac{1}{2})\right]$, because $1 \in f(B)$ for each $B \in \mathcal{A} \downarrow A$. Therefore $A \in \mathcal{G} \subseteq \mathcal{A} \downarrow A \subseteq \mathcal{A}$, so that $\alpha \leq \gamma$. □

Corollary 2.2. *If X is completely regular and $\alpha \subseteq \kappa(X)$ is hereditary, then (2.1) holds for $Z = \mathbb{R}$.*

The *grill* of a family $\mathcal{A}$ of subsets of X is the family $\mathcal{A}^{\#} := \{B \subseteq X : \forall A \in \mathcal{A}, A \cap B \neq \varnothing\}$. Note that if $\mathcal{A} = \mathcal{O}(\mathcal{A})$, then

$$A \in \mathcal{A} \Longleftrightarrow A^c \notin \mathcal{A}^{\#}.$$

If $\mathcal{A} \in \kappa(X)$ and C is a closed subset of X such that $C \in \mathcal{A}^{\#}$ then the family

$$\mathcal{A} \vee C := \mathcal{O}\left(\{A \cap C : A \in \mathcal{A}\}\right),$$

called *section of* $\mathcal{A}$ by C, is a compact family on X [3]. A collection α of families of open subsets of a given set is *sectionable* if $\mathcal{A} \vee C \in \alpha$ whenever $\mathcal{A} \in \alpha$ and C is a closed set in $\mathcal{A}^{\#}$. It was shown in [5, Theorem 2.9] that $C_\kappa(X,Z)$ is completely regular whenever Z is. A simple modification of the proof leads to the following generalization.

Theorem 2.3. *If Z is completely regular and $\alpha \subseteq \kappa(X)$ is sectionable, then $C_\alpha(X,Z)$ is completely regular.*

As $r[\mathcal{A}, U] = [\mathcal{A}, rU]$ for all $r \neq 0$, it is immediate that inversion for $+$ is always continuous in $C_\alpha(X,\mathbb{R})$. More generally, the proof of the joint continuity of scalar multiplication in $C_\kappa(X,\mathbb{R})$ [5, Proposition 2.10] can be adapted to the effect that:

Proposition 2.4. *If $\alpha \subseteq \kappa(X)$ is hereditary, then multiplication by scalars is jointly continuous for $C_\alpha(X,\mathbb{R})$.*

Corollary 2.5. *Let $\alpha \subseteq \kappa(X)$ be hereditary. If $C_\alpha(X,\mathbb{R})$ is a topological group then it is a topological vector space.*

3. Self-joinable collections and joint continuity of addition at the zero function

As usual, if A and B are subsets of an additive group, $A+B := \{a+b : a \in A, b \in B\}$ and if $\mathcal{A}$ and $\mathcal{B}$ are two families of subsets, $\mathcal{A}+\mathcal{B} := \{A+B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

As we have mentioned, a topology on an additive group is a group topology if and only if inversion and translations are continuous, and $\mathcal{N}(o)+\mathcal{N}(o) \geq \mathcal{N}(o)$, where o is the neutral element. First, we investigate the latter property, that is,

$$\mathcal{N}_\alpha(\overline{0})+\mathcal{N}_\alpha(\overline{0}) \geq \mathcal{N}_\alpha(\overline{0}), \tag{3.1}$$

for the space $C_\alpha(X,\mathbb{R})$, where $\overline{0}$ denotes the zero function.

If α and γ are two subsets of $\kappa(X)$, we say that α is *γ-joinable* if for every $\mathcal{A} \in \alpha$, there is $\mathcal{G} \in \gamma$ such that $\mathcal{G} \vee \mathcal{G} \subseteq \mathcal{A}$, where

$$\mathcal{G} \vee \mathcal{G} := \{G_1 \cap G_2 : G_1, G_2 \in \mathcal{G}\}.$$

A subset α of $\kappa(X)$ is *self-joinable* if it is α-joinable. A family $\mathcal{A}$ is called *joinable* if $\{\mathcal{A}\}$ is $\kappa(X)$-joinable. In [5], a space X is called *infraconsonant* if every compact family is joinable, that is, $\kappa(X)$ is self-joinable. [5, Theorem 3.1] shows that among completely regular spaces X,

$$\mathcal{N}_\kappa(\overline{0})+\mathcal{N}_\kappa(\overline{0}) \geq \mathcal{N}_\kappa(\overline{0})$$

if and only if X is infraconsonant. More generally,

Theorem 3.1. *Let X be a completely regular space. Then $\alpha \subseteq \kappa(X)$ is self-joinable if and only if*

$$\mathcal{N}_\alpha(\overline{0})+\mathcal{N}_\alpha(\overline{0}) \geq \mathcal{N}_\alpha(\overline{0}). \tag{3.2}$$

Proof. Let $\mathcal{A} \in\alpha$ and $V \in \mathcal{N}_\mathbb{R}(0)$. Because α is self-joinable, there exist a compact family $\mathcal{B}$ in α such that $\mathcal{B} \vee \mathcal{B} \subseteq \mathcal{A}$. If $W \in \mathcal{N}_\mathbb{R}(0)$ such that $W+W \subseteq V$, then $[\mathcal{B},W]+[\mathcal{B},W] \subseteq [\mathcal{A},V]$, which proves (3.2).

Conversely, assume that $\alpha = \alpha^\cap$ is not self-joinable. Let $\mathcal{A}$ be a family of α such that $\mathcal{B} \vee \mathcal{B} \nsubseteq \mathcal{A}$ for every $\mathcal{B} \in \alpha$. Note that $\mathcal{B} \vee \mathcal{C} \nsubseteq \mathcal{A}$ for every pair of families $\mathcal{B}$ and $\mathcal{C}$ in α for otherwise $\mathcal{D} = \mathcal{B} \cap \mathcal{C}$ would be a family of α such that $\mathcal{D} \vee \mathcal{D} \subseteq \mathcal{A}$. Let $V = \left(-\frac{1}{2}, \frac{1}{2}\right)$. We claim that for any pair $(\mathcal{B},\mathcal{C}) \in \alpha^2$ and any pair (U,W) of $\mathbb{R}$-neighborhood of 0, $[\mathcal{B},U]+[\mathcal{C},W] \nsubseteq [\mathcal{A},V]$. Indeed, there exist $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $B \cap C \notin \mathcal{A}$. Then $B^c \cup C^c \in \mathcal{A}^\#$. Moreover, $B^c \notin \mathcal{B}^\#$ so that by [5, Lemma 2.5], there exist $B_1 \in \mathcal{B}$ and $f \in C(X,\mathbb{R})$ such that $f(B_1) = \{0\}$ and $f(B^c) = \{1\}$. Similarly, $C^c \notin \mathcal{C}$ so that there exist $C_1 \in \mathcal{C}$ and $g \in C(X,\mathbb{R})$ such that $g(C_1) = \{0\}$ and $g(C^c) = \{1\}$. Then $f+g \in [\mathcal{B},U]+[\mathcal{C},W]$ but $1 \in (f+g)(A)$ for all $A \in \mathcal{A}$ so that $f+g \notin [\mathcal{A},V]$. □

Note that the collection $k(X)$ (of compactly generated families) is self-joinable, and that a union of self-joinable collections is self-joinable. Therefore, there is a largest self-joinable subset $\Lambda(X)$ of $\kappa(X)$. If α is self-joinable, so is $\alpha^\cap$. Therefore $\Lambda(X)$ is stable for finite intersections. In fact, $\Lambda(X)$ is a topology on $C(X,\$^*)$ and

$$k(X) \subseteq \Lambda(X) \subseteq \kappa(X). \tag{3.3}$$

Corollary 3.2. *Let X be completely regular. The largest subcollection α of $\kappa(X)$, for which (3.2) holds is $\alpha = \Lambda(X)$. In particular, a completely regular space X is infraconsonant if and only if $\kappa(X) = \Lambda(X)$.*

The collection $\Lambda(X)$ is sectionable ([4]), but in general it is not hereditary. We will construct now a largest hereditary collection of compact families for which (3.2) holds.

If α and γ are two subsets of $\kappa(X)$, we say that α is *hereditarily γ-joinable* if for every $\mathcal{A} \in \alpha$, and every $A \in \mathcal{A}$, there is $\mathcal{G} \in \gamma$ such that $A \in \mathcal{G}$ and $\mathcal{G} \vee \mathcal{G} \subseteq \mathcal{A}$. A subset α of $\kappa(X)$ is *hereditarily self-joinable* if it is hereditarily α-joinable. A family $\mathcal{A}$ is called *hereditarily joinable* if $\{\mathcal{A}\}$ is hereditarily $\kappa(X)$-joinable. There exists a largest hereditarily self-joinable subset $\Lambda^{\downarrow}(X)$ of $\kappa(X)$. Notice that $\Lambda^{\downarrow}(X)$ is also the largest self-joinable and hereditary collection of compact families, and that $\Lambda^{\downarrow}(X)$ is sectionable.

Corollary 3.3. *Let X be completely regular. The largest hereditary subcollection α of $\kappa(X)$, for which (3.2) holds is $\alpha = \Lambda^{\downarrow}(X)$. In particular, a completely regular space X is infraconsonant if and only if $\kappa(X) = \Lambda^{\downarrow}(X)$.*

Of course, $\Lambda^{\downarrow}(X)$ is a topology, and $\Lambda^{\downarrow}(X) \subseteq \Lambda(X)$. The inclusion can be strict. In fact, we have:

Proposition 3.4. *A regular space X is infraconsonant if and only if $\Lambda^{\downarrow}(X) = \Lambda(X)$ if and only if $\kappa(X) = \Lambda^{\downarrow}(X)$.*

Proof. If X is not infraconsonant, there is a non-joinable family $\mathcal{A}$ on X. For any $x \in X \setminus \bigcap \mathcal{A}$, the family $\mathcal{O}(x) \cup \mathcal{A}$ belongs to $\Lambda(X)$ but not to $\Lambda^{\downarrow}(X)$. If X is regular and infraconsonant, then by [5, Lemma 3.2] $\kappa(X) = \Lambda^{\downarrow}(X)$. Finally if $\kappa(X) = \Lambda^{\downarrow}(X)$ then $\Lambda^{\downarrow}(X) = \Lambda(X)$, because $\Lambda(X)$ is between $\Lambda^{\downarrow}(X)$ and $\kappa(X)$. □

Examples of non-infraconsonant spaces are provided in [5], so that both inclusions in (3.3) can be strict simultaneously.

4. Self-splittable collections and continuity of translations

A collection α is *γ-splittable* if for every $\mathcal{A} \in \alpha$ and for every open subsets U_1 and U_2 of X such that $U_1 \cup U_2 \in \mathcal{A}$ there exist families $\mathcal{G}_i = \mathcal{G}_i \downarrow U_i$ in γ, $i \in \{1, 2\}$, such that $\mathcal{G}_1 \cap \mathcal{G}_2 \subseteq \mathcal{A}$. A collection α is *self-splittable* if it is α-splittable. A compact family $\mathcal{A}$ on X is *splittable* if $\{\mathcal{A}\}$ is $\kappa(X)$-splittable. An immediate induction shows that if α is self-splittable, then for every $\mathcal{A} \in \alpha$ and every finite collection $\{U_1, \ldots, U_n\}$ of open subsets of X such that $\bigcup_{i=1}^{n} U_i \in \mathcal{A}$, there are families $\mathcal{C}_i \in \alpha$ with $\mathcal{C}_i = \mathcal{C}_i \downarrow U_i$ such that $\bigcap_{i=1}^{n} \mathcal{C}_i \subseteq \mathcal{A}$.

In [10], F. Jordan calls a topological space *compactly splittable* if every compact family is splittable. A topological space with at most one non-isolated point is said to be *prime.* A modification of the proof of [9, Theorem 18] shows:

Proposition 4.1. *Prime spaces are compactly splittable.*

It follows from [10, Theorem 2] and [5, Corollary 4.2] that translations are continuous in the Isbell topological space $C_\kappa(X, \mathbb{R})$ if X is compactly splittable. More generally, we have:

[4]Indeed, if α is self-joinable, so is $\{\mathcal{A} \vee C : \mathcal{A} \in \alpha, C \in \mathcal{A}^{\#}, C \text{ closed}\}$ because if $\mathcal{A} \in \alpha$, there is $\mathcal{B} \in \alpha$ such that $\mathcal{B} \vee \mathcal{B} \subset \mathcal{A} \subset \mathcal{A} \vee C$. By maximality, $\Lambda(X)$ is sectionable.

Proposition 4.2. *If $\alpha \subseteq \kappa(X)$ is self-splittable, then translations are continuous for $C_\alpha(X,\mathbb{R})$.*

Proof. We show continuity of the translation by f_0 at g_0. Let $\mathcal{A} \in \alpha$ and $U \in \mathcal{O}_\mathbb{R}$ such that $f_0 + g_0 \in [\mathcal{A}, U]$. There is $A_0 \in \mathcal{A}$ such that $(f_0+g_0)(A_0) \subseteq U$. For each $x \in A_0$, there exists $V_x = -V_x \in \mathcal{O}_\mathbb{R}(0)$ such that $f_0(x) + g_0(x) + 2V_x \subseteq U$. Moreover, by continuity of f_0 and g_0, there is $W_x \in \mathcal{O}_X(x)$ such that $f_0(W_x) \subseteq f_0(x) + V_x$ and $g_0(W_x) \subseteq g_0(x) + V_x$. As $\bigcup_{x\in A_0} W_x \in \mathcal{A}$ and $\mathcal{A}$ is compact, there is a finite subset F of A_0 such that $W := \bigcup_{x\in F} W_x \in \mathcal{A}$. Because α is self-splittable, there exists, for each $x \in F$, a compact family $\mathcal{C}_x = \mathcal{C}_x \downarrow W_x$ of α such that $\bigcap_{x\in F} \mathcal{C}_x \subseteq \mathcal{A}$. Note that by construction $g_0 \in \bigcap_{x\in F}[\mathcal{C}_x, g_0(x) + V_x]$. Moreover, if $g \in \bigcap_{x\in F}[\mathcal{C}_x, g_0(x)+V_x]$, then for each $x \in F$ there is $C_x \in \mathcal{C}_x$, $C_x \subseteq W_x$, such that $g(C_x) \subseteq g_0(x) + V_x$. If $y \in \bigcup_{x\in F} C_x \in \mathcal{A}$, then $y \in C_x$ for some x and

$$f_0(y) + g(y) \in f_0(x) + V_x + g_0(x) + V_x \subseteq U,$$

so that $f_0 + \bigcap_{x\in F}[\mathcal{C}_x, g_0(x)+V_x] \subseteq [\mathcal{A}, U]$. □

[3, Example 4.9] shows that even on a compactly splittable space (like the Arens space), there exists $\alpha \subseteq \kappa(X)$ such that translations are not continuous for $C_\alpha(X,\mathbb{R})$, so that α is not self-splittable.

Note that in a regular space X, the collection $k(X)$ is self-splittable, and a union of self-splittable collections is self-splittable. Therefore, there is a largest self-splittable subset $\Sigma(X)$ of $\kappa(X)$. If α is self-splittable, so is $\alpha^\cap$, hence that $\Sigma(X)$ is a topology on $C(X, \$^*)$, and

$$k(X) \subseteq \Sigma(X) \subseteq \kappa(X). \tag{4.1}$$

Moreover, $\Sigma(X)$ is clearly hereditary, and sectionable ([5]).

Both inequalities in (4.1) can be strict. Examples of non-compactly splittable spaces are provided in [9], so that $\Sigma(X)$ can be strictly included in $\kappa(X)$. On the other hand, in view of Proposition 4.1, if X is prime and not consonant (e.g., the Arens space), then $k(X)$ is strictly included in $\Sigma(X)$.

Theorem 4.3. *Let X be regular. If $\alpha \subseteq \kappa(X)$ is self-joinable, hereditary, and sectionable, then α is self-splittable.*

Proof. By way of contradiction, assume that α is not self-splittable. Let $\alpha_1 = \alpha \cup \{\mathcal{O}(X)\}$. Using that $\emptyset \in \mathcal{O}(X)$, one can easily show that α_1 is self-joinable, hereditary, and sectionable. It is also easy to check that α_1 is not self-splittable.

Let $\mathcal{C} \in \alpha_1$ witness that α_1 is not self-splittable. There exist $\mathcal{C} \in \alpha_1$ and open sets U_1, U_2 such that $U_1 \cup U_2 \in \mathcal{C}$, but for any $\mathcal{B}_1, \mathcal{B}_2 \in \alpha_1$ with $U_1 \in \mathcal{B}_1$ and $U_2 \in \mathcal{B}_2$ we have $\mathcal{B}_1 \cap \mathcal{B}_2 \not\subseteq \mathcal{C}$.

By regularity, there is an open cover $\mathcal{V}$ of $U_1 \cup U_2$ such that $\mathrm{cl}(V) \subseteq U_1$ or $\mathrm{cl}(V) \subseteq U_2$ for every $V \in \mathcal{V}$. Since $U_1 \cup U_2 \in \mathcal{C}$, there exist a finite $\mathcal{V}_1 \subseteq \mathcal{V}$ such that $\bigcup \mathcal{V}_1 \in \mathcal{C}$. Let $W_1 = \bigcup\{V \in \mathcal{V}_1 : \mathrm{cl}(V) \subseteq U_1\}$ and $W_2 = \bigcup\{V \in \mathcal{V}_1 : \mathrm{cl}(V) \subseteq U_2\}$.

[5]Indeed, if α is self-splittable, so is $\{\mathcal{A} \vee C : \mathcal{A} \in \alpha, C \in \mathcal{A}^\#, C \text{ closed}\}$. Indeed, if $\bigcup_{i=1}^n U_i \in \mathcal{A} \vee C$, then $\bigcup_{i=1}^n U_i \cup C^c \in \mathcal{A}$, so that there are families $\mathcal{C}_i = \mathcal{C}_i \downarrow U_i$ and $\mathcal{C}_c = \mathcal{C}_c \downarrow C^c$ in α such that $\bigcap \mathcal{C}_i \cap \mathcal{C}_c \subseteq \mathcal{A}$. Therefore

$$\bigcap_i \mathcal{C}_i \vee C = \left(\bigcap_i \mathcal{C}_i \cap \mathcal{C}_c\right) \vee C \subseteq \mathcal{A} \vee C.$$

By maximality, $\Lambda(X)$ is sectionable.

Notice that $W_1 \cup W_2 \in \mathcal{C}$, $\mathrm{cl}(W_1) \subseteq U_1$, and $\mathrm{cl}(W_2) \subseteq U_2$. Let $\mathcal{C}_1 = \mathcal{C} \downarrow (W_1 \cup W_2)$. Since α_1 is hereditary, $\mathcal{C}_1 \in \alpha_1$. By self-joinability of α_1 there is a $\mathcal{D} \in \alpha_1$ such that $\mathcal{D} \bigvee \mathcal{D} \subseteq \mathcal{C}_1$. Notice that $\mathcal{D} \subseteq \mathcal{C}_1$.

Suppose there is a $D \in \mathcal{D}$ such that $D \cap W_1 = \emptyset$. Since $D \in \mathcal{D} \subseteq \mathcal{C}_1$, there is an $E \in \mathcal{C}$ such that $E \subseteq D \cap (W_1 \cup W_2)$. Notice that $E \subseteq W_2 \subseteq U_2$. So, $U_2 \in \mathcal{C}$. Since α_1 is hereditary, $\mathcal{C} \downarrow U_2 \in \alpha_1$. Notice that $U_2 \in \mathcal{C} \downarrow U_2 \in \alpha_1$, $U_1 \in \mathcal{O}(X) \in \alpha_1$, and $\mathcal{C} \downarrow U_2 \cap \mathcal{O}(X) = \mathcal{C} \downarrow U_2 \subseteq \mathcal{C}$, which contradicts our choice of U_1 and U_2. So, we may assume that $\mathcal{D} \# W_1$. Similarly, we may assume that $\mathcal{D} \# W_2$.

For each $i \in \{1, 2\}$ let $\mathcal{D}_i = \mathcal{D} \bigvee \mathrm{cl}(W_i)$. Since α_1 is sectionable, $\mathcal{D}_1, \mathcal{D}_2 \in \alpha_1$. Notice that $U_i \in \mathcal{D}_i$ for every i. Let $P \in \mathcal{D}_1 \cap \mathcal{D}_2$. For every $i \in \{1, 2\}$ there exist $D_i \in \mathcal{D}$ such that $D_i \cap \mathrm{cl}(W_i) \subseteq P$. Since $D_1, D_2 \in \mathcal{D}$, $D_1 \cap D_2 \in \mathcal{C}_1$. There is an $E \in \mathcal{C}$ such that $E \subseteq D_1 \cap D_2 \cap (W_1 \cup W_2)$. Now,

$$E \subseteq (D_1 \cap D_2) \cap \mathrm{cl}(W_1 \cup W_2) \subseteq (D_1 \cap \mathrm{cl}(W_1)) \cup (D_2 \cap \mathrm{cl}(W_2)) \subseteq P.$$

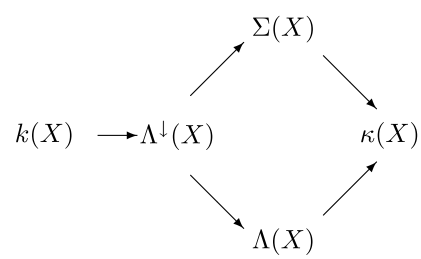
So, $P \in \mathcal{C}$. Thus, $\mathcal{D}_1 \cap \mathcal{D}_2 \subseteq \mathcal{C}$, contradicting our choice of U_1 and U_2. □

Corollary 4.4. *Let X be regular. If X is infraconsonant then X is compactly splittable.*

Proof. If X is infraconsonant, then $\kappa(X)$ is self-joinable. Since $\kappa(X)$ is hereditary and sectionable, $\kappa(X)$ is self-splittable, by Theorem 4.3. Since $\kappa(X)$ is self-splittable, X is compactly splittable. □

Note that Corollary 4.4 provides a negative answer to [5, Problem 1.2]. The converse of Corollary 4.4 is not true. For instance, the Arens space is compactly splittable because it is prime, but it is not infraconsonant [5, Theorem 3.6].

In the diagram below, X is a regular space, and arrows represent inclusions. We have already justified that all of these inclusions may be strict, except for $k(X) \subseteq \Lambda^{\downarrow}(X)$. We will see in the next section that it may be strict.



In view of Proposition 2.4, Theorem 3.1, Proposition 4.2 and Theorem 4.3, we obtain:

Corollary 4.5. *Let X be completely regular. Then $C_{\Lambda^{\downarrow}}(X, \mathbb{R})$ is a topological vector space.*

As a consequence, we can extend [5, Theorem 5.3] from prime spaces to general completely regular spaces to the effect that:

Corollary 4.6. *Let X be completely regular. The following are equivalent:*

(1) *X is infraconsonant;*
(2) $\kappa(X) = \Lambda(X)$;
(3) $\kappa(X) = \Lambda^{\downarrow}(X)$;
(4) $\Lambda(X) = \Lambda^{\downarrow}(X)$;

(5) $C_\kappa(X,\mathbb{R})$ *is a topological vector space;*
(6) $C_\kappa(X,\mathbb{R})$ *is a topological group;*
(7) $\mathcal{N}_\kappa(\overline{0})+\mathcal{N}_\kappa(\overline{0})\geq\mathcal{N}_\kappa(\overline{0})$;
(8) $\cap : C_\kappa(X,\$^*)\times C_\kappa(X,\$^*)\to C_\kappa(X,\$^*)$ *is jointly continuous.*

Proof. Equivalences between (1) through (7) follow immediately from Theorem 3.1, Corollary 3.2 and Proposition 3.4. The equivalence with (8) follows from [5, Proposition 3.3]. □

5. A vector space topology strictly finer than the compact-open topology

A finite measure μ is called τ-*additive* if for every family $\mathcal{P}\subset\mathcal{O}_X$, and for every $\varepsilon>0$ there is a finite subfamily $\mathcal{P}_\varepsilon\subset\mathcal{P}$ such that $\mu(\bigcup\mathcal{P}_\varepsilon)\geq\mu(\bigcup\mathcal{P})-\varepsilon$. Hence, if μ is a τ-additive measure on X, then for each $r>0$, the family

$$\mathcal{M}_r^\mu:=\{O\in\mathcal{O}_X:\mu(O)>r\}$$

is compact. A topological space X is called *pre-Radon* if every finite τ-additive measure μ on X is a Radon measure, that is, $\mu(B)=\sup\{\mu(K):K\subset B, K \text{ compact}\}$ for each Borel subset B of X.

Lemma 5.1. *Let μ be a τ-additive finite measure on a space X. Then $\gamma_\mu:=\{\mathcal{M}_r^\mu\downarrow A : A\in\mathcal{M}_r^\mu, r>0\}$ is self-splittable and hereditarily self-joinable.*

Proof. Proof of 1. γ_μ is hereditary and self-joinable (hence hereditarily self-joinable) because if $U\in\mathcal{M}_r^\mu$ and $m=\frac{r+\mu(U)}{2}$ then

$$(\mathcal{M}_m^\mu\downarrow U)\vee(\mathcal{M}_m^\mu\downarrow U)\subseteq\mathcal{M}_r^\mu\downarrow U. \tag{5.1}$$

Indeed, if O_1 and O_2 are elements of $\mathcal{M}_m^\mu\downarrow U$, we can assume that $\mu(O_1\cup O_2)\leq\mu(U)$ so that

$$\mu(O_1\cap O_2)=\mu(O_1)+\mu(O_2)-\mu(O_1\cup O_2)>2m-\mu(U)=r.$$

Self-splittability follows from the fact ([6]) that if $U_1\cup U_2\in\mathcal{M}_r$, for $d:=\min(\mu(U_1),\mu(U_2),\mu(U_1\cup U_2)-r)>0$, $m_1:=\mu(U_1)-\frac{d}{2}$, $m_2:=\mu(U_2)-\frac{d}{2}$, we have

$$\left(\mathcal{M}_{m_1}^\mu\downarrow U_1\right)\cap\left(\mathcal{M}_{m_2}^\mu\downarrow U_2\right)\subseteq\mathcal{M}_r. \tag{5.2}$$

Indeed, if $A_i\in\mathcal{M}_{m_i}^\mu\downarrow U_i$ for $i\in\{1,2\}$, then

$$\begin{aligned} &\mu(A_1\cup A_2)=\mu(A_1)+\mu(A_2)-\mu(A_1\cap A_2)\\ \geq\ &\mu(A_1)+\mu(A_2)-\mu(U_1\cap U_2)\\ >\ &\mu(U_1)+\mu(U_2)-\mu(U_1\cap U_2)-d\\ >\ &r. \end{aligned}$$

□

Theorem 5.2. *If X is a (Hausdorff) completely regular space but is not pre-Radon, then*

$$C_{\Lambda^\downarrow}(X,\mathbb{R})>C_k(X,\mathbb{R}).$$

[6]Note that γ_μ is not sectionable, so that Theorem 4.3 is not sufficient to deduce self-splittability.

Proof. The proof of [2, Proposition 3.1] shows that if X is a Hausdorff non pre-Radon space, then there is a τ-additive finite measure μ and an $r > 0$ such that $\mathcal{M}_r$ is compact but not compactly generated. In view of Lemma 5.1, $\gamma_\mu \subseteq \Lambda^{\downarrow}(X)$ so that $C_k(X, \$^*) < C_{\Lambda^{\downarrow}}(X, \$^*)$. In view of Proposition 2.2, $C_k(X, \mathbb{R}) < C_{\Lambda^{\downarrow}}(X, \mathbb{R})$ because $\Lambda^{\downarrow}(X)$ is hereditary. □

Note that we have shown that the inclusion $k(X) \subseteq \Lambda^{\downarrow}(X)$ may be strict.

For instance, if X is the Sorgenfrey line, which is not pre-Radon, then $C_{\Lambda^{\downarrow}}(X, \mathbb{R})$ is a topological vector space and is strictly finer than the compact-open topology.

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