THE ASTONISHING OBLIVION OF PEANO'S MATHEMATICAL LEGACY (I)

YOUTHFUL ACHIEVEMENTS, FOUNDATIONS, ARITHMETIC, VECTOR SPACES

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Abstract. The formulations that Peano gave to many mathematical notions at the end of 19th century were so perfect and modern that they have become standard today. A formal language of logic that he created, enabled him to perceive mathematics with great precision and depth. He described mathematics axiomatically basing the reasoning exclusively on logical and set-theoretic primitive terms and properties, which was revolutionary at that time. Yet numerous Peano's contributions remain either unremembered or underestimated.

Ask a mathematician about Peano's achievements and you would probably hear about Peano's continuous curve that maps the unit interval onto a square and about Peano's axioms of natural numbers. One might have heard of Peano series and Peano remainder.

Figure 1. Giuseppe Peano (1858-1932)

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It is unlikely that he/she would mention *"Zermelo" axiom of choice, "Borel-Lebesgue" theorem, "Borel" theorem on smooth functions, "Fréchet" derivative, "Bouligand" tangent cone, "Grönwall" inequality, "Banach" operator norm, "Kuratowski" upper and lower limits of sequences of sets, "Choquet" filter grill* or *"Mamikon" sweeping-tangent theorem*, in spite of the fact that Peano anticipated these notions or proved these theorems well before, and often in a more accomplished and general form, than those who granted them their names.

It is plausible that you won't be told that he was at the origin of many mathematical symbols (like $\in, \cup, \cap, \subset, \exists$), of reduction of all mathematical objects and properties to sets, of axiomatic approach to Euclidean space (with vectors and scalar product), of the theory of linear systems of differential equations (with matrix exponential and resolvents), of modern necessary optimality conditions, of derivation of measures, of definition of surface area and of many others.

At the time when Peano appeared on the mathematical scene, mathematical discourse was in general vague and approximative ([1]). Peano's writings impress by their clarity, precision, elegance and abstraction. Although Cauchy was praised for having given solid bases to mathematics, he was not exempt from errors that would be characterized today as elementary. Nor had the new rigor of Weierstrass put an end to vagueness.

The simplicity and ease with which Peano grasped the essence of things is astonishing and contrasts sharply with a tortuous style prevailing in contemporaneous mathematical texts.

In this essay consisting of three parts (c.f., [7], [8]) we will recall what is more or less forgotten about the importance of this great scientist. In doing so, we will exploit many facts gathered in the papers commemorating the 150th anniversary of the birth of Peano by Dolecki, Greco [4, 5, 6], Greco, Pagani [20, 21], Greco, Mazzucchi, Pagani [18, 19], Bigolin, Greco [2], Greco [14, 12, 13] and Greco, Mazzucchi [15, 17, 16].

From a methodological point of view, we are focused mainly on primary sources. Therefore our account of historical writings is far from being exhaustive.

The reader may judge whether the perceived oblivion is astonishing. We discuss plausible reasons of this oblivion at the end of the last part of this essay [8].

1. Short biography

Giuseppe Peano studied at the University of Turin from 1876 till 1880 and remained therein on the faculty until his death in 1932. After graduation

[1] For example, historians of mathematics agree that the first rigorous proof that a function is constant provided that its derivative is null, was given by H. A. Schwarz in 1870. Much later, in 1946, J. H. Pearce, a reviewer of a textbook "The Theory of Functions of Real Variables" by L. M. Graves, stresses that Rolle's Theorem has a *correct* proof, "a comparative rarity in books of this kind".

he became an assistant of Angelo Genocchi in the academic year 1881/82. In charge of exercises to Genocchi's calculus course, he soon took over the course, because Genocchi fell ill. Peano compared lecture notes of Genocchi's lessons with all the principal calculus textbooks of his time. He realized that in the contemporaneous mathematical literature numerous definitions and proofs were flawed and many theorems had overabundant hypotheses, which led him to rework and rectify them. This work resulted in many supplements by Peano to the lecture notes, so that when Genocchi saw the result, that is, *Calcolo differenziale e integrale* [9, (1884)], he disclaimed his contribution, stating that "everything [in the book] was due to that outstanding young man Dr. Giuseppe Peano".

In a celebrated *Encyclopädie der Mathematischen Wissenschaften* [41, (1899)], [47, (1899)] *Calcolo differenziale e integrale* and his another book *Lezioni di analisi infinitesimale* [34, (1893)] are cited among most influential treatises of infinitesimal calculus together with those of Euler (1748) and Cauchy (1821).

Peano reached the summit of fame at the turn of the century, when he took part in Paris in the International Congress of Philosophy and the International Congress of Mathematicians. Bertrand Russell reported that *Peano was always more precise than anyone else in discussions and invariably got the better of any argument upon which he embarked* [45, (1967), p. 218].

2. Youthful achievements

Most of major achievements of Peano were realized or prefigured before he turned thirty. They are collected in the already mentioned *Calcolo differenziale e integrale*, in *Applicazioni geometriche* [29, (1887)] and in *Calcolo geometrico* [30, (1888)]. Hence among youthful accomplishments we will consider those carried out a couple of years from graduation in 1880.

In 1882 he discovered that the definition of surface area, presented by Serret in his *Cours d'Analyse* [46, (1868), p. 296], was incorrect. Peano gave an example in which Serret's definition led to a contradiction. It turned out that Schwarz gave the same example two years before (see [8, §1]). A few years later, Peano proposed another definition of surface area that was compatible with the Lagrange formula in case of smooth surfaces.

In 1884 Peano observed that the proof of the mean value theorem formulated by Jordan in [22, (1882)] was faulty. In an exchange of messages with Jordan, the young Peano showed much deeper understanding of the subject than the famous Jordan (see [7, §1]).

In 1884 Peano showed that the Dirichlet function, conceived as an example of a function that is out of reach of analytic constructions, is in fact the double limit of a continuous function of two variables (see Section 3.1).

Also in [9, (1884)] Peano proved that for every sequence of real numbers $\{c_n\}_{n\in\mathbb{N}}$ there exists a C^∞-function $f:\mathbb{R}\to\mathbb{R}$ such that $f^{(n)}(0)=c_n$ for each $n \in \mathbb{N}$, the result later rediscovered by Borel [3, (1895)] and commonly called *Borel's theorem* (see Besenyei [1, (2014)]).

2.1. **Some of Peano's counter-examples.** As we have said, Peano encountered numerous inaccuracies and errors in mathematical literature and provided, with astonishing ease, a long list of counter-examples. He remains perhaps the champion of counter-examples in the mathematical world. Of course, it is natural that errors happen to (almost) everyone and papers of numerous great mathematicians contain, sometimes fecund, errors. Peano's rigor was, however, quite exceptional; Bertrand Russell comments in *The Principles of Mathematics* [44, (1903), p. 241] that Peano had a rare immunity from error. We list below some of Peano's counter-examples from *Calcolo differenziale e integrale* of 1884, to sundry statements of Cauchy, Lagrange, Serret, Bertrand, Todhunter, Sturm, Hermite, Schlömilch and others ([2]).

A. *The order of partial derivation cannot be altered in general*: if

$$f(x,y) := \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, \text{ if } x^2+y^2>0 \\ 0, \text{ if } x=y=0 \end{cases}$$

then $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ ([3]).

B. *The existence of partial derivatives is not sufficient for the mean value theorem in two (and more) variables:* Peano shows that for

$$f(x,y) := \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, \text{ if } x^2+y^2>0 \\ 0, \text{ if } x=y=0 \end{cases}$$

the mean value formula does not hold ([4]).

[2] In *Calcolo differenziale e integrale* Peano cites numerous authors, which is exceptional for textbooks. In order of appearance: Dini, Dedekind, Pasch, Cantor, Harnack, Lipschitz, Du Bois-Reymond, Serret, Jordan, Euclides, Leibniz, Joh. Bernoulli, Euler, Dirichlet, Cauchy, Gilbert, Darboux, Weierstrass, Schwarz, Heine, Poisson, Abel, Mac-Laurin, Wiener, Sturm, Waring, Tardy, Hoppe, Schlömilch, Most, Gatting, Bertrand, Terquem, Tardy, Mossa, Teixeira, Fergola, Faà di Bruno, Bolzano, Stolz, Rausenberger, Novi, Catalan, Oliver, Taylor, Lagrange, König, Amigues, Newton, Liouville, Hermite, N. Mercator, Gregory, Machin, Lindemann, Gergonne, Ampère, Jacobi, Bellavitis, Genocchi, Peano, Wronski, Trudi, Frobenius, Duhamel, Realis, Laurent, Lacroix, Cayley, Donkin, Baltzer, Hesse, Sylvester, Rouquet, Todhunter, Pringsheim, Mertens, Hoüel, Könisberger, Crelle, Clausen, Riemann, Volterra, Archimede, De Zolt and Wallis.

[3] Peano mentions other, more complicated counter-examples, for instance, that of Dini and of Schwarz (1873):

$$f(x,y) := \begin{cases} x^2 \arctan \frac{y}{x} - y^2 \arctan \frac{x}{y} \text{ if } xy \neq 0 \\ 0, \text{ if } xy = 0. \end{cases}$$

[4] Indeed, both partial derivatives are null at $(0,0)$, otherwise

$$f_x(x,y) = \frac{y^3}{(x^2+y^2)^{3/2}} \text{ and } f_y(x,y) = \frac{x^3}{(x^2+y^2)^{3/2}}.$$

C. *On the formula of de l'Hôpital:* $f(0) = 0 = g(0)$ and $\lim_{x\to 0} \dfrac{f(x)}{g(x)}$ exists without the existence of $\lim_{x\to 0} \dfrac{f'(x)}{g'(x)}$. Peano proposes a counter-example, defining $f[x] := x^2 \sin \frac{1}{x}$ (for $x \neq 0$) and $g(x) := x$.

D. *A function can attain, on each straight line passing through* $(0,0)$, *a local minimum at* $(0,0)$, *without attaining its local minimum at* $(0,0)$ ([5]). Consider

$$f(x,y) := (y - x^2)(y - 2x^2).$$

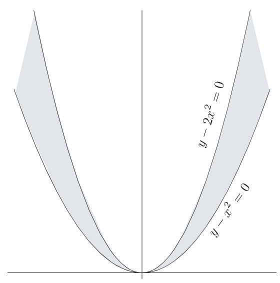


FIGURE 2. Zero sub-level of f.

The value $f(x)$ is positive if $y \geq 2x^2$ and if $y \leq x^2$; it is negative if $x^2 \leq y \leq 2x^2$. Therefore $(0,0)$ is neither a local minimum nor a local maximum. Each straight line L passing through $(0,0)$ remains in the positivity area of f on an open interval around $(0,0)$, that is, f attains a local minimum on L.

3. Foundations

The formal language of logic that Peano developed, enabled him to perceive mathematics with great precision and depth. Actually he described mathematics axiomatically basing the reasoning exclusively on logical and set-theoretic primitive terms and properties, which was revolutionary at that time. Logic was for him the common part of all theories.

For every real number t,

$$f(t,t) = \frac{|t|}{\sqrt{2}}, \quad f_x(t,t) = f_y(t,t) = \frac{(\mathrm{sgn}(t))^3}{2\sqrt{2}} \in \{-\frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}\}.$$

Therefore, for $x_0 := y_0 := -1$ and $h := k := 3$, we have

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = \frac{|x_0 + h| - |x_0|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

and $h\frac{\partial f}{\partial x}(x_0 + \vartheta h, y_0 + \vartheta k) + k\frac{\partial f}{\partial y}(x_0 + \vartheta h, y_0 + \vartheta k) = 2h\frac{\partial f}{\partial x}(x_0 + \vartheta h, x_0 + \vartheta h) = 3\frac{(\mathrm{sgn}(x_0+\vartheta h))^3}{\sqrt{2}} \in \{-\frac{3}{\sqrt{2}}, 0, \frac{3}{\sqrt{2}}\}$ for every real number $\vartheta \in [0,1]$; hence, the mean value formula does not hold.

[5] Peano constructs this counter-example of a related statement of Serret: "if $df(x,y) = 0$, and $d^4 f(x,y)(h,k) > 0$ for each (h,k) such that $d^2 f(x,y)(h,k) = d^3 f(x,y)(h,k) = 0$, then (x,y) is a local minimum".

It should be emphasized that the formal language conceived and used by Peano was not a kind of shorthand adapted for a mathematical discourse, but a collection of ideographic symbols and syntactic rules with unambiguous set-semantics, which produced precise mathematical propositions, as well as inferential rules that ensured the correctness of arguments.

Semantics was, for him, inherent to syntax, a mathematical point of view as opposed to that of logicians. In the figure below, we report an excerpt of Peano's presentation (in latino sine flexione ([6]) and Peano's symbolic language) of the following Leibniz theorem from 1694: *If g, h are functions from $\mathbb{R}$ to $\mathbb{R}$, then a function f from $\mathbb{R}$ to $\mathbb{R}$ verifies the equality $f' = g\,f + h$ if and only if*

$$f(x) = e^{\int_0^x g}\left(f(0) + \int_0^x e^{-\int_0^u g} h(u)\mathrm{d}u\right)$$

for every $x \in \mathbb{R}$.

$\cdot 2$ $\quad g,h\,\varepsilon\,\mathrm{qFq}\ .\supset:\ f\varepsilon\,\mathrm{qFq}\ .\ \mathrm{D}f = [(gx\times fx+hx)|x,\mathrm{q}]\ .=.$
$f = \{\mathrm{e}\upharpoonleft\mathrm{S}(g;\,0,x)[f0+\mathrm{S}\{[\mathrm{e}\upharpoonleft-\mathrm{S}(g;\,0,x)]\times hx|x;\,0,x\}]|x,\mathrm{q}\}$

{ LEIBNIZ A.Erud. a.1694 Math.S. t.5 p.313 }

Dato duo functione g et h, æquatione, identitate in numero reale x, conditione in functione f:

$$\mathrm{D}fx = gx\times fx+hx$$

vocare in f « æquatione differentiale lineare ». Si gx es constante, et hx es nullo, resulta æquatione considerato in p.323.

Calculo de functione f: $\quad x\varepsilon\mathrm{q}\ .\supset_x .\ \mathrm{D}fx = gx\,fx - hx$
Transporta: $\quad x\varepsilon\mathrm{q}\ .\supset_x .\ \mathrm{D}fx - gx\,fx = hx$
Multiplica per $\mathrm{e}\upharpoonleft-\mathrm{S}(g;0,x)$:
$x\varepsilon\mathrm{q}\ .\supset_x .\ \mathrm{D}[\mathrm{e}\upharpoonleft-\mathrm{S}(g;0,x)\times fx|x,\mathrm{q}]x = \mathrm{e}\upharpoonleft-\mathrm{S}(g;\,0,x)\times hx$
Integra ab 0 ad x:
$x\varepsilon\mathrm{q}\ .\supset_x .\ \mathrm{e}\upharpoonleft-\mathrm{S}(g;0,x)\times fx - f0 = \mathrm{S}[\mathrm{e}\upharpoonleft-\mathrm{S}(g;0,x)\times hx|x;0,x]$
Trahe fx:
$x\varepsilon\mathrm{q}\ .\supset_x .\ fx = \mathrm{e}\upharpoonleft\mathrm{S}(g;0,x)\{f0+\mathrm{S}[\mathrm{e}\upharpoonleft-\mathrm{S}(g,0,x)\times hx|x;0,x]\}$
Unde nos deduce $f =$ secundo membro, ubi varia x in campo de numeros reale.

FIGURE 3. Formal statement and proof of Leibniz theorem from Peano's *Formulario mathematico* [37, (1908) p. 431].

3.1. **Dirichlet function.** Dirichlet was probably the first to conceive functions as arbitrary assignments, which need not be expressed analytically, that is, by algebraic operations, in terms of elementary functions and their limits. To show the extent of his new concept, he gave in [26, (1829) p. 132], as an example, the celebrated *Dirichlet function*, which is the characteristic function of the irrational numbers $\chi_{\mathbb{R}\setminus\mathbb{Q}}$ ([7]).

[6] *Latino sine flexione* (Latin without inflections), a simplified Latin conceived, for universal scientific communication, by Peano [35, (1903)].

[7] that assigns 0 to the rational numbers and 1 to the irrational numbers.

In [9, (1884)] Peano shows that, surprisingly, the Dirichlet function is analytically expressible as double limit of elementary functions

$$\chi_{\mathbb{R}\setminus\mathbb{Q}}(x) = \lim_{m\to\infty} \varphi\left(\sin\left(m!\pi x\right)\right),$$

where

$$\varphi(y) := \lim_{t\to 0} \frac{y^2}{y^2+t^2} = \begin{cases} 1 & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Indeed, if x is rational, say $x = \dfrac{p}{q}$, then $\sin\left(m!\pi x\right) = 0$ for each natural $m \geq q$, so that $\lim_{m\to\infty} \varphi\left(\sin\left(m!\pi x\right)\right) = 0$; on the other hand, if x is irrational, then $m!x$ is irrational for each natural m and thus $\sin(m!\pi x) \neq 0$, so that $\lim_{m\to\infty} \varphi\left(\sin\left(m!\pi x\right)\right) = 1$.

Peano adopts Dirichlet's definition of function in [9, (1884)]; in [37, (1908)] he defines *functions* and, more generally, *relations* as subsets of Cartesian products. In [38, (1911)], commenting on the freshly published *Principia Mathematica*, where relations are primary notions, Peano reiterates his preference to consider *set* as a primitive notion and defines functions as particular relations, as it is commonly done today.

3.2. **Reduction of mathematics to sets.** Until nineteenth century, there was a great variety of mathematical objects: numbers, lines, surfaces, figures, all considered as entities. The language of mathematics was constituted of a mixture of symbols and of common language. Because of semantic ambiguity of natural languages, mathematical facts expressed with their aid are not always univocal.

With Dedekind and Cantor, sets became mathematical objects, while Peano reduced all the objects and properties to sets. Relations became subsets of Cartesian products, functions became particular relations and operations were expressed by means of functions. All this constituted a conceptual revolution.

In this new framework, two objects x and y are equal if and only if $x \in X$ is equivalent to $y \in X$ for every set X. In other words, Peano implements the principle of Aristotle "*Nam quaecumque de uno praedicatur, ea etiam de altero praedicari debent*", of Saint Thomas Aquinas "*Quaecumque sunt idem, ita se habent, quod quidquid praedicatur de uno, praedicatur et de alio*" and of Leibniz "*Eadem sunt quorum unum in alterius locum substitui potest, salva veritate*", that we quote after Peano [39, (1915)] and [40, (1916)].

Peano understood the urgent necessity of an unequivocal formal language to refound mathematics on a solid basis. Starting from 1889, he formalized a significant part of mathematics of his times.

Mathematical objects are accessible through symbols. Peano introduced symbols he needed for his formalism of mathematics. Among them

$$\in, \cup, \cap, \subset, \exists,$$

that are now universal. He denoted the sets of natural numbers with N_0, of integer numbers with n, of rational numbers with r (for *rational*), of real numbers with q (for *quantity*), of numerical finite-dimensional Euclidean space with q_n and so on ([8]), and formed all mathematical expressions using two primitive propositions $x \in y$ and $x = y$. Therefore he kept the distinction between $\in$ and $\subset$, hence between an element x and the corresponding singleton $\{x\}$.

A relevant subject of research activity of Peano and his School ([9]) concerned *definitions* in Mathematics, a subject which received and still receives more attention from philosophers than from mathematicians. Peano used formal expressions to announce mathematical facts and formal inferential transformations to prove them. Peano's symbolic propositions were not stenographic, but organic, with precise univocal semantic values. Thanks to this absolute precision of his formalism, Peano could detect errors and pitfalls, and see the necessity of hypotheses or axioms. For Peano, mathematical facts were precisely those that can be expressed in terms of set-theoretic and logical symbols; therefore in [36, (1906)] Peano rejected the paradox of Richard [43, (1905)], as pertaining to linguistics and not to mathematics.

3.3. **Axiom of choice.** Peano realized that the *principle of infinitely many arbitrary choices* was not guaranteed by the "axioms" traditionally used in mathematics, when he elaborated a proof of existence of solutions to systems of ordinary differential equations [33, (1890)] (c.f. [7, §6.2]). His discovery, that made emerge the Axiom of Choice from mathematical unconsciousness, was possible because of his logical set-theoretic ideography, conceived and accomplished for the above proof of existence ([10]). Peano recognized as legitimate only the *principle of determined choices* in the case of infinitely many sets; where, for Peano, "determining an element of a set" meant "establishing a property which holds only for one element of the set". Thus for the specific problem of selecting elements from infinitely many closed bounded sets of Euclidean space, he chose the greatest element with respect to a lexicographic order ([11]).

[8] Capital letters R and Q are used by Peano to denote positive rational numbers and positive real numbers, respectively.

[9] The name *the School of Peano* was given to a group of about fifty Peano's followers, among whom were his pupils, assistants and other mathematicians committed to Peano's projects. Among most devoted were Giovanni Vailati, Filiberto Castellano, Cesare Burali-Forti, Alessandro Padoa, Giovanni Vacca, Mario Pieri, Tommaso Boggio and Ugo Cassina (see Kennedy [24, (1980)], [25, (2006)]).

[10] Ingenuous disinvoltura that reigned in the matter before Peano's discovery, should not be confused with prefiguration of the Axiom of Choice.

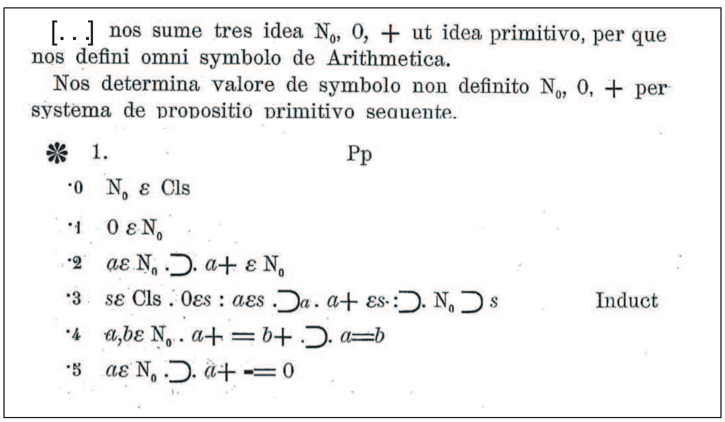
[11] For every natural number $n \geq 1$ and non-empty compact subset K of $\mathbb{R}^n$, Peano defines recursively an element $\omega_n(K)$ of K in the following way. Let $\omega_1(K) := \max K$ and let $\omega_{n+1}(K)$ be the element $(a, b) \in \mathbb{R} \times \mathbb{R}^n$ such that $a := \omega_1(\{x \in \mathbb{R} : \text{there is } y \in \mathbb{R}^n \text{ such } (x, y) \in K\})$ and $b := \omega_n(\{y \in \mathbb{R}^n : (a, y) \in K\})$. Clearly, $\omega_n(K)$ is the greatest element of K with respect to a lexicographic order on $\mathbb{R}^n$ given by "$(x_1, \dots, x_n) \ll (y_1, \dots, y_n)$ if and only if $(x_1, \dots, x_n) \neq (y_1, \dots, y_n)$ and $x_j < y_j$ for $j := \min\{i : x_i \neq y_i\}$".

After the rediscovery of the *axiom of choice* by Zermelo in [48, (1904)] ([12]), the pertinence of this axiom was discussed by the mathematical community, among whom Russell and Poincaré had their say, but Peano's contribution was forgotten. A promise of Zariski in 1924, to reestablish Peano's priority was apparently not kept (see [5, p. 321] for details).

4. Arithmetic

Peano proposed *six axioms* to define *natural numbers*. We list them as follows: the primitive notions $\mathbb{N}_0$, 0 and an operation σ fulfill the following axioms:

P0. $\mathbb{N}_0$ is a set,
P1. $0 \in \mathbb{N}_0$,
P2. $\sigma(n) \in \mathbb{N}_0$ for every $n \in \mathbb{N}_0$,
P3. if S is a set, $0 \in S$ and $\sigma(S) \subset S$, then $\mathbb{N}_0 \subset S$,
P4. σ is injective,
P5. $\sigma(n) \neq 0$ for every $n \in \mathbb{N}_0$.



[...] nos sume tres idea N_0, 0, $+$ ut idea primitivo, per que nos defini omni symbolo de Arithmetica.

Nos determina valore de symbolo non definito N_0, 0, $+$ per systema de propositio primitivo sequente.

✻ 1. Pp

·0 $N_0 \; \varepsilon \; \mathrm{Cls}$

·1 $0 \, \varepsilon \, N_0$

·2 $a \varepsilon N_0 \,.\!\supset\!.\, a+ \,\varepsilon\, N_0$

·3 $s \varepsilon \mathrm{Cls} \,.\, 0 \varepsilon s : a \varepsilon s \,.\!\supset_a .\, a+ \,\varepsilon s \,.\!:\supset\!.\, N_0 \supset s$ Induct

·4 $a, b \varepsilon N_0 \,.\, a+ = b+ \,.\!\supset\!.\, a = b$

·5 $a \varepsilon N_0 \,.\!\supset\!.\, a+ -\!= 0$

Figure 4. Here are Peano's six axioms of arithmetic as they appear in *Formulario mathematico* [37, (1908), p. 27] (Peano's comments are in *latino sine flexione*).

It follows from the axioms of Peano that $\mathbb{N}_0$ is infinite in the sense of Peirce and Dedekind (since the map $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ is injective but not surjective) and that $\mathbb{N}_0$ is a minimal infinite set (because of the induction principle P3). In the introduction to *Arithmetices principia, nova methodo exposita* [32, (1889)] Peano writes ([13])

[12] In *Zermelo's Axiom of Choice* [27, (1982) pp. 2, 80] Moore writes: "...in 1904 Ernst Zermelo first formulated the Axiom of Choice" and "Peano [1890] was the first to *reject* the use of [the principle of] infinitely many arbitrary choices." Moore seems not to see a temporal contradiction in a rejection of something that was formulated 14 years later.

[13] Translation from Latin is taken from Kennedy [23, (1973)].

> Questions pertaining to the foundations of mathematics, although treated by many these days, still lack a satisfactory solution. The difficulty arises principally from the ambiguity of ordinary language. For this reason it is of the greatest concern to consider attentively the words we use. [...]
>
> I have indicated by signs all the ideas which occur in the fundamentals of arithmetic, so that every proposition is stated with just these signs. The signs pertain either to logic or to arithmetic.

Following *Lehrbuch der Arithmetic* of Grassmann [10, (1861)], Peano extended by induction the operation σ to those of addition and multiplication. He was then in a position to extend arithmetic to integers, rational numbers and real numbers.

Some claim that Peano was beholden to Dedekind for his foundation of arithmetic. This is, however, not the case, because Peano proceeded axiomatically, proving, by the way, the independence of his axioms, while Dedekind proved everything, even unprovable, like the existence of infinite set. Peano used a completely formal and coherent language, while Dedekind was often vague (he did not distinguish *membership* from *set inclusion*).

5. Vector spaces

5.1. **Affine and vector spaces.** Peano firmly maintained the distinction between points and vectors and so on. He applied the geometric calculus of Grassmann and refounded axiomatically affine spaces and Euclidean geometry, based on the primitive notions of *point*, *vector* (i.e., difference of points) and *scalar product*.

In *Calcolo geometrico* [30, (1888)] Peano provided a modern definition of vector space structured by *addition* and *multiplication by scalars*, which fulfill

$$\text{(comutativity)} \qquad a+b=b+a,$$

$$\text{(associativity)} \qquad a+(b+c)=(a+b)+c,\ m(na)=(mn)a,$$

$$\text{(distributivity)} \qquad m(a+b)=ma+mb,\ (m+n)a=ma+na,$$

$$\text{(normalization)} \qquad 1a=a,\ 0a=\mathbf{0},$$

for every vectors a, b, c, and scalars m, n. That concept was implicit in the work of Grassmann [11, (1862)] and based on the notions of *sum*, *difference* and *multiplication by scalars*.

5.2. **Norms.** In [31, (1888)] Peano introduced numerical *Euclidean* space $\mathbb{R}^n$ for arbitrary natural n ([14]). He recognized the equivalence of the *Euclidean norm* in $\mathbb{R}^n$ with the l_∞-norm. Subsequently he defined the norm of

[14] Reid [42, (1955)], reviewing Murray and Miller [28, (1954)] writes: "To the reviewer it appears highly regrettable that the authors have not seen fit to introduce vector and matrix notation prior to the last chapter, where such is used in a limited fashion; certainly

linear maps F between Euclidean spaces by

$$\|F\| := \max_{x \neq 0} \frac{\|Fx\|}{\|x\|},$$

which constituted the first occurrence of the *Banach operator norm.* Furthermore he established its basic properties and its compatibility with the linear operator algebra, for example,

$$\|GF\| \leq \|G\| \, \|F\| \, .$$

He compared it with the Euclidean norm in the corresponding space of matrices and relates it to the eigenvalues of $F^T F$, where F^T is the transposed operator of F. He also gave the *Liouville formula*:

$$\det(e^A) = e^{\operatorname{tr} A}.$$

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