THE ASTONISHING OBLIVION OF PEANO'S MATHEMATICAL LEGACY (II)

ANALYSIS AND GEOMETRY

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Abstract. The formulations that Peano gave to many mathematical notions at the end of 19th century were so perfect and modern they have become standard today. A formal language of logic that he created, enabled him to perceive mathematics with great precision and depth. He described mathematics axiomatically basing the reasoning exclusively on logical and set-theoretic primitive terms and properties, which was revolutionary at that time. Yet numerous Peano's contributions remain either unremembered or underestimated.

This paper is a continuation of [8] in which we commenced to delineate what is more or less ignored about Peano's heritage.

Figure 1. Giuseppe Peano (1858-1932)

1. Dispute about the mean value theorem

In the nineteenth century *Nouvelles Annales de Mathématiques* published letters and short notes, offering a forum to the mathematical community. In a letter [31, (1884)] to Nouvelles Annales, Peano observed that the proof of the *mean value theorem* given by Jordan in his *Cours d'Analyse* [22,

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(1882)] was faulty. Mind that, at that time, Peano was a young assistant, while Jordan was a famous professor almost twice as old as Peano. On the other hand, it is impressive that an easy basic fact, which is today taught in freshmen calculus courses, constituted a difficulty for a great mathematician like Jordan. The mean value theorem is usually formulated as follows: *If a real function f is continuous in $[a,b]$ and differentiable in (a,b), then there is $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a).$$

In his proof, Jordan divides the interval to subintervals $a = a_0 < a_1 < \ldots < a_{n-1} = b$, of diameter tending to 0, and claims that the differences

$$\frac{f(a_r) - f(a_{r-1})}{a_r - a_{r-1}} - f'(a_{r-1}) \tag{1}$$

tend also to 0. Peano's example $f(x) := x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(x) := 0$ for $x = 0$,

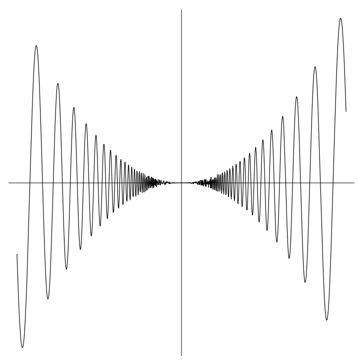

FIGURE 2. $f(x) := x^2 \sin \frac{1}{x}$ if $x \neq 0$.

with appropriately chosen a_r and a_{r-1} tending to 0, shows that (1) does not hold in general.

Peano indicated that Jordan's claim was true under the assumption of continuous differentiability of f and added that the mean value theorem could be easily proved without that assumption. Jordan replied that Peano's objections were founded, and that, [Jordan] implicitly assumed that

$$\frac{f(x+h) - f(x)}{h} \to_h f'(x) \text{ uniformly as } h \text{ tends to } 0 \tag{2}$$

in the interval $[a,b]$. Moreover he asked Peano to provide a proof without the continuity of the derivative, as he did not know a satisfactory one ([1]). In [32,

[1] At this point, Ph. Gilbert of Louvain intervenes in the exchange, saying that the request of professor Jordan was done with archness, because the mean value theorem without the continuity of derivative is false. The example he proposes to support his claim is (of course!) wrong. In his answer [32, (1884)], Peano gives the (today standard) proof of Bonnet, using the theorems of Weierstrass and Rolle, and mentions that satisfactory proofs can be found in the books of Serret, Dini, Harnack and Pasch.

(1884)] Peano remarked that (2) amounted to the continuous differentiability of f and that a correct proof of the mean value theorem was due to Bonnet.

2. Differentiability

The definition of *derivative* at a point x of a real-valued function defined on a subset of Euclidean space appeared already in *Applicazioni geometriche* [34, (1887)] and was generalized in *Formulario mathematico* [47, (1908)] to a function valued in Euclidean space. A function $f : U \to \mathbb{R}^m$, where U is a subset of $\mathbb{R}^n$ and x is an accumulation point of U, is said to be *differentiable at* x if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ (said, *total differential*) such that

$$\lim_{U \ni y \to x} \frac{f(y) - f(x) - L(y-x)}{\|y - x\|} = 0. \tag{3}$$

It was Thomae who pointed out in [60, (1875)] that differentiability was not equivalent to partial differentiability. After Thomae, the existence of a total differential was assured by the existence and continuity of partial derivatives, that is, by strict differentiability (see below). Peano's definition set the derivative free from a particular coordinate system, thus allowed it to pass from one coordinate system to another.

It should be stressed that Peano's definition appeared in a rigorous modern form (3). On defining differentiability Peano referred to the concepts of Grassmann [12, (1862)] and of Jacobi [21, (1841)], but in fact they were more rudimentary (radial derivative and Jacobian matrix). As the domain of f need not be the whole of Euclidean space, in general the linear operator L in (3) is not unique; in the case of uniqueness Peano called L the *derivative* of f at x and denoted it by $Df(x)$. This definition contrasted with the standard language of mathematical definitions in that epoch, which was usually informal and often vague.

Today (3) is universally used in the case where x in the interior of U (so that L is unique) and $Df(x)$ is commonly called the *Fréchet derivative* of f at x, although Fréchet gave its informal (geometric) definition only in [10, (1911)]. ([2])

In Peano [41, (1892)] a function f from an interval U to $\mathbb{R}$ is said to be *strictly differentiable* at $x \in U$ if

$$\lim_{U \ni y,z \to x} \frac{f(y) - f(z)}{y - z} = f'(x), \tag{4}$$

and from which it is observed that strict differentiability in U amounts to continuous differentiability.

[2] One month later Fréchet published another note [11, (1911)], acknowledging contributions of Stolz (1893), of Pierpoint (1905) and of W. H. Young (1910) (but not that of Peano), the authors who apparently also ignored Peano's contribution and considered merely the case where x is an interior point of U.

In [35, (1888) p. 133] Peano gave the following *mean value theorem* for *vector-valued functions* f of one variable: if f has an $(n+1)$-st derivative $f^{(n+1)}$ on $[t, t+h]$, then there exists an element $k \in \mathrm{cl\,conv}\, f^{(n+1)}([t, t+h])$ ([3]) such that

$$f(t+h) = f(t) + hf'(t) + \cdots + \frac{h^n}{n!} f^{(n)}(t) + \frac{h^{n+1}}{(n+1)!} k.$$

Here is another surprise, because the concept of convex hull has usually been attributed to Minkowski [26, (1896)]. ([4])

Moreover, Peano extended the notion of derivative by replacing the limit of the difference quotient (of y at t) by its adherence ([5])

$$\mathrm{Dg}\, y(t) := \mathrm{adh}_{h \to 0} \frac{y(t+h) - y(t)}{h},$$

and he employed it to define approximate solutions of

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{5}$$

in the proof of the existence of solutions of a system of differential equations that appeared in *Formulario mathematico* [47, (1908)] (see next Section 6.2). The *extended derivative* $\mathrm{Dg}\, y(t)$ is a set, and if y is differentiable it is a singleton.

A vector function y is called an *approximate solution* of (5) (on $[t_0, t_1]$) if there exists an $\varepsilon > 0$ such that

$$\mathrm{Dg}\, y(t) \subset f(t, y(t)) + B(0, \varepsilon) \text{ and } \limsup_{h \to 0} \left\| \frac{y(t+h) - y(t)}{h} \right\| < \infty \tag{6}$$

for each $t_0 \leq t \leq t_1$. Peano's proof of the existence of solutions of (5) used the following *mean value property* of approximate solutions:

$$\frac{y(t_1) - y(t_0)}{t_1 - t_0} \in \mathrm{cl\,conv} \left(\bigcup_{t \in [t_0, t_1]} \mathrm{Dg}\, y(t) \right).$$ ([6])

The introduction of extended derivative and its application in the context of differential equations attests to a remarkable ease with which Peano was able to introduce concepts in order to adequately approach mathematical problems.

In Peano [40, (1892)] a polynomial function $a_0 + a_1(x - x_0) + \ldots + a_n(x - x_0)^n$ is called a *development* of f of *rank* n with respect to powers of $x - x_0$ if

$$\lim_{x \to x_0} \frac{f(x) - (a_0 + a_1(x - x_0) + \ldots a_n(x - x_0)^n)}{(x - x_0)^n} = 0 \tag{7}$$ ([7]).

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[3] Here "conv" stands for the *convex hull* and "cl conv" for the *closed convex hull*.

[4] In 1889 Peano [37] introduced the modern notion of convex set to axiomatize geometry.

[5] $\mathrm{adh}_{h \to 0}\, \varphi(h) := \left\{ v : \underset{\{h_k\}_k}{\exists} (\lim_{n \to \infty} h_k = 0 \text{ and } v = \lim_{n \to \infty} \varphi(h_k)) \right\}$

[6] Therefore, if f in (5) is continuous, then each approximate solution is locally Lipschitz.

[7] Peano gave the rules for the developments of sums, products and compositions.

This equality can be rewritten in such a way that the n-th coefficient is given by

$$a_n = \lim_{x\to x_0} \frac{f(x) - (a_0 + a_1(x-x_0) + \dots a_{n-1}(x-x_0)^{n-1})}{(x-x_0)^n},$$

which leads to the *Peano generalized derivative of order* n, that is,

$$a_n n!.$$

If $f^{(n)}(x_0)$ exists, then $a_n n! = f^{(n)}(x_0)$, but even a discontinuous function can have a development. For example,

$$f_n(x) := \begin{cases} x^{n+1}\theta\left(\frac{1}{x}\right), \text{ if } x \neq 0, \\ 0, \text{ if } x = 0, \end{cases}$$

where $\theta(t)$ is the fractional part of t, has a development of rank n, and

$$f_\infty(x) := \begin{cases} \exp(-\frac{1}{x^2})\theta\left(\frac{1}{x}\right) \text{ if } x \neq 0, \\ 0, \text{ if } x = 0, \end{cases}$$

has developments of arbitrary rank; they both have discontinuities in each neighborhood of 0 ([8]).

If f is n times differentiable on $[x_0, x_0+h]$ and $f^{(n+1)}(x_0)$ exists, Peano [38, (1889)] proved that there exists $\xi \in [x_0, x_0+h]$ such that the remainder of rank $n+1$ of the *Taylor formula* is

$$\Big(\frac{f^{(n)}(\xi) - f^{(n)}(x_0)}{\xi - x_0} - f^{(n+1)}(x_0)\Big)\frac{h^{n+1}}{(n+1)!}.$$

3. Lower and upper limits of variables sets

Generalizing the notions of limit of straight lines, planes, circles and spheres (that depend on a parameter) considered as sets, Peano defined two limits of variable figures (in particular, curves and surfaces).

A *variable figure* (or *set*) is a family, indexed by the reals, of subsets A_λ of an affine Euclidean space X, and the *lower limit of a variable figure* was given in [34, (1887)] by

$$\mathrm{Li}_{\lambda\to+\infty} A_\lambda := \{y \in X : \lim_{\lambda\to+\infty} \mathrm{d}(y, A_\lambda) = 0\}.$$

In the last two editions of *Formulario mathematico* [46, (1903)], [47, (1908)] the *upper limit of a variable figure* was defined by

$$\mathrm{Ls}_{\lambda\to+\infty} A_\lambda := \{y \in X : \liminf_{\lambda\to+\infty} \mathrm{d}(y, A_\lambda) = 0\},$$

and also expressed as

$$\mathrm{Ls}_{\lambda\to\infty} A_\lambda = \bigcap_{n\in\mathbb{N}} \mathrm{cl} \bigcup_{\lambda\geq n} A_\lambda.$$

[8] Indeed, $\lim_{x\to 0} \frac{f_n(x)}{x^k} = 0$ for $0 \leq k \leq n$, because $0 \leq \theta\left(\frac{1}{x}\right) < 1$, so that $a_0 = a_1 = \ldots = a_n = 0$, so that (7) holds with $x_0 = 0$. Similarly for f_∞. On the other hand, the function $t \longmapsto \theta\left(\frac{1}{t}\right)$ is discontinuous at every $t \in \{\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\}$.

Peano used these notions to define lower and upper tangent cones (see next Section 4) and in the theory of differential equations (see next Section 6).

4. Tangency

The notion of tangent to a circle can be found already in Euclid's work and to a curve in *Géométrie* of Descartes (1637). Until the time of Peano, several definitions of tangent set to arbitrary figures ([9]) were formulated, for example,

(α) *a tangent plane to a surface S at a point p is a plane that contains the tangent straight line at p of every curve traced on the surface S and passing through p;*

(β) *a tangent plane to S at p is a plane that contains the tangents at p to each curve on S that has a tangent straight line and pass through p.*

These and other then accepted definitions, however, led to controversial results (see Dolecki, Greco [7, (2011)] for historical details). In *Applicazioni Geometriche* [34, (1887)] Peano gave a metric definition of tangent straight line and of tangent plane and, finally, introduced a unifying notion, that of *affine tangent cone* of a set A at a point x:

$$\operatorname{tang}(A, x) := x + \operatorname{Li}_{\lambda \to +\infty} \lambda(A - x). \tag{8}$$

Later, in *Formulario mathematico* [47, (1908)], Peano introduced another type of tangent cone, namely

$$\operatorname{Tang}(A, x) := x + \operatorname*{Ls}_{\lambda \to +\infty} \lambda(A - x). \tag{9}$$

To distinguish the two notions above, we shall call the first *lower affine tangent cone* and the second *upper affine tangent cone*.

As usual, after abstract investigation of a notion, Peano considered significant special cases; he calculated the upper affine tangent cones for several important figures and for curves and surfaces parametrized in a regular way.

5. Optimality conditions

A well-known necessary condition of maximality of a function at a point is presently formulated in terms of derivative of the function and of tangent cone of the constraint at that point. Consider a real-valued function $f : X \to \mathbb{R}$, where X is a Euclidean affine space, and a subset A of X.

Regula (of optimality) *If f is differentiable at $x \in A$ and $f(x) = \max\{f(y) : y \in A\}$, then*

$$\langle Df(x), y - x \rangle \leq 0 \text{ for every } y \in \operatorname{Tang}(A, x), \tag{10}$$

where $Df(x) : X \to \mathbb{R}$ is the derivative of f at x and $\operatorname{Tang}(A, x)$ is the upper affine tangent cone of A at x.

This condition was known to Peano already in [34, (1887)] and in the very form (10) in [47, (1908)] and was applied by Peano to numerous optimization

[9] That is, subsets of Euclidean space.

problems, in particular, to those of minimizing the sum of distances of a point from one or several designated points or figures (see Dolecki, Greco [6, (2007)] for further details).

6. Differential equations

6.1. **Linear systems of differential equations.** In [36, (1888)] Peano introduced the exponential of linear operators and the so called *Peano series* to represent the solution of a general linear system of differential equations that he transformed into a vector equation

$$x' = Ax, \tag{11}$$

with a linear operator $A(t) : \mathbb{R}^n \to \mathbb{R}^n$ continuously depending on t. Starting with a constant $x_0 \in \mathbb{R}^n$, he defined

$$x_{n+1}(t) := \int_{t_0}^{t} Ax_n \, ds, \tag{12}$$

showed the existence of M such that $|x_n(t)| \leq M^n \|x_0\| (t - t_0)^n / n!$ for each n, so that the *Peano series*

$$x := x_0 + x_1 + x_2 + \ldots$$

and its derivative $Ax = Ax_0 + Ax_1 + Ax_2 + \ldots$ converge uniformly. Clearly x is a solution of (11) such that $x(t_0) = x_0$. On using (12), Peano defined the *resolvent* operator of (11) ([10])

$$R_{t_0}^{t} := \left(I + \int_{t_0}^{t} A\, ds + \int_{t_0}^{t} A\, ds \int_{t_0}^{t} A\, ds + \ldots\right),$$

and thus the solution to (11) with the initial condition $x(t_0) = x_0$, given by $x(t) = R(t_0, t)\, x_0$. In the case of a constant A, he represented the resolvent by

$$e^{A} := I + A + \tfrac{1}{2}A^2 + \ldots + \tfrac{1}{n!}A^n + \ldots,$$

so that, in this case,

$$x(t) = e^{A(t - t_0)} x_0.$$

Finally, he gave a solution of a non-homogeneous equation

$$x' = Ax + b$$

in the form

$$x(t) = R_{t_0}^{t} x_0 + R_{t_0}^{t} \int_{t_0}^{t} R_{s}^{t_0}\, b(s)\, ds.$$

In this short paper, Peano preceded this illuminating theory by a theory of linear operators, their matrix representations, their norms and convergent series of operators, in particular, the exponential of operators.

[10] It is called *Peano-Baker series* in Baake, Schlägel [1, (2012)]. Here $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity operator.

Peano's view was unprecedented in that epoch. As Garret Birkhoff observed in [2, (1973)], these foreshadowed the modern theory of Banach spaces and algebras.

6.2. **Nonlinear differential equations and differential inequalities.** In [33, (1886)] Peano proved the existence of solutions in the small of an initial value problem

$$x' = f(t,x), \quad x(t_0) = x_0, \tag{13}$$

for a continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and, consequently, the uniqueness for f having, in addition, a bounded partial derivative with respect to x. To prove uniqueness of solutions, Peano used an argument that amounts to the following *Grönwall type differential inequality* ([11]): if c is a real number and $u'(t) \leq c\,u(t)$ for each $t \geq t_0$, then

$$u(t) \leq u(t_0)\, e^{c\,(t-t_0)} \tag{14}$$

for $t \geq t_0$. Moreover, Peano used (14) in [44, (1897)] to prove continuous dependence of solutions with respect to their initial values, whenever a Lipschitz condition holds.

The proof of the existence of solutions to (13) in [33, (1886)] is based on iterated use of the following comparison properties (see Greco, Mazzucchi [13, 15] for details): If $\varphi_1, \varphi_2 : [t_1, t_2] \to \mathbb{R}$ satisfy one of the two following conditions:

(i) $\varphi_1'(t) > f(t, \varphi_1(t))$ and $\varphi_2'(t) \leq f(t, \varphi_2(t))$ for each $t \in [t_1, t_2]$,

(ii) $\varphi_1'(t) \geq f(t, \varphi_1(t))$ and $\varphi_2'(t) < f(t, \varphi_2(t))$ for each $t \in [t_1, t_2]$,

then $\varphi_1(t_2) > \varphi_2(t_2)$ provided that $\varphi_1(t_1) \geq \varphi_2(t_1)$. ([12])

Klein asked Peano to generalize his theorem from the scalar case to systems of differential equations in view of a publication in Mathematische Annalen (see Segre [57, (1997)]). Peano replied that passing from a scalar equation to a system of equations would considerably complicate the quest, but a few years later he presented a paper [39, (1890)], in which he solved the problem, which can be stated in the same terms, the only difference being that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ for a natural $n \geq 1$. Peano [39, (1890)] showed that the Lipschitz condition was not necessary for uniqueness, and also gave

[11] In [16, (1919)] Grönwall introduced his inequalities to establish differentiability with respect to a parameter of the solutions of a system of differential equations.

[12] In the inequalities (*i*) and (*ii*) we maintain the convention adopted by Peano [39, (1890)], that is, we assume the existence of both the left and the right derivatives fulfilling the same inequalities.

several examples of non-uniqueness, for instance,

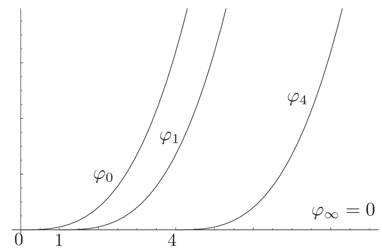

FIGURE 3. Here we show four out of infinitely many solutions: $\varphi_0, \varphi_1, \varphi_4$ and $\varphi_\infty \equiv 0$.

for every $r, s \in [0, \infty)$, the null function and the functions $\varphi_r(t) := \varphi(t-r)$, $\psi_s(t) := -\varphi(s-t)$ and $h_{s,r}(t) := \psi_s(t) + \varphi_r(t)$ where

$$\varphi(t) := \begin{cases} 0, & \text{if } t \leq 0, \\ t^3, & \text{if } t > 0, \end{cases}$$

are the solutions of

$$x' = 3x^{\frac{2}{3}}, \; x(0) = 0. \tag{15}$$

A year later, Picard published a paper [50, (1891)] proving the existence of a solution (but not uniqueness) of a vector equation of type (13) under a Lipschitz condition. Of course, Picard's assumption was gratuitous in view of the result of Peano, who published in [42, (1892)] a comment in this vein.

Although Peano's existence theorem for systems of differential equations became famous, both the proofs given by Peano in [39, (1890)] and in *Formulario mathematico* [47, (1908)] have always been ignored (see Greco, Mazzucchi [14] for details); the few who talked about them formulated judgements in evident contrast with what Peano affirmed. We shall give here an outline of Peano's principal definitions and propositions so that a reader might appreciate the originality of his ideas.

Let $f : [t_0 - s, t_0 + s] \times \overline{B}_r(x_0) \to \mathbb{R}^n$ be a continuous function, $s > 0$ and $\overline{B}_r(x_0) \subset \mathbb{R}^n$ a closed ball centered at x_0 with radius $r > 0$. Let M be the maximum of f on $[t_0 - s, t_0 + s] \times \overline{B}_r(x_0)$ and let $a \in \overline{B}_r(x_0)$ and $t_1, t_2 \in [t_0 - s, t_0 + s]$ so that $t_1 \neq t_2$. We denote by $\mathrm{Sol}(a, t_1, t_2)$ the set of solutions of

$$x' = f(t, x), \quad x(t_1) = a, \tag{16}$$

on $[t_1, t_2]$ and by

$$\mathrm{F}(a, t_1, t_2) := \{\gamma[t_2] : \gamma \in \mathrm{Sol}(a, t_1, t_2)\} \tag{17}$$

their sections. We denote by $\mathrm{Sol}_\varepsilon(a,t_1,t_2)$ the set of ε-approximated solutions ([13]) of (16) on $[t_1,t_2]$, and by

$$\mathrm{F}_\varepsilon(a,t_1,t_2) := \{\gamma[t_2] : \gamma \in \mathrm{Sol}_\varepsilon(a,t_1,t_2)\}, \tag{18}$$

their sections and by

$$\mathrm{A}(a,t_1,t_2) := \bigcap_{\varepsilon>0} \mathrm{F}_\varepsilon(a,t_1,t_2) \tag{19}$$

their intersections. Let $t \in [t_0 - s, t_0 + s]$ with $t \neq t_0$. The principal steps of Peano's proof are:

(i) $\overline{\mathrm{F}_\varepsilon(a,t_1,t_2)} \subset \mathrm{F}_{\varepsilon+h}(a,t_1,t_2)$ for each $h > 0$;
(ii) $\mathrm{A}(a,t_1,t_2)$ is a compact set;
(ii) $\mathrm{A}(a,t_1,t_2) = \mathrm{F}(a,t_1,t_2)$;
(iv) $\mathrm{F}_\varepsilon(x_0,t_0,t) \neq \emptyset$ for each $\varepsilon > 0$, if $|t_0 - t|M \leq r$;
(v) $\mathrm{A}(x_0,t_0,t) \neq \emptyset$ if $|t_0 - t|M \leq r$.

In doing so, Peano realized that the existence of a selection of a multivalued map $t \mapsto \mathrm{A}(x_0,t_0,t)$ was not granted by the axioms of the set theory, but in the specific problem, with which he was confronted, he could get around the obstacle by picking the least element of the compact set $\mathrm{A}(x_0,t_0,t)$ with respect to a lexicographic order of $\mathbb{R}^n$. He observed that a *principle of infinite arbitrary choices* was not granted by the axioms of set theory.

Having in mind the indicated phases (i)-(v) of proof, it is surprising that Kennedy [23, (1969)] affirms that

> In 1890 he [Peano] extended his theorem to systems of first order differential equations, using an entirely different method of proof (successive approximations).

Equally bizarre is an opinion of Mahwin [24, (1988)], [25, (2001)]:

> The existence of at least one solution to Cauchy's problem for a system with a continuous right-hand side is proved in 1890 by Peano, by combining Euler-Cauchy's approximation method with a compactness theorem of Ascoli and Arzelà.

Let us also cite Flett [9, (1980), p. 158]:

> Peano's proof is both long and arduous, since what is essentially a proof of the Ascoli-Arzelà theorem is intricately embedded in it.

7. Integral representation of remainders and Schwartz's distributions

Clark McGranery informed us recently of repeated citations (without bibliographic references) of Peano in Laurent Schwartz's autobiography [56,

[13] Peano defines ε-approximated solutions in [47, (1908)], as we have seen in (6), while in [39, (1890)] a function $\gamma : [t_1,t_2] \to \overline{B}_r(x_0)$ is said to be an ε-approximate solution of (16), if $\gamma(t_1) = a$ and $\|\gamma'(t) - f(t,\gamma(t))\| < \varepsilon$ for each $t \in [t_1,t_2]$.

(2001), pp. 212, 229, 230]. In chapter VI " The Invention of Distributions" Schwartz wrote: ([14])

> The mathematician Peano wrote in 1912 [sic] on the difficulties of differentiation: "I am sure that something must be found. There must exist a notion of generalized functions which are to functions what the real numbers are to the rationals." This was a marvelous intuition, and it arose long before 1944. But the mathematical knowledge of the time did not make it possible for Peano to find the generalized functions, or even to conceive of them; at that time it would have been a superhuman task.

In fact, in *Resto nelle formule di quadratura, espresso con un integrale finito* [48, (1913), p. 569] Peano defined a Heaviside function φ of one real variable by $\varphi(x) = 0$ for $x < 0$, $\varphi(x) = 1$ for $x > 0$ and $\varphi(0) = \frac{1}{2}$, and commented:

> The function φ has null derivative everywhere except for 0, where φ is discontinuous with a jump $+1$, so that, by the usual definition, the derivative is infinite, what halted the analysts, but the electricians continued to advance; Maxwell, Heaviside, and, more recently, Giorgi [. . .] introduced an *impulsive function,* that I will indicate with Ux, which is null for all $x \neq 0$ and is infinite for $x = 0$, however in such away that $\int_{-\infty}^{+\infty} U(x)dx = 1$. Consequently,
>
> $$\int_{-\infty}^{x} U(z)dz = \varphi(x) \text{ for each } x \in \mathbb{R}.$$
>
> As in algebra, after having studied integers, [...] one introduces the rational numbers which are not numbers considered before, but they belong to a wider category than the integers, so the *impulsive function* is not a function, as those which are defined in analysis, but it belongs to a wider category of entities ([15]).

Nowadays, the *impulsive function* U is known as the *Dirac delta function* ([16]).

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14 An other citation of Peano in the same chapter of [56, (2001), p. 229]: "It was necessary to do what Peano had said to do in 1912, but had not done himself (and which I had never heard of). To generalize these functions, it was necessary to overcome a powerful inhibition. But like Peano, I knew by heart the generalization of the rationals to the reals!".

15 In [49, (1914)] Peano derived the equalities $\int_{-\infty}^{+\infty} U(x)dx = 1$ and $\int_{-\infty}^{x} U(z)dz = \varphi(x)$, by defining, for every real number x, the function $z \mapsto U(z-x)$ as equal to $\frac{1}{dx}$ on an infinitesimal interval $(x, x+dx)$ and null otherwise.

16 This function was reintroduced by Dirac in 1926 for use in quantum mechanics; see Schwartz [56, (2001), p. 214] and Dirac [5, (1958), pp. 58-61].

In [48, (1913)], as well as in [49, (1914)], Peano gave a rule for integral representation of remainders (*a*) of quadrature formulas and (*b*) of approximation by a polynomial of degree less than n. More precisely, Peano showed that if $R(f)$ denotes the remainder related to a real function f of one real variable, then, under appropriate hypotheses,

$$R(f) = \int_{-\infty}^{+\infty} g(x) D^n f(x) dx, \tag{20}$$

where the "kernel" g (which is independent of f) fulfills $g(x) = R(p_x)$ with the function $p_x : \mathbb{R} \to \mathbb{R}$ defined by

$$p_x(z) := \frac{(z-x)^{n-1}}{(n-1)!} \frac{1}{2} \mathrm{sgn}(z-x). \tag{21}$$

Appropriate hypotheses are: (*i*) linearity of R, (*ii*) the natural number n must be such that the remainder R of polynomial functions of degree less than n is null, (*iii*) the remainder's formula (20) concerns functions n-times differentiable. Nowadays, the theorems assuring the representation (20) of a linear remainder in terms of (21) are called "Peano kernel Theorems"; they form a powerful tool in numerical anlysis ([17]).

To provide "the kernel g as remainder of the function (21)", whenever (20) holds, Peano adopted a heuristic approach based on functions of Heaviside and Dirac. First of all, he observed that the function $\varphi_x : \mathbb{R} \to \mathbb{R}$, defined by

$$\varphi_x(z) := \frac{(z-x)^{n-1}}{(n-1)!} \varphi(z-x), \tag{22}$$

and the function p_x given by (21) have the same remainder ([18]), since, being $\varphi - \frac{1}{2}\mathrm{sgn} = \frac{1}{2}$, their difference $\varphi_x - p_x$ is a polynomial of degree less than n. Secondly, he observed that the n-th derivative of φ_x is the Dirac delta function U at x, i.e., $z \mapsto U(z-x)$. Therefore, by (20),

$$R(\varphi_x) = \int_{-\infty}^{+\infty} g(z) D^n \varphi_x(z) dz = \int_{-\infty}^{+\infty} g(z) U(z-x) dz = g(x). \tag{23}$$

Talking about the Heaviside function φ (called *discontinuity factor* in [49, (1914)]), Peano wrote (in Latin without flexions) the following comment, which makes us think of the distributions, viewed as generalized functions or as functions beyond themselves.

> Cultores de analysi infinitesimale considera solo functiones determinato et finito, ut φz, que pote es espresso per symbolos de analysis, et non considera functione impellente, que es comodo, sed non necessario.

[17] See, for example, Sard [54, (1948)], [55, (1963)], Davis [4, (1963)], Stroud [58, (1974)] and Powell [51, (1981)].

[18] That is, $R(\varphi_x) = R(p_x)$.

On evoking in [56, (2001), p. 230] the moment of discovery of distributions, Schwartz recollects Peano:

> The spark shot forth one night in early November 1944 - I no longer remember exactly which. To find generalized solutions of partial differential equations, it was necessary to generalize the notion of function! And I immediately found how to generalize it; the very notion which Peano had vainly searched far in 1912 [sic].

8. Axiomatizing Geometry

In his modern axiomatic approach to geometry in *Vorlesungen über neuere Geometrie* [29, (1882)] Pasch banned any reference to geometric signification of objects during deductive processes in order to avoid errors due to a contamination of purely logical and mathematical inference by non-mathematical intuition.

In *Principii di geometria logicamente esposti* [37, (1889)] Peano carried out this purist conception of deductive method to its extreme consequences by evicting the use of common language. In order to develop fundamentals of affine geometry he exclusively employed his logical set-theoretic ideography. He rigorously followed the rule of using only completely determined terms and making unequivocal what is meant by definition and proof. Peano used two primitive terms (*point* and *segment*) and 17 axioms, the first 11 of which nearly coincided with Pasch's axioms ([19]).

It is remarkable that he defined *convex sets* and studied operations that preserve convexity, privileging the use of convexity in geometrical construction. In particular, Peano's axiom XVII is an *axiom of continuity* which states that every segment joining an internal and an external point of a convex set is divided by a point into an internal and an external segment (see fig. 5). The axiom of continuity was used by Peano [37, (1889), p. 38-39] to assert Euclidean parallel axiom in terms of half-lines (see footnote 22). There is no axiom of continuity neither in Pasch [29, (1882)] nor in the first

[19] In Whitehead [63, (1907), p. 3-7] and in Torretti [61, (1978), p. 220-221] the reader can find all Peano's axioms translated in English.

edition of Hilbert's *Grundlagen der Geometrie* [17, (1899)]. ([20])

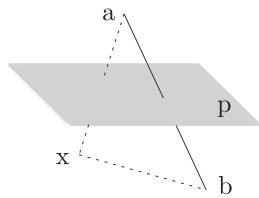

FIG. 4: Peano's axiom XVI

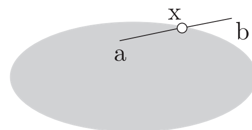

FIG. 5: Peano's axiom XVII

In *Sui fondamenti della geometria* [43, (1894)] Peano axiomatized metric geometry by adding a primitive term of *motion* and other 8 axioms, and, which is most remarkable, in *Analisi della teoria dei vettori* [45, (1898)] he axiomatized geometry with the aid of the theory of vectors, adopting as three primitive terms *point*, *vector* and *internal product*.

He wrote in [43, (1894)] that a first scientific question that arises in a deductive theory is that of *independence of axioms*, hence their minimality, and moreover described in [45, (1898)] a method of verification of independence by assigning different interpretations, under which some of the axioms are fulfilled and the other are not. In [43, (1894)] he wrote:

> To prove the independence of n postulates, it would be necessary to give n examples of interpretation of the undefined signs, each of which satisfies $n - 1$ postulates, and not the remaining one.

He also pursued the objective of minimality of primitive terms. Following Peano, a member of his school, Padoa [27, (1900)] put forward a method for proving the independence of primitive terms.

In *Grundlagen der Geometrie* [17, (1899)] of Hilbert, the axiomatic approach to geometry is very different from that of Peano. Let us give an example of this difference in the axiom of continuity which for Hilbert is composed of both the axiom of Archimedes and an axiom of completeness; the latter reads [19, (1902), p. 15]:

> To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.

[20] Hilbert's axiom of continuity includes an axiom of completeness, appeared for the first time in the French translation [18, (1900), p. 123]. Hilbert wrote [19, (1902), p. 15-16]: "[The axiom of completeness] although not of a purely geometrical nature, merits particular attention from a theoretical point of view. [. . .] However, in what is to follow, no use will be made of the *axiom of completeness*."

As Axiom XVII of continuity, Peano's Axioms XIII, XIV and XVI (see fig. 6, 7, 4) were completely original and fecund.

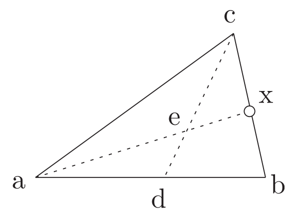

Fig. 6: Peano's axiom XIII

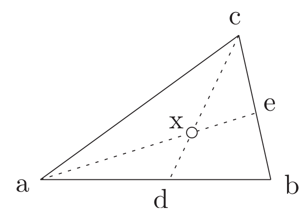

Fig. 7: Peano's axiom XIV

Let xy denote the segment (endpoints excluded) joining two points x and y. Peano [37, 43] defines and denotes by $x'y$ the *shadow* of a point y illuminated from a point x, as the set of the points c such that $y \in xc$ ([21]). More generally, if X and Y are set of points, he defines the *segment-join*

$$XY := \bigcup\{xy : x \in X, y \in Y\}$$

and the *shadow*

$$X'Y := \bigcup\{x'y : x \in X,\ y \in Y\}. \quad ({}^{22})$$

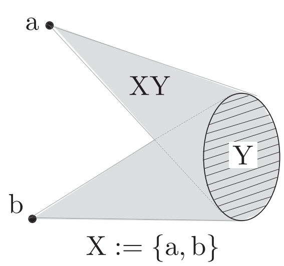

Fig. 8: the segment join XY

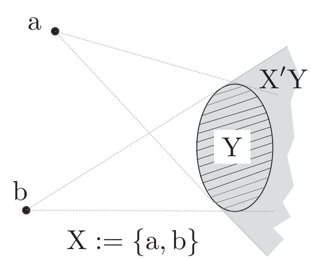

Fig. 9: the shadow $X'Y$

Axiom XIII: *Let a, b, c be non-collinear points. For every point $d \in ab$ and every point $e \in cd$ there exists a point $x \in cb$ such that $e \in ax$.*

Axiom XIV: *Let a, b, c be non-collinear points. Then for every point $d \in ab$ and every point $e \in cb$ there exists a point $x \in cd \cap ae$.*

Axiom XVI: *Let p be a plane and let a and b be points such that $b \in a'p$. Then for every point x either $x \in p$ or $xa \cap p \neq \emptyset$ or $xb \cap p \neq \emptyset$.*

[21] Peano also refers to $x'y$ as the "prolongation of the segment xy beyond y". This binary operation, denoted by the apostrophe $'$, is introduced by Peano for expliciting the variables y and z in the relation $x \in yz$; in fact $x \in yz \iff z \in y'x \iff y \in z'x$.

[22] These two operations of "segment-join" and "shadow" were used by Peano to "construct" classical sets of points: straight-lines, half straight-lines (rays), parallel rays, planes, half-planes, half-spaces, angles, triangle, tetrahedron, and so on. For example, let a, b, c, d be points, then $a(bc)$ is a triangle; $(ab)(cd)$ is a tetrahedron; $a'(bc)$ is the subset of a plane delimited by the segment bc and by the rays $a'b$ and $a'c$; $(ab)'(ab)$ is a straight-line through a and b, whenever $a \neq b$; analogously, $(abc)'(abc)$ is a plane through a, b and c, whenever a, b and c are non-collinear. Two rays $a'b$ and $c'd$ are said to be parallel, if $d(a'b) = b(c'd)$ (see [37, (1889), p. 38].

The geometric axioms of Peano (and above all, Axioms XIII and XIV) and the operations of "segment-join" and "shadow" regain importance nowadays in the setting of *convex geometries* (see Van de Vel [62, (1993)]), of *linear geometries* (see Coppel [3, (1998)]) and of *join geometries* and *geometric interval spaces* (see Prenowitz and Jantosciak [52, (1979)]), *order geometry* (see Pambuccian [28, (2011)], Retter [53, (2013)]).

Axioms XIII, XIV and XVI are related to the following famous Pasch axiom [29, (1882), p. 21]: *If a line in the plane of a triangle does not pass through any of its vertices but meets one of its sides, then it also meets another of its sides.* Axioms XIII, XIV and XVI are absent both in Pasch's *Vorlesungen* [29, (1882)] ([23]) and in Hilbert's *Grunglagen*; although these axioms as well as the operations of segment-join and shadow, are investigated in recent literature, it is extremely rare they are attributed to Peano. For example, inner and outer Pasch axioms, discussed in [59, (1999), p. 179-180] by Tarski, coincide with Peano's Axiom XIV and XIII, respectively; Peano's Axiom XV coincides with the "space order axiom" which, as reported in Bernays's Supplement of [20, (1971), p. 200] to Hilbert's *Grundlagen*, Van der Waerden proposed as a substitute for the Pasch axiom. An other example is that the operations of join and extension in Van de Vel [62, (1993), p. 77] and in Prenowitz and Jantosciak [52, (1979), p. 49, 160], as well as the operations of product and quotient in Coppel [3, (1998), p. 93, 97] are precisely Peano's operations of segment-join and shadow, respectively.

References

[1] M. Baake and U. Schlagel. The Peano-Baker series. *Proc. Steklov Inst. Math.*, **275**:167–171, 2011.

[2] G. Birkhoff and U. Merzbach. *A sourcebook in classical analysis*. Harvard University Press, Cambridge, 1973.

[3] W. A. Coppel. *Foundations of convex geometry*. Cambridge University Press, Cambridge, 1998.

[4] P. J. Davis. *Interpolation and approximation*. Dover Publications, New York, 1975.

[5] P. A. M. Dirac. *The principles of quantum mechanics*. Oxford University Press, London, 1958 (4th ed.).

[6] S. Dolecki and G. H. Greco. Towards historical roots of necessary conditions of optimality: Regula of Peano. *Control and Cybernetics*, **36**:491–518, 2007.

[7] S. Dolecki and G. H. Greco. Tangency vis-à-vis differentiability in the works of Peano, Severi and Guareschi. *J. Convex Analysis*, **2**:301–339, 2011.

[8] S. Dolecki and G. H. Greco. The astonishing oblivion of Peano's mathematical legacy (I): Youthful achievements, foundations, arithmetic, vector spaces. *Forthcoming*, 2015.

[9] T.M. Flett. *Differential analysis. Differentiation, differential equations and differential inequalities*. Cambridge University Press, Cambridge, 1980.

[10] M. Fréchet. Sur la notion de différentielle. *C.R.A.Sc. Paris*, **152**:845–847, 1911. 27 March 1911.

[11] M. Fréchet. Sur la notion de différentielle. *C.R.A.Sc. Paris*, **152**:1950–1951, 1911. 18 April 1911.

[23] The statement asserted by Theorem 13 of the second edition of Pasch's *Vorlesungen* [30, (1926), p. 25] is Peano's Axiom XIII.

[12] H. G. Grassmann. *Extension theory*. Translation by Lloyd C. Kannenberg of *Die Ausdehnungslehre* (1862). American Mathematical Society, Providence, 2000.
[13] G. H. Greco and S. Mazzucchi. Peano's 1886 paper on existence theorem for scalar differential equations: a review. *Forthcoming*, 2015.
[14] G. H. Greco and S. Mazzucchi. Peano's 1890 existence theorem for systems of differential equations: a revisited proof. *Forthcoming*, 2015.
[15] G. H. Greco and S. Mazzucchi. The originality of Peano's 1886 existence theorem for scalar differential equations. *Forthcoming*, 2015.
[16] T. H. Grönwall. Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Annals of Mathematics*, **20**:292–296, 1919.
[17] D. Hilbert. *Grundlagen der Geometrie*. Teubner, Leipzig, 1st edition, 1899.
[18] D. Hilbert. Les principes fondamentaux de la géométrie. *Annales Sci. l'École Norm. Sup.*, **17**:103–209, 1900.
[19] D. Hilbert. *The foundations of geometry*. Open Court Publishing Co., La Salle, 1st edition, 1902.
[20] D. Hilbert. *The foundations of geometry*. Open Court Publishing Co., La Salle, 2nd edition, 1971.
[21] C. Jacobi. De determinantibus functionalibus. *Journal für die reine und angewandte Mathematik*, **22**:319–359, 1841.
[22] C. Jordan. *Cours d'Analyse de l'École Polytechnique* (3 vol.). Gauthier-Villars, Paris, 1882-87.
[23] H. C. Kennedy. Is there an elementary proof of Peano's existence theorem for first order differential equations? *American Mathematical Monthly*, **76**:1043–1045, 1969.
[24] J. Mawhin. Problèmes de Cauchy pour les équations différentielles et théories de l'intégration: influences mutuelles. *Cahiers du séminaire d'histoire des mathématiques*, **9**:231–246, 1988.
[25] J. Mawhin. De La Vallée Poussin's contributions to the fundamental theory of ordinary differential equations. In *De La Vallée Poussin, Charles-Jean: Collected Works.* Vol. II, pp. 301-314. Académie Royale de Belgique, 2001.
[26] H. Minkowski. *Geometrie der Zahlen*. Teubner, Leipzig, 1896.
[27] Alessandro Padoa. Essai d'une théorie algébrique des nombres entiers précédée d'une introduction logique à une théorie déductive quelconque. In *Bibliothèque du congrès international de philosophie,* Paris 1900, Vol. III, pp. 309-365. Colin, Paris, 1901.
[28] V. Pambuccian. The axiomatics of order geometry. I. Ordered incidence spaces. *Espositiones Mathematicae*, **29**:24–66, 2011.
[29] M. Pasch. *Vorlesungen über neuere Geometrie*. Teubner, Leipzig, 1st edition, 1882.
[30] M. Pasch. *Vorlesungen über neuere Geometrie*. Teubner, Leipzig, 2nd edition, 1926.
[31] G. Peano. Extrait d'une lettre. *Nouvelles Annales de Mathématiques*, **3**:45–47, 1884.
[32] G. Peano. Réponse à Ph. Gilbert. *Nouvelles Annales de Mathématiques*, **3**:252–256, 1884.
[33] G. Peano. Sull'integrabilità delle equazioni differenziali di primo ordine. *Atti R. Acc. Scienze Torino*, **21**:677–685, 1886.
[34] G. Peano. *Applicazioni geometriche del calcolo infinitesimale*. Fratelli Bocca, Torino, 1887.
[35] G. Peano. *Calcolo geometrico secondo Ausdehnungslehre di H. Grassmann.* Fratelli Bocca, Torino, 1888.
[36] G. Peano. Intégration par séries des équations différentielles linéaires. *Mathematische Annalen*, **32**:450–456, 1888.
[37] G. Peano. *I principii di geometria logicamente esposti*. Fratelli Bocca, Torino, 1889.
[38] G. Peano. Une nouvelle forme du reste dans la formule de Taylor. *Mathesis*, **9**:182–183, 1889.
[39] G. Peano. Démonstration de l'intégrabilité des équations différentielles ordinaires. *Mathematische Annalen*, **37**:182–228, 1890.

[40] G. Peano. Sulla formula di Taylor. *Atti Accad. Scienze Torino*, **27**:40–46, 1892.
[41] G. Peano. Sur la définition de la dérivée. *Mathesis*, **2**:12–14, 1892.
[42] G. Peano. Sur le théorème général relatif à l'existence des intégrales des équations différentielles ordinaires. *Nouvelles Annales de Mathématiques*, **11**:79–82, 1892.
[43] G. Peano. Sui fondamenti della geometria. *Rivista di Matematica*, **4**:51–90, 1894.
[44] G. Peano. Generalità sulle equazioni differenziali ordinarie. *Atti R. Acc. Sc. Torino*, **33**:9–18, 1897.
[45] G. Peano. Analisi della teoria dei vettori. *Atti R. Acc. Scienze Torino*, **33**:513–534, 1897-98.
[46] G. Peano. *Formulaire Mathématique*. Fratelli Bocca, Torino, 4th edition, 1902-3.
[47] G. Peano. *Formulario Mathematico*. Fratelli Bocca, Torino, 5th edition, 1908.
[48] G. Peano. Resto nelle formule di quadratura espresso con un integrale definito. *Atti della Reale Accademia dei Lincei, Rendiconti*, **22**:562–569, 1913.
[49] G. Peano. Residuo in formulas de quadratura. *Mathesis*, **4**:5–10, 1914.
[50] E. Picard. Sur le théorème général relatif à l'existence des intégrales des équations différentielles ordinaires. *Nouvelles Annales de Mathématiques*, **10**:197–201, 1891.
[51] M. J. D. Powell. *Approximation theory and methods*. Cambridge University Press, Cambridge, 1981.
[52] W. Prenowitz and J. Jantosciak. *Join geometries. A theory of convex sets and linear geometry*. Springer-Verlag, New York, 1979.
[53] W. Retter. *Topics in abstract order geometry*. PhD thesis, Technischen Universität Hamburg-Harburg, 2013.
[54] A. Sard. Integral representation of remainders. *Duke Math. J.*, **15**:333–345, 1948.
[55] A. Sard. *Linear approximations*. American Mathematical Association, Providence, 1963.
[56] L. Schwartz. *A mathematician grappling with his century*. Birkhäuser Verlag, Basel, 2001.
[57] M. Segre. Le lettere di Giuseppe Peano a Felix Klein. *Nuncius*, **12**:109–122, 1997.
[58] A. H. Stroud. *Numerical quadrature and solutions of ordinary differential equations*. Springer-Verlag, New York, 1974.
[59] A. Tarski and S. Givant. Tarski's system of geometry. *Bulletin of Symbolic Logic*, **5**:175–214, 1999.
[60] J. Thomae. *Einleitung in die Theorie der bestimmten Integrale*. Nebert, Halle, 1875.
[61] R. Torretti. *Philosophy of geometry from Riemann to Poincaré*. Reidel Publishing Company, Dordrecht, 1978.
[62] M. L. J. van de Vel. *Theory of convex structures*. North-Holland, Amsterdam, 1993.
[63] A.N. Whitehead. *The axioms of descriptive geometry*. Cambridge University Press, Cambridge, 1907.

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