THE ASTONISHING OBLIVION OF PEANO'S MATHEMATICAL LEGACY (III)

MEASURE THEORY AND TOPOLOGY

SZYMON DOLECKI AND GABRIELE H. GRECO

Abstract. The formulations that Peano gave to many mathematical notions at the end of 19th century were so perfect and modern they have become standard today. A formal language of logic that he created, enabled him to perceive mathematics with great precision and depth. He described mathematics axiomatically basing the reasoning exclusively on logical and set-theoretic primitive terms and properties, which was revolutionary at that time. Yet numerous Peano's contributions remain either unremembered or underestimated.

This is the last of our three papers (c.f., [8],[9]) about Peano's forgotten heritage.

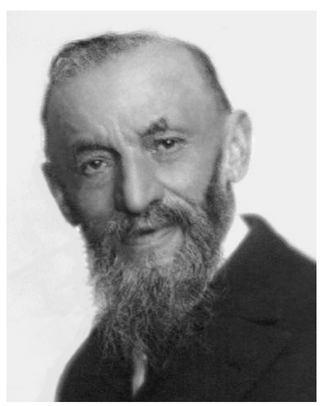


Figure 1. Giuseppe Peano (1858-1932)

1. Surface area

Serret defined in [38, (1880) p. 296] the area of a surface as a limit of the areas of inscribed polyhedral surfaces. In 1882 Peano observed that this definition is ambiguous and that in the case of the lateral surface of a circular cylinder (of height H and of radius R), it is possible to choose a sequence

Date: February 23, 2015. Submitted to *The Mathematical Intelligencer*.
The authors are grateful to Professors David L. Russell (Virginia Polytechnic Institute and State University) and Iwo Labuda (University of Mississippi, Oxford) for their precious advice on presentation of this paper.

of polyhedral surfaces fulfilling Serret's condition such that the mentioned limit is *anything* greater than or equal to $2\pi RH$ (see Peano [31, (1890)] and Lebesgue [22, (1902)]).

Here is a description of Peano's construction [35, (1902-03), pp.300-301]. Given integers n and m, divide the circular cylinder into n circular cylinders of height $\frac{H}{n}$ and inscribe in each circular base of these cylinders a regular m-gon in such a way, that the vertices of each polygon are rotated by $\frac{\pi}{m}$ with respect to the adjacent ones. Consider the triangles, the vertices of which are two consecutive vertices of the polygon inscribed in a base and one vertex is that of a polygon inscribed in a neighboring base, lying vertically in between the first two vertices. These $2mn$ isosceles triangles are the facets of a polyhedron. The circular cylinder is approximated by this inscribed polyhedron that resemble a traditional Venetian lantern.

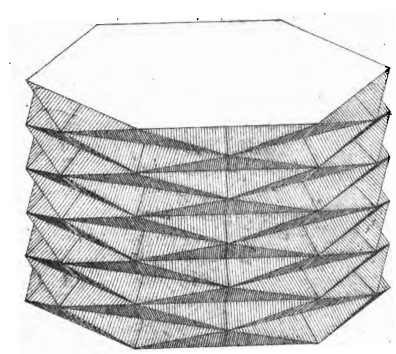


FIGURE 2. A Venetian lantern taken from a course of C. Hermite [15, (1883) p. 36].

The length of the bases of these isosceles triangles is $2R\sin\frac{\pi}{m}$ and the altitude is

$$\sqrt{R^2(1-\cos\frac{\pi}{m})^2+\frac{H^2}{n^2}}.$$

As $1-\cos\frac{\pi}{m}=2\sin^2\frac{\pi}{2m}$, the area of the polyhedral surface of $2mn$ facets is equal to

$$A\,(n,m) := 2mnR\sin\frac{\pi}{m}\sqrt{4R^2\sin^4\frac{\pi}{2m}+\frac{H^2}{n^2}}. \tag{1}$$

When n and m tend to ∞, the limit of $A(n,m)$ depends on the ratio of n and m. For example, if $n=m$, then $\lim_{m\to\infty} A\,(m,m)=2\pi RH$, but if, for instance, $n=m^3$, then $\lim_{m\to\infty} A(m^3,m)=\infty$.

Peano identified a principal error of Serret's method, that is, that a variable plane passing through three non-collinear points of a surface S, does not necessarily tend to the tangent plane of S at a point x, when these three points tend to x (see Peano [31, (1890)]).

When Peano reported his discovery to Genocchi, he was told that a similar construction had already been discovered by H. A. Schwarz two years earlier (see Kennedy [20, (1980)], [21, (2006)]).

Peano did not rest on his laurels. In *Applicazioni geometriche* [30, (1887)] he proposed a definition of surface area that coincides with the Lagrange area formula in case of regular Cartesian surfaces ([1]):

$$\iint_D \sqrt{1+\|\nabla f(x,y)\|^2}\,\mathrm{d}x\,\mathrm{d}y. \tag{2}$$

Peano's construction was the following: fix a plane L and, for an arbitrary finite partition of the surface S, move arbitrarily but rigidly each element of the partition and project it orthogonally onto L. Then take the sum of so obtained plane areas. This sum depends on the partition and on the positions of its elements after the transport. The supremum of so obtained sums over all the partitions and all the positions, defines the area of S (see Greco, Mazzucchi, Pagani [11])

2. Concept of plane measure

In [29, (1883)] Peano presented concepts of *external* and *internal area* and, what is most considerable, that of *measurability* ([2]) for planar sets, ten years before the work of Jordan [17, (1892)]. In introducing the inner and outer area of planar sets as well as in defining surface area, Peano was also influenced by Archimedes's approach to calculus of area, length and volume of convex figures. At that time a concept of measure was commonly used, but was not defined. ([3])

Peano was the first to prove that a positive function f of one variable is *integrable* if and only if the *positive hypograph*

$$\operatorname{hypo}^+ f := \{(x,r) : 0 \le r \le f(x)\}$$

of f is *measurable*. If this is the case, Peano showed that the integral of f is equal to the area of $\operatorname{hypo}^+ f$.

Peano considers finite unions of polygons that cover a given planar set A and finite unions of polygons that are included in A. Denote by $\mathbb{P}$ the collection of the finite families of polygons. The infimum over $\mathcal{P} \in \mathbb{P}$ of

$$\sum\nolimits_{P\in\mathcal{P}} \operatorname{area}(P) \tag{3}$$

such that $\bigcup_{P\in\mathcal{P}} P \supset A$, defines the *external area* of A, and the supremum over $\mathcal{P} \in \mathbb{P}$ of (3) such that $\bigcup_{P\in\mathcal{P}} P \subset A$ and $\mathcal{P}$ consists of non-overlapping polygons, defines the *internal area* of A. If these two values coincide, A is said to be *measurable*, and the common value is called the *area* of A.

---

[1] That is, the graphs of C^1-functions f

[2] Peano does not use the term *measurability.*

[3] After 1883, the *Inhalt* (content), which corresponds to external measure, was introduced in works by Stolz [39, (1884)], Cantor [1, (1884)], Harnack [12, (1885)].

This choice of Peano of using polygonal sets in the definitions of both external and internal area, enables one to immediately infer that the measure is isometrically invariant and does not depend on coordinate systems. The corresponding construction of Jordan, using grills of rectangles, requires a proof of such invariance.

3. Measure theory

The interest of Peano in measure theory was rooted in his criticism of the definition of *area* (1882), of *integral* (1883) and of *derivative* (1884).

This criticism led him to an innovative measure theory, which was exposed systematically and fully in a chapter of *Applicazioni geometriche* [30, (1887)], where he refounded the Riemann integral by means of inner and outer measures, as he anticipated in his juvenile work [29, (1883)]. Peano in [30, (1887)] and Jordan in [17, (1892)] and in the second edition of *Cours d'Analyse* [18, (1893)] developed well known concepts of classical measure theory, like measurability and change of variables and proved several fundamental theorems, with some methodological differences between them.

The mathematical tools employed by Peano were really advanced at that time (and perhaps are even now), both on a geometrical and a topological level. Peano used extensively the geometric vector calculus introduced by Grassmann. The geometric notions included oriented areas and volumes (called geometric forms).

Peano's measure theory was based on solid grounds of logic, set theory and topology. For example, he introduced interior and exterior points in connection with internal and external measures.

3.1. **Abstract measures.**

The most innovative ingredient of the approach of Peano is the introduction of *abstract measures* and their *differentiation.* We use the term *abstract measure* to designate a *"distributive" set function* of Peano, that is, a mapping $\mu : \mathcal{M} \to \mathbb{R}_+$ where $\mathcal{M}$ is a distributive family of subsets of a given set X (see next Section 4.2) such that

$$\mu(A_0 \cup A_1) = \mu(A_0) + \mu(A_1),$$

provided that $\mu(A_0 \cap A_1) = 0$, in particular if μ is *finitely additive.*

Among distributive set functions he considered *outer measure, inner measure, upper integral* and *lower integral*, which he defined as, respectively, the least upper bound of the upper Riemann sums and the greatest lower bound of the lower Riemann sums. Then Peano defined the integral with respect to a *finitely additive* set-function and the derivative of a measure with respect to another measure (see Greco, Mazzucchi, Pagani [10, (2010)]).

Peano observes that the outer measure of a set is the sum of the inner measure of this set and of the outer measure of its boundary

$$\mu_+(A) = \mu_-(A) + \mu_+(\partial A),$$

and thus A is measurable whenever $\mu_+(\partial A) = 0$.

Thomas Hawkins wrote in [14, (1975)]: "The [Peano measure] theory is surprisingly elegant and abstract for a work of 1887 and strikingly modern in its approach."

4. Topology

Peano's interest in general topology was considerably motivated by its founding role in measure theory.

4.1. **Interior and closure.** The notions of interior, exterior and boundary points of subsets of Euclidean space existed informally in mathematical literature before 1887, but were precisely defined for the first time in *Applicazioni Geometriche* [30, (1887)], where x is said to be an *interior point* of a subset A of Euclidean space X if there is $r > 0$ such that $B(x, r) \subset A$; an x is called an *exterior point* of A if it is an interior point of $X \setminus A$.

An x is a *boundary point* ([4]) if it is neither exterior nor interior. Subsequently, Peano defines the *interior* $\operatorname{int} A$ of A as the set of interior points, and the *closure* of A by

$$\operatorname{cl} A := \{x \in X : \operatorname{dist}(x, A) = 0\},$$

and relates it to the notion of *closed* set of Cantor [1, (1884)], that is, $\operatorname{cl} A$ is the least closed set that includes A.

These fundamental topological concepts reappeared several years later in the second edition of *Cours d'Analyse* [18, (1893)] of Jordan.

4.2. **Distributive and antidistributive families.** Miscellaneous distributive properties were studied in *Applicazioni geometriche* [30, (1887)]. Peano formalized the notion of distributive family, used implicitly by Cantor in [1, (1884)]. A family $\mathcal{H}$ of subsets of X is called *distributive* on X if

$$\text{(D)} \qquad H_0 \cup H_1 \in \mathcal{H} \Longleftrightarrow H_0 \in \mathcal{H} \text{ or } H_1 \in \mathcal{H}.$$

Among examples of distributive families given by Peano were the family of infinite sets and that of unbounded subsets of Euclidean space.

He called a family $\mathcal{A}$ (of subsets of X) *antidistributive* on X if

$$\text{(I)} \qquad A_0 \cup A_1 \in \mathcal{A} \Longleftrightarrow A_0 \in \mathcal{A} \text{ and } A_1 \in \mathcal{A}.$$

Such families are nowadays called *ideals*. Peano's examples comprise the family of finite sets and that of bounded subsets of Euclidean space.

A family $\mathcal{F}$ of subsets of a set X is called a *filter* on X (see H. Cartan [3, 2, (1937)]) if

$$\text{(F)} \qquad F_0 \in \mathcal{F} \text{ and } F_1 \in \mathcal{F} \Longleftrightarrow F_0 \cap F_1 \in \mathcal{F}$$

We include here neither the usual non-degeneracy condition: $\varnothing \notin \mathcal{F}$ (the only filter $\mathcal{F}$ on X fulfilling $\varnothing \in \mathcal{F}$ is the power set 2^X of X) nor the non-emptiness: $\mathcal{F} \neq \varnothing$.

---

[4] In Peano terminology, *limit point.*

Denote by $\mathbb{D}$ the class of distributive families on X, by $\mathbb{I}$ the class of antidistributive families on X and by $\mathbb{F}$ the class of filters on X. In order to relate the properties (D), (I) and (F), consider three unary operations $2^X \to 2^X$, namely, for $\mathcal{A} \subset 2^X$,

$$\text{(grill)} \qquad \mathcal{A}^{\#} := \{H \subset X : \underset{A\in\mathcal{A}}{\forall} H \cap A \neq \varnothing\},$$

$$\text{(complementary)} \qquad \mathcal{A}^{c} := \{H \subset X : H \notin \mathcal{A}\},$$

$$\text{(complementation)} \qquad \mathcal{A}^{\flat} := \{X \setminus A : A \in \mathcal{A}\}.$$

Then

$$\mathbb{D} = \{\mathcal{F}^{\#} : \mathcal{F} \in \mathbb{F}\} = \{\mathcal{A}^{c} : \mathcal{A} \in \mathbb{I}\} \tag{4}$$

and

$$\mathbb{I} = \{\mathcal{H}^{c} : \mathcal{H} \in \mathbb{D}\} = \{\mathcal{F}^{\flat} : \mathcal{F} \in \mathbb{F}\},\ \mathbb{F} = \{\mathcal{H}^{\#} : \mathcal{H} \in \mathbb{D}\} = \{\mathcal{A}^{\flat} : \mathcal{A} \in \mathbb{I}\}.$$

The *grill* was introduced by Choquet [5, (1947)], who noticed that $\mathcal{H}$ is the grill of a filter if and only if (D) holds, and so rediscovered the distributive property.

4.3. **Compactness.** In [30, (1887)] Peano cites Cantor [1, (1884) p. 454] for the following theorem ([5])

Theorem 1. *Let S be a closed bounded non-empty set of Euclidean space X. If $\mathcal{H}$ is a distributive family of subsets of X and $S \in \mathcal{H}$, then there exists a point $x \in S$, such that each neighborhood of x belongs to $\mathcal{H}$.*

Peano restates Theorem 1 in terms of antidistributive families:

Theorem 2. *Let S be a closed bounded non-empty subset of Euclidean space X. If $\mathcal{A}$ is an antidistributive family of subsets of X and for each $x \in S$ there is a neighborhood of x belonging to $\mathcal{A}$, then $S \in \mathcal{A}$.*

Let $\mathcal{Q}$ be an arbitrary family of subsets of an Euclidean space X. Clearly, the family of the subsets of finite unions of elements of $\mathcal{Q}$ is an antidistributive family on X. From Theorem 2 it follows that for a closed bounded set S such that $S \subset \bigcup_{A\in\mathcal{Q}} \operatorname{int}_X A$, there is a finite subfamily of $A_1, A_2, \ldots, A_n$ of $\mathcal{Q}$ such that $S \subset A_1 \cup A_2 \cup \ldots \cup A_n$. Hence Theorem 2 amounts to the Heine-Borel theorem:

Theorem 3. *Let S be a closed bounded subset of Euclidean space X. Each open cover of S has a finite subcover of S.*

Let us give a proof of the equivalence of Theorems 1 and 2.

[5] In order to simplify the formulation, we add the assumption that S is closed. In his proof, Peano, like Cantor, considers successive partitions of S of diameter tending to 0; he mentions that this method was used by Cauchy.

Proof. Let $\mathcal{N}(x)$ stand for the neighborhood filter of x and let S be a closed bounded non-empty set. Theorem 1 says that

$$\underset{\mathcal{H}\in\mathbb{D}}{\forall}\,(\, S\in\mathcal{H} \Longrightarrow \underset{x\in S}{\exists}\,\mathcal{N}(x)\subset\mathcal{H}\,). \tag{5}$$

By (4), $\mathbb{D}=\{\mathcal{A}^c : \mathcal{A}\in\mathbb{I}\}$; hence (5) becomes

$$\underset{\mathcal{A}\in\mathbb{I}}{\forall}\,(\, S\in\mathcal{A}^c \Longrightarrow \underset{x\in S}{\exists}\,\mathcal{N}(x)\subset\mathcal{A}^c\,),$$

that is,

$$\underset{\mathcal{A}\in\mathbb{I}}{\forall}\big((\underset{x\in S}{\forall}\ \underset{V\in\mathcal{N}(x)}{\exists}\ V\in\mathcal{A}) \Longrightarrow S\in\mathcal{A}\big). \tag{6}$$

On the other hand, Theorem 2 amounts to (6). ∎

By (4), $\mathbb{D}=\{\mathcal{F}^{\#} : \mathcal{F}\in\mathbb{F}\}$; hence (5) yields

$$\underset{\mathcal{F}\in\mathbb{F}}{\forall}(S\in\mathcal{F}^{\#} \Longrightarrow \underset{x\in S}{\exists}\,\mathcal{N}(x)\subset\mathcal{F}^{\#}). \tag{7}$$

On recalling that, by definition, $\operatorname{adh}\mathcal{F}$ consists of $x\in X$ such that "$Q\cap F\neq\varnothing$ for every $Q\in\mathcal{N}(x)$ and $F\in\mathcal{F}$" (in other words, $\mathcal{N}(x)\subset\mathcal{F}^{\#}$), (7) becomes:

Theorem 4. *Let S be a closed bounded non-empty set of Euclidean space X. If $\mathcal{F}$ is a filter on X and $S\in\mathcal{F}^{\#}$, then $S\cap\operatorname{adh}\mathcal{F}\neq\varnothing$.*

It is amazing that in the eighties of the nineteenth century Peano routinely used as a matter of fact two dual properties of abstract compactness, one of which he got from Cantor [1, (1884)]. The definition of "compactness" by Heine (1872) came earlier, while those of Borel (1895), Lebesgue (1902), Vietoris (1921) and Alexandrov and Urysohn (1923) were subsequent to Cantor and Peano.

Zermelo seems to be the only one who recognized at that time the importance of Peano's distributive and antidistributive families in the context of compactness [41, (1927)].

Peano considered the *Weierstrass theorem* as an immediate corollary of Theorem 1.

Corollary 5. *A continuous real-valued function on a closed bounded set attains its minimum and maximum.*

It is interesting to see Peano's use of distributive families in its proof.

Proof. The case of maximum. Let f be a real function continuous on a closed bounded set S. Consider the family $\mathcal{H}$ of subsets of S such that $H\in\mathcal{H}$ whenever $\sup f(H)=\sup f(S)$ and notice that $\mathcal{H}$ is distributive. By Theorem 1, there exists $x\in S$ such that

$$\sup f(S)=\inf_{V\in\mathcal{N}(x)}\sup f(V)\,.$$

As the function f is continuous, its upper limit at each point is equal to its value at that point, that is, $\inf_{V\in\mathcal{N}(x)}\sup f(V)=f(x)$. ∎

Although the framework remains that of Euclidean space, the method is valid for a continuous function on a compact subset of a topological space.

Peano applied the corollary above for $f = \mathrm{dist}(\cdot, X \setminus O)$ in a metric space, obtaining the following

Proposition 6. *If F is a compact set and O is an open set such that $F \subset O$, then there exists $r > 0$ such that $B(F, r) := \{x : \mathrm{dist}(x, F) < r\} \subset O$.*

5. Differentiation of measures

By retracing research on "grandeurs coexistentes" (*coexistent magnitudes*) by Cauchy [4, (1841)], in *Applicazioni geometriche del calcolo infinitesimale* [30, (1887)] Peano defined the "density" (strict derivative) of a "mass" (a distributive set function) with respect to a "volume" (a positive distributive set function), proved its continuity (whenever the strict derivative exists) and showed the validity of the *mass-density paradigm*: "mass" is recovered from "density" by integration with respect to "volume".

It is remarkable that Peano's strict derivative provided a consistent foundation for the concept of "infinitesimal ratio" between two magnitudes, successfully used since Kepler. In this way the classical (pre-Lebesgue) measure theory reached a complete and definitive form in Chapter V of Peano's *Applicazioni geometriche* [30, (1887)] ([6]).

In order to grasp the essence of Peano's contribution and to compare it with analogous results by Cauchy, Lebesgue, Radon and Nikodym, we present it in a particular significant case.

Peano's strict derivative of a set function (for instance, the "density" of a "mass" μ with respect to the "volume") at a point $\bar{x}$ is computed, when it exists, as the limit of the quotient of the "mass" with respect to the "volume" of a cube Q, when $Q \to \bar{x}$ (that is, the supremum of the distances of the points of the cube Q from $\bar{x}$ tends to 0). In formula, Peano's strict derivative $g_P(\bar{x})$ of a mass μ at $\bar{x}$ is given by:

$$g_P(\bar{x}) := \lim_{Q \to \bar{x}} \frac{\mu(Q)}{\mathrm{vol}_n(Q)}. \tag{8}$$

On the other hand, the *Cauchy derivative* $g_C(\bar{x})$ of a mass μ at $\bar{x}$ [4, (1841)] is obtained as the limit of the ratio between "mass" and "volume" of a cube Q when $Q \to \bar{x}$ and $\bar{x} \in Q$, that is,

$$g_C(\bar{x}) := \lim_{\substack{Q \to \bar{x} \\ \bar{x} \in Q}} \frac{\mu(Q)}{\mathrm{vol}_n(Q)}. \tag{9}$$

[6] A pioneering role of that book is remarked by J. Tannery [40, (1887)]: "Chapter V is titled: Geometric magnitudes. This chapter is probably the most relevant and interesting, the one that marks the difference of the Book of Peano with respect to other classical Treatises: definitions concerning *sets of points*, exterior, interior and limit points of a given set, distributive functions (coexistent magnitudes in the sense of Cauchy), exterior, interior and proper length (or area or volume) of a set, the extension of the notion of integral to a set, are stated in an abstract, very precise and very clear way."

Observe that (9) is analogous to derivative, (8) to strict derivative [9].

Lebesgue's derivative of set functions was computed *à la Cauchy.* Lebesgue considered finite σ-additive and absolutely continuous measures as "masses", while Peano contemplated distributive set functions. Lebesgue's derivative exists (i.e., the limit (9) exists for *almost every* $\bar{x}$), it is measurable and the reconstruction of a "mass" as the integral of the derivative is assured by absolute continuity of the "mass" with respect to volume. On the contrary, Peano's strict derivative need not exist, but when it does, it is continuous and the mass-density paradigm holds:

$$\mu(Q) = \int_Q g_P \, d \operatorname{vol}_n .$$

The constructive approaches to differentiation of set functions corresponding to the two limits (8) and (9) are opposed to the approach given by Radon [37, (1913)] and Nikodym [28, (1930)], who define the derivative in a more abstract and wider context than those of Lebesgue and Peano. As in the case of Lebesgue, the existence of a Radon-Nikodym derivative is assured by assuming absolute continuity and σ-additivity of the measures.

6. Peano's filling curve

Hausdorff wrote in *Grundzüge der Mengenlehre* [13, (1914)] of Peano's filling curve: *this is one of the most remarkable facts of set theory, the discovery of which we owe to G. Peano* ([7]). Today this fact is considered as topological and is a consequence of the Hahn-Mazurkiewicz theorem (1913-14) saying that *each compact connected locally connected metric space is a continuous image of the unit interval.*

Invited by Felix Klein to publish in Mathematische Annalen, Peano submitted [32, (1890)], in which he proved the existence of a continuous map from the interval $[0, 1]$ onto the square $[0, 1] \times [0, 1]$.

Figure 3. The figure representing the second approximations of Peano's curve as it appears in Peano's Formulario mathematico [36, (1908) p. 240].

In order to construct such a map, he used the ternary representation of each element t of $[0, 1]$ and transformed it into ternary representations of $x(t) \in [0, 1]$ and $y(t) \in [0, 1]$, that is, of an element of $[0, 1] \times [0, 1]$. Because

[7] *Das ist eine der merkwürdigsten Tatsachen der Mengenlehere, deren Entdeckung wir G. Peano verdanken.*

the sought map needs to be continuous, Peano's construction is necessarily more sophisticated than that of Cantor, which established a bijection between $[0,1]$ and $[0,1]\times[0,1]$.

He defined first an involution $\boldsymbol{k}:\{0,1,2\}\to\{0,1,2\}$ by $\boldsymbol{k}(0):=2$, $\boldsymbol{k}(1):=1$, $\boldsymbol{k}(2):=0$. In particular, $\boldsymbol{k}^n(a)=a$ if n is even and $\boldsymbol{k}^n(a)=\boldsymbol{k}(a)$ if n is odd. Then he defined his curve explicitly by

$$\begin{aligned} b_n &= \boldsymbol{k}^{a_2+a_4+\cdots+a_{2n-2}}(a_{2n-1}), \\ c_n &= \boldsymbol{k}^{a_1+a_3+\cdots+a_{2n-1}}(a_{2n}). \end{aligned}$$

The vertices of the three polygonal lines *inscribed* in Peano's curve in the figure below, are calculated at $0,1$ and, respectively, at $a_1a_2\overline{1}$, $a_1a_2a_3a_4\overline{1}$ and $a_1a_2a_3a_4a_5a_6\overline{1}$, where $a_1,a_2,a_3,a_4,a_5,a_6\in\{0,1,2\}$ and $\overline{1}$ stands for the periodic 1.

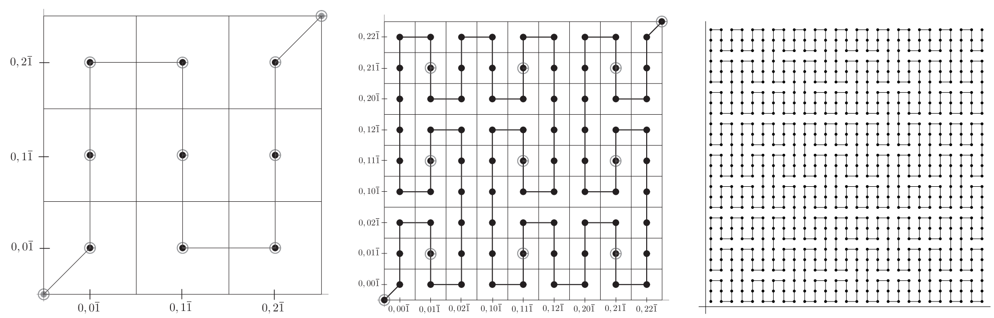


FIGURE 4. The first three approximations of Peano's curve corresponding to first digits of the ternary representations.

Peano observes that his curve is (and even its components are) nowhere differentiable. This fact is obvious, because the image of each segment of the n-th subdivision of $[0,1]$ by Peano's curve is equal to the corresponding square of the n-th subdivision of $[0,1]\times[0,1]$.

Peano's original construction was not illustrated by any figure. Solicited by Klein, David Hilbert published a note [16, (1891)] on Peano's curve (see the figure below), presenting a variant based on binary representations. He described a Cauchy sequence of polygonal lines (for the uniform convergence) of curves, hence convergent to a continuous map, the image of which is dense by construction. On the other hand, it is also closed (hence surjective), as the image of a compact set by continuous map.

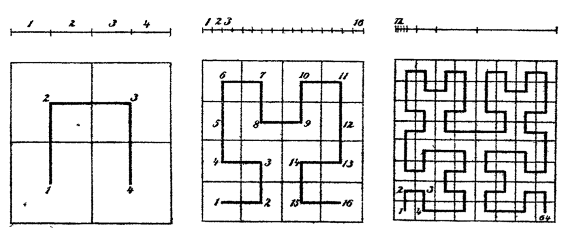


FIGURE 5. The first three approximations of Hilbert's, as it appears in Hilbert [16, (1891)]].

Peano himself concedes that he conceived the filling curve as a counterexample to commonly diffused ideas of curve, for instance, that the area of a curve is null ([8]).

7. Sweeping-tangent theorem

The fashionable sweeping-tangent theorem of Mamikon says that *the area of a tangent sweep of a curve is equal to the area of its corresponding tangent cluster* (see Apostol, Mnatskanyan [25, (2008)], [23, (2002)], [24, (2002)] and the figures below).

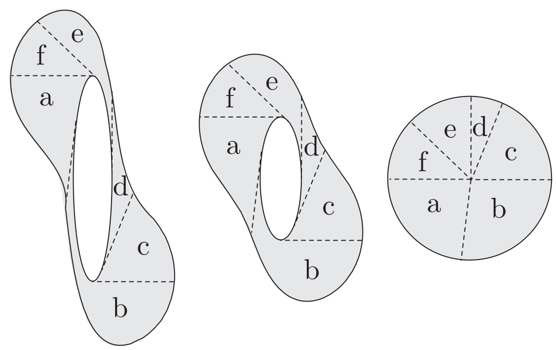


Figure 6. The three figures have the same area, because the first two are swept by the same tangent vector to the inner ellipsis and have the third figure as tangent cluster. The areas marked by the same letter have the same area as well.

This theorem, first published by Mamikon A. Mnatskanyan in [26, (1981)], has numerous applications, as it enables one to obtain the areas of complicated figures almost without calculation, by reducing the problem to that of the area of some simple figures.

In [11] Greco, Mazzucchi and Pagani discovered with surprise that in *Applicazioni geometriche* [30, (1887) p. 242] (see the figure below) and in *Lezioni di analisi infinitesimali* [33, (1893) pp. 225-226] Peano considerably generalizes Mamikon's theorem. In fact, Peano uses the Grassmann external algebra to give a formula for the area of plane figures that are *described by a segment AB of variable length that never passes twice through the same point*, and, consequently, Peano analyzes the following four special cases (of which Mamikon's theorem corresponds to 3):

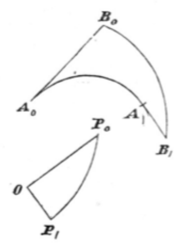
(1) A moves along a straight line and the angle of AB with that line is constant;

[8] Peano had occasional epistolary exchange with Jordan: two letters from Peano to Jordan (from 1884 and 1894) are known, while no letter from Jordan to Peano has been found. In spite of their familiarity, in 1894 in *L'intermédiare des mathématiciens* [19, (1894)] Jordan asks if there exists a curve of undetermined area. Peano replies in [34, (1896)] that if one joins the ends of his curve with those of a rectifiable curve lying outside the square, then the difference between the outer and inner measures of the set inside the curve is equal to the area of the square.

(2) A is fixed;
(3) AB is tangent at A to the curve described by A (see the figure below);
(4) AB is of constant length and normal to the curve described by its midpoint.

— 242 —

Se la retta AB si muove in guisa da riuscire sempre tangente in A alla curva descritta dal punto A, sarà $\mathrm{AB}\,.\,d\mathrm{A}=0$, e l'area u si riduce in virtù della (3), a



$$u=\frac{1}{2}\int \mathrm{AB}\,.\,d\mathrm{AB};$$

d'altra parte se da un punto fisso O si conduce un segmento variabile $\mathrm{OP}\equiv\mathrm{AB}$, l'area v descritta da OP sarà data da

$$v=\frac{1}{2}\int \mathrm{OP}\,.\,d\mathrm{P};$$

ma siccome $\mathrm{OP}\equiv\mathrm{AB}$, e $d\mathrm{P}\equiv d\mathrm{OP}\equiv d\mathrm{AB}$, si deduce $u=v$, ossia l'area descritta dal segmento AB è eguale all'area descritta dal segmento equipollente OP.

FIGURE 7. The two figures have the same area, as they appear in Peano's *Applicazioni geometriche* [30, (1887) p. 242].

Se il segmento AB si muove in guisa da riuscire sempre tangente in A alla curva descritta dal punto A, sarà UdA = 0, e la formula (3) diventa:

$$\mathrm{d}u=\frac{1}{2}\,\mathrm{UdU}.$$

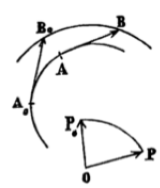


Fig. 44.

D'altra parte, se da un punto fisso O si conduce il vettore P — O = B — A, l'area dv descritta da OP vale:

$$\mathrm{d}v=\frac{1}{2}\,(\mathrm{P}-\mathrm{O})\,\mathrm{dP}$$

ma siccome P — O = U, e dP = dU, si avrà

$$\mathrm{d}v=\frac{1}{2}\,\mathrm{UdU}=\mathrm{d}u$$

ossia le aree infinitesime, e quindi le aree finite, descritte dal segmento AB e dal segmento equipollente OP di origine fissa O, sono eguali.

FIGURE 8. The two figures have the same area, as they appear in Peano's *Lezioni di analisi matematica* [30, (1893) pp. 225-6].

8. Ubi maior minor cessat

It is a typical attitude of a mathematician to distinguish important from unimportant and, definitively, not to investigate nor to refer to what is judged unimportant, although such a judgement is often unjustified. Therefore some works are shadowed by others that appear more fashionable. For example, axiomatic and formal refoundation of geometry started by Pasch, continued by Peano and his follower Pieri, was forgotten, because Hilbert's accomplishment of Euclid's geometry appeared easier and thus more attractive ([9]). Similarly, although Peano's differentiation of "measures" (1887) preceded that of Lebesgue (1910), Peano's achievement is practically ignored. Lebesgue did not mention Peano's theory, although he must have been aware of it, because he quoted works of Vitali and Fubini who amply used it (see [10, (2010)]). Frege would probably remained unknown for a long time without the work of foundation and diffusion of mathematical language (i.e., logical and set-theoretic ideography) by Peano. And even if Russell refers to Peano as his inspiration many times, the contribution of Peano is either unremembered or underestimated.

Some influential mathematicians writing about history did disservice to the memory of Peano, principally because they did not know well enough the things on which they reported. For example, Dieudonné in [6, (1983)] denied any logical value of Peano's definitions concerning limits and sets (see [10, (2010)]). This was also the case of several historians of mathematics, who are in fact historiographers, writing without knowledge of primary sources. We have already mentioned (see Section 6.2 of [9]) those debating about Peano's proofs in the theory of differential equation apparently beyond their grasp.

Other historians, by labeling certain mathematicians as "opposed to progress", discourage one from studying their point of view, as "the right historical perspective" is that of progress. For instance, Moore in [27, (1982)] eulogizes Zermelo (1904) who "first formulated the Axiom of Choice" and stigmatizes Peano (1890) who was "the first to reject the principle of infinitely many arbitrary choices" (that is, the Axiom of Choice).

In preceding writings (see [7, (2011)]), we mentioned the denigration of Peano by his colleagues and their followers.

It looks like the perceived oblivion is due to a conjugation of several factors, as the natural role of fashion, the difficulty in grasping his innovatory thoughts, the laxity of those reporting on his work, and denigration.

References

[1] G. Cantor. Über unendliche lineare Punktmannigfaltigkeiten. *Mathematische Annalen*, **23**:454–488, 1884.

[2] H. Cartan. Filtres et ultrafiltres. *C. R. Acad. Sc. Paris*, **25**:777–779, 1973.

[9] Neither Pasch nor Hilbert ever cited Peano in the revised editions of their works that were posterior to those of Peano. It was also the case of Dedekind in regard of axiomatization of numbers.

[3] H. Cartan. Théorie des filtres. *C. R. Acad. Sc. Paris*, **205**:595–598, 1973.
[4] A. Cauchy. Mémoire sur le rapport différentiel de deux grandeurs qui varient simultanément. *Exercices d'analyse et de physique mathématique*, **2**:188–129, 1841.
[5] G. Choquet. Sur les notions de filtre et de grille. *C. R. Acad. Sci. Paris*, **224**:171–173, 1947.
[6] Jean Dieudonné. Review of "F. A. Medvedev, Set functions according to G. Peano". *Amer. Math. Soc. Mathematical Reviews*, MR801411 (86h:01046), 1983.
[7] S. Dolecki and G. H. Greco. Tangency vis-à-vis differentiability in the works of Peano, Severi and Guareschi. *J. Convex Analysis*, **2**:301–339, 2011.
[8] S. Dolecki and G. H. Greco. The astonishing oblivion of Peano's mathematical legacy (I): Youthful achievements, foundations, arithmetic, vector spaces. *Forthcoming*, 2015.
[9] S. Dolecki and G. H. Greco. The astonishing oblivion of Peano's mathematical legacy (II): Analysis and Geometry. *Forthcoming*, 2015.
[10] G. H. Greco, S. Mazzucchi, and E. Pagani. Peano on derivative of measures: strict derivative of distributive set functions. *Rendiconti Lincei, Matematica e Applicazioni*, **21**:305 –339, 2010.
[11] G. H. Greco, S. Mazzucchi, and E. Pagani. Peano on definition of surface area. *Forthcoming*, 2015.
[12] A. Harnack. Über den Inhalt von Punktmengen. *Mathematische Annalen*, **25**:241–250, 1885.
[13] F. Hausdorff. *Grundzüge der Mengenlehre*. Verlag von Veit, Leipzig, 1914.
[14] T. Hawkins. *Lebesgue's theory of integration. Its origin and developments*. AMS Chelsea Publishing, Providence, 1975.
[15] C. Hermite. *Cours de M. Hermite professé pendant le 2e semestre 1881-82*. Hermann, Paris, 1883.
[16] D. Hilbert. Über die stetige Abbildung einer Linie auf ein Flächenstück. *Mathematische Annalen*, **38**:459–460, 1891.
[17] C. Jordan. Remarques sur les intégrales définies. *J. Math. Pures Appl.*, **8**:69–99, 1892.
[18] C. Jordan. *Cours d'Analyse de l'École Polytechnique* (3 vol.). Gauthier-Villars, Paris, 2nd edition, 1893-96.
[19] C. Jordan. Question n. 60. *L'intermédiare des mathématiciens*, **1**:23, 1894.
[20] H. C. Kennedy. *Life and Works of Giuseppe Peano*. D. Reidel, Dordrecht, 1980.
[21] H. C. Kennedy. *Life and Works of Giuseppe Peano*. Peremptory Edition, definitive edition, 2006.
[22] H. Lebesgue. Intégrale, longueur, aire. *Annali di Matematica Pura e Applicata*, **7**(231–359), 1902.
[23] Apostol T. M and Mamikon A. Mnatskanyan. Subtangents. an aid to visual calculus. *American Mathematical Monthly*, **109**:525–533, 2002.
[24] Apostol T. M and Mamikon A. Mnatskanyan. Tangents and subtangents used to calculate areas. *American Mathematical Monthly*, **109**:900–908, 2002.
[25] Apostol T. M and Mamikon A. Mnatskanyan. The method of sweeping tangents. *Mathematical Gazette*, **92**:97–102, 2008.
[26] Mamikon A. Mnatskanyan. On the area of the region on a developable surface. *Dokl. Akad. Nauk Armenian. SSR*, **2**:97–102, 1981.
[27] Gregory H. Moore. *Zermelo's Axiom of Choice. Its Origins, Development, and Influence*. Springer-Verlag, New York, 1982.
[28] O. Nikodym. Sur une généralisation des intégrales de M. J. Radon. *Fundamenta Mathematicae*, **15**:131–179, 1930.
[29] G. Peano. Sull'integrabilità delle funzioni. *Atti R. Acc. Scienze Torino*, **18**:439–446, 1883.
[30] G. Peano. *Applicazioni geometriche del calcolo infinitesimale*. Fratelli Bocca, Torino, 1887.

[31] G. Peano. Sulla definizione dell'area d'una superficie. *Atti R. Acc. Lincei, Rendiconti*, **6** (1st sem.):54–57, 1890.
[32] G. Peano. Sur une courbe, qui remplit toute une aire plane. *Mathematische Annalen*, **36**:157–160, 1890.
[33] G. Peano. *Lezioni di analisi infintesimale* (2 vol.). Candeletti, Torino, 1893.
[34] G. Peano. Réponse n. 60 (C. Jordan) Courbe dont l'aire soit indéterminée. *L'intermédiare des mathématiciens*, **3**:39, 1896.
[35] G. Peano. *Formulaire Mathématique*. Fratelli Bocca, Torino, 4th edition, 1902-3.
[36] G. Peano. *Formulario Mathematico*. Fratelli Bocca, Torino, 5th edition, 1908.
[37] J. Radon. Theorie und Anwendungen der absolut additiven Mengenfunktionen. *Sitzungsberichte der Akademie der Wissenschaften in Wien. Mathematisch-naturwissenschaftliche Klasse. Abteilung IIa*, **112**:1295–1438, 1913.
[38] J.-A. Serret. *Cours de calcul différentiel et intégral* (vol. II). Gauthier-Villars, Paris, 1880.
[39] O. Stolz. Über einen zu einer unendlichen Punktmenge gehörigen Grenzwerth. *Mathematische Annalen*, **24**:152–156, 1884.
[40] J. Tannery. Review of "Peano G.: Applicazioni geometriche del calcolo infinitesimale". *Bull. Sc. Math.*, **11**:237–239, 1887.
[41] E. Zermelo. Über das Maß und die Diskrepanz von Punktmengen. *Journal für die reine und angewandte Mathematik*, **158**:154–167, 1927.

Institut de Mathématiques de Bourgogne, Université de Bourgogne, B. P. 47870, 21078 Dijon, France
E-mail address: dolecki@u-bourgogne.fr

Independent researcher, professor of Mathematical Analysis, Trento, Italy.
E-mail address: gabriele.h.greco@gmail.com, https://sites.google.com/site/ghgreco/