ADHERENCE OF MINIMIZERS FOR DUAL CONVERGENCES

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Dedicated to Professor Stefan Rolewicz on His 75-th Birthday

Abstract. It is proved that many known convergences (e.g., continuous convergence, Isbell topology, compact-open topology, pointwise convergence) on the space of continuous maps (valued in a topological space) can be represented as the dual convergences with respect to collections of families of sets, and that they can be characterized in terms of the corresponding hyperspace convergences of the inverse images of closed sets. As a result, the convergence of real-valued functions for a dual convergence implies the convergence of their sets of minima on the corresponding hyperspace.

1. Introduction

It is a well-known and simple fact (see e.g., [5]) that the *continuous convergence* of real-valued functions entails the *upper Kuratowski convergence* of the corresponding sets of minimizers:

$$f_0 \in \lim_{[X,\mathbb{R}]} \mathcal{F} \Longrightarrow \operatorname{Min} f_0 \in \lim_{[X,\$]} \operatorname{Min} \mathcal{F}.$$

If X and Z are topological spaces, then the continuous convergence $[X, Z]$ is the coarsest convergence on the set $C(X, Z)$ of continuous functions for which the coupling map $\langle x, f\rangle = f(x)$ is (jointly) continuous. As the usual topology of the real line is the supremum of the upper and the lower topologies inherited from the extended line $\bar{\mathbb{R}}$, the continuous convergence is the supremum of the upper and the lower continuous convergences. On the other hand, the upper Kuratowski convergence is the continuous convergence on $C(X, \$)$, where $\$$ stands for the *Sierpiński topology* $\{\varnothing, \{1\}, \{0, 1\}\}$ on $\{0, 1\}$. The space $C(X, \$)$ can be identified with the *hyperspace* of closed subsets of X. Therefore the discussed stability result follows from the following two observations: the continuous convergence with respect to the lower topology is equivalent to the upper Kuratowski convergence of the corresponding lower level sets; the continuous convergence with respect to the upper topology implies the upper convergence of the corresponding infima.

Similarly, numerous topologies (and convergences) defined on the set $C(X, \mathbb{R})$ with the aid of families of subsets of X, like *compact-open topology* or *pointwise convergence*, entail the convergence of the sets of minima with respect to the corresponding topologies on the hyperspace of closed subsets of X. We observe that this is still true for the *Isbell topology*, that is, the topology defined on $C(X, \mathbb{R})$ with the aid of the collection of compact families of subsets of X [14][15]. It follows from [9] that the Isbell topology of $C(X, \$)$ is homeomorphic to the *upper Kuratowski topology*, that is, to the topological reflection of the upper Kuratowski convergence.

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In this paper I show that all the mentioned convergences and topologies on $C(X,\mathbb{R})$ can be represented as dual with respect to some collections of families of subsets of X. In particular, continuous convergence is dual with respect to a subcollection of compact families. Similarly the *Isbell topology* is dual with respect to all compact families, the *compact-open topology* is dual with respect to compact sets, and the *pointwise convergence* is dual with respect to finite sets. Moreover dual convergences can be represented in terms of the corresponding hyperspace convergences of the inverse images of closed sets [10]. In our special setting of lower and upper topologies on $C(X,\bar{\mathbb{R}})$, this specializes to the corresponding convergences of lower and upper levels. As a consequence, many results (known and new) on stability of minima appear as corollaries of a single general fact. For example,

(1) *The continuous convergence of functions implies the upper Kuratowski convergence of their minima;*
(2) *The convergence of functions in the Isbell topology implies the convergence of their minima in the topologization of the upper Kuratowski convergence;*
(3) *The convergence of functions in the compact-open topology implies the convergence of their minima in the upper Wijsman topology;*
(4) *The pointwise convergence of functions implies the convergence of their minima in the upper set-theoretic convergence;*
(5) *The convergence of functions in the closed-open topology implies the convergence of their minima in the upper Vietoris topology.*

2. Continuous convergence

Although our framework is that of topological spaces, we are confronted with non-topological convergences as soon as we investigate spaces of (continuous) maps. Indeed, as mentioned in the Introduction, even if X, Z are topological spaces, the least structure on $C(X,Z)$, for which the canonical coupling is continuous, is, in general, non-topological. Actually the emergence of non-topological convergence theory [1] was motivated by this fact.

A *convergence* on Y is a relation between the filters $\mathcal{F}$ on Y and the elements y of Y, denoted by $y \in \lim \mathcal{F}$ $(= \lim_Y \mathcal{F})$ provided that $\mathcal{F} \subset \mathcal{G} \Longrightarrow \lim \mathcal{F} \subset \lim \mathcal{G}$ and $y \in \lim \mathcal{N}_\iota(y)$ for every $y \in Y$, where $\mathcal{N}_\iota(y)$ is the principal ultrafilter determined by y (see [7]). A set O in a convergence space Y is *open* if $O \cap \lim_Y \mathcal{F} \neq \varnothing$ implies $O \in \mathcal{F}$. The family of open subsets of a convergence space fulfills all the axioms open sets of a topology. This topology is called the *topologization* of the convergence (the *topological reflection* in terms of category theory).

If X, Z are convergence spaces, then $C(X,Z)$ stands for the subset of Z^X consisting of all the maps continuous from X to Z. The *continuous convergence* $[X,Z]$ (of X with respect to Z) is the coarsest among the convergences on $C(X,Z)$ for which the evaluation is continuous from $X \times C(X,Z)$ to Z. The convergence $[X,Z]$ exists for arbitrary spaces X and Z. Let us describe explicitly the continuous convergence in the case of topological spaces X and Z. If $\mathcal{G}$ is a filter on X, and $\mathcal{F}$ is a filter on $C(X,Z)$, then $\langle \mathcal{G}, \mathcal{F} \rangle$ stands for the filter generated by $\{\bigcup_{f \in F} f(G) : G \in \mathcal{G}, F \in \mathcal{F}\}$. Then

$$f \in \lim\nolimits_{[Y,Z]} \mathcal{F}$$

if and only if $f(x) \in \lim_Z \langle \mathcal{N}(x), \mathcal{F} \rangle$ for every $x \in X$ (where $\mathcal{N}(x)$ is the neighborhood filter of x); in other terms, if for each $x \in X$ and every open subset O

of Z such that $f(x) \in O$ there is a neighborhood W of x and $F \in \mathcal{F}$ such that $\bigcup_{f\in F} f(W) \subset O$.

The convergence $[X, Z]$ is Hausdorff (that is, two filters that converge to distinct elements do not mesh) provided that Z is Hausdorff. If however Z is the Sierpiński topological space $\$$, then $[X, \$]$ is the upper Kuratowski convergence on the space of X-closed sets, and if D is a closed set such that $D \supset A$, and $A \in \lim_{[X,\$]} \mathcal{F}$, then also $D \in \lim_{[Y,\$]} \mathcal{F}$. Moreover $[X, \$]$ is *hypercompact*, which means that every filter converges (in this case, to the whole of X).

3. Dual topologies

Let X, Z be topological spaces. A family $\mathcal{A}$ of open subsets of X is *openly isotone* if $O \supset A \in \mathcal{A}$ implies that $O \in \mathcal{A}$. If $\mathcal{A}$ is openly isotone and O is an open subset of Z, then let

$$[\mathcal{A}, O] := \{f \in C(X, Z) : f^{-1}(O) \in \mathcal{A}\}. \tag{3.1}$$

We denote by $\mathcal{O}_X$ the family of open subsets of X, and by $\mathcal{O}_X(A) := \{O \in \mathcal{O}_X : A \subset O\}$. If $\mathcal{A} = \mathcal{O}_X(A)$ then (3.1) is abridged to

$$[A, O] := \{f \in C(X, Z) : A \subset f^{-1}(O)\}.$$

It is straightforward that

$$[\bigcup_{i\in I} \mathcal{A}_i, O] = \bigcup_{i\in I} [\mathcal{A}_i, O], \tag{3.2}$$

$$[\mathcal{A}_0 \wedge \mathcal{A}_1, O] = [\mathcal{A}_0, O] \cap [\mathcal{A}_1, O], \tag{3.3}$$

where $\mathcal{A}_0 \wedge \mathcal{A}_1 := \{A_0 \cup A_1 : A_0 \in \mathcal{A}_0, A_1 \in \mathcal{A}_1\}$. In general, $[\mathcal{A}, O_0] \cap [\mathcal{A}, O_1]$ is not equal to $[\mathcal{A}, O_0 \cap O_1]$.

If α is a collection of openly isotone families $\mathcal{A}$ of subsets of X, and $\mathcal{O}_Z$ is the family of open subsets of Z, then

$$\{[\mathcal{A}, O] : \mathcal{A} \in \alpha, O \in \mathcal{O}_Z\} \tag{3.4}$$

is a subbase for a topology on $C(X, Z)$. This topology will be denoted $\alpha(X, Z)$ and the space of continuous maps endowed with it by $C_\alpha(X, Z)$. If, in particular, $\alpha = \{\mathcal{O}_X(D) : D \in \mathcal{D}\}$, where $\mathcal{D}$ is a family of subsets of X, then (3.4) is a base for a topology on the function space, sometimes denoted by $C_{\mathcal{D}}(X, Z)$.

Many classical topologies on spaces of continuous maps are defined with the aid of bases of the form $[D, O]$, where $D \in \mathcal{D}$ and O is open.

Example 3.1. *If we consider the family $X^{<\aleph_0}$ of finite subsets of X, then*

$$\{[F, O] : F \in X^{<\aleph_0}, O \in \mathcal{O}_Z\}$$

is a base of a topological space denoted by $C_p(X, Z)$; the corresponding topology is that of pointwise convergence*.*

Example 3.2. *If $\mathcal{K} = \mathcal{K}_X$ stands for the family all compact subsets X, then*

$$\{[K, O] : K \in \mathcal{K}_X, O \in \mathcal{O}_Z\}$$

is a base a topological space $C_k(X, Z)$, called the compact-open topology*.*

Example 3.3. *If $\mathcal{Z} = \mathcal{Z}_X$ stands for the family all closed subsets X, then*

$$\{[Z, O] : Z \in \mathcal{Z}_X, O \in \mathcal{O}_Z\}$$

is a base a topological space $C_z(X, Z)$, called the closed-open topology*. It is a very strong topology (see Example 4.3).*

A family $\mathcal{A}$ of open subsets of a topological space X is called *compact* [8] if whenever a family $\mathcal{P}$ of open sets fulfills $\bigcup \mathcal{P} \in \mathcal{A}$, then there exists a finite subfamily $\mathcal{P}_0$ such that $\bigcup \mathcal{P}_0 \in \mathcal{A}$. The family $\mathcal{O}(x)$, of open neighborhoods of x, is compact; if K is a compact set, then the family $\mathcal{O}(K)$ is compact. Denote by $\kappa = \kappa(X)$ the collection all (openly isotone) compact families on X. We notice that

$$(\forall_{i \in I}\ \mathcal{A}_i \in \kappa(X)) \Longrightarrow \bigcup_{i \in I} \mathcal{A}_i \in \kappa(X),$$

hence for every family $\mathcal{C}$ of compact subsets of X, the the family $\bigcup_{K \in \mathcal{C}} \mathcal{O}_X(K)$ is compact.[1]

Example 3.4. *The topology for which*

$$\{[\mathcal{A}, O] : \mathcal{A} \in \kappa(X), O \in \mathcal{O}_Z\}$$

is a subbase, is called the Isbell topology *(see, e.g.,* [10]*).*

If α and γ are two collections of families on X, and let Z be a topological space. Obviously, if $\gamma \subset \alpha$ then $\gamma(X, Z) \leq \alpha(X, Z)$, because there are more open sets for α than for γ. In particular, the Isbell topology is finer than the compact-open topology, which is finer than the pointwise convergence. In fact,

$$p(X, Z) \leq k(X, Z) \leq \kappa(X, Z) \leq [X, Z],$$

and all the inequalities can be strict (see [10]).

As for the closed-open topology, it is finer than the compact-open topology, provided that X is Hausdorff (because in this case, each compact subset of X is closed). In Section 5, regularity of X is shown to be sufficient for $z(X, Z)$ to be finer than $[X, Z]$.

The collection $o(X) := \{\mathcal{O}_X(x) : x \in X\}$ (consisting of all the families of open neighborhoods of elements of X) is a subclass of $\kappa(X)$.[2] Therefore the topology generated by $\{[\mathcal{A}, O] : \mathcal{A} \in o(X), O \in \mathcal{O}_Z\}$ is coarser than the Isbell topology. This topology is closely related to the continuous convergence. Namely,

Theorem 3.1. *$f_0 \in \lim_{[X,Z]} \mathcal{F}$ if and only if $f_0 \in [\mathcal{O}_X(x), O]$ implies that there is $W \in \mathcal{O}_X(x)$ such that $[W, O] \in \mathcal{F}$ for every $x \in X$ and each $O \in \mathcal{O}_Z$.*

Proof. By definition, $f_0 \in \lim_{[X,Z]} \mathcal{F}$ if $\langle x, f_0 \rangle \in \lim_\sigma \langle \mathcal{O}_X(x), \mathcal{F} \rangle$ for every $x \in X$, that is, whenever for every $O \in \mathcal{O}_Z(f_0(x))$ there is $W \in \mathcal{O}_X(x)$ and $F \in \mathcal{F}$ such that $f \in [W, O]$ for each $f \in F$. In other words, if for every open set O such that $f_0 \in [x, O]$ and there is $W \in \mathcal{O}_X(x)$ such that $[G, O] \in \mathcal{F}$. Now, $f_0 \in [x, O]$ if and only if $f_0 \in [\mathcal{O}_X(x), O]$, because f_0 is continuous. □

More is true: it was proved in [10] (in full generality) that

Theorem 3.2. *$f_0 \in \lim_{[X,Z]} \mathcal{F}$ if and only if $f_0 \in [\mathcal{A}, O]$ implies that there is $A \in \mathcal{A}$ such that $[A, O] \in \mathcal{F}$ for every $\mathcal{A} \in \kappa(x)$ and each $O \in \mathcal{O}_Z$.*

[1] A topological space is called *consonant* if every compact family is of that form [9].

[2] A general case of arbitrary convergences is considered in [10].

4. Hyperspaces and function spaces

We have said that if $\$$ is the Sierpiński topology (on a two-element set), then $C(X,\$)$ can be identified with the hyperspace of all the closed subsets of X. The dual topologies with respect to collections α, also admit specializations to that case, as does the continuous convergence. The following proposition is a special case a general fact in convergence spaces (see, e.g., [7]):

Proposition 4.1. *The continuous convergence* $[X,\$]$ *is homeomorphic to the upper Kuratowski convergence.*

Proof. Let $\psi_{A_0} \in C(X,\$)$ and $\mathcal{F}$ be a filter on $C(X,\$)$. By definition $\psi_{A_0} \in \lim_{[X,\$]} \mathcal{F}$ if and only if $\psi_{A_0}(x) = 1$ implies that there exists $W \in \mathcal{O}_X(x)$ such that $[W,\{1\}] \in \mathcal{F}$. Now, the hypothesis is equivalent to $x \notin A_0$ and the thesis means that there is $F \in \mathcal{F}$ and $W \in \mathcal{O}_X(x)$ such that $\psi_A(W) = \{1\}$ for every $A \in F$, that is, $W \cap \bigcup_{A\in F} A = \varnothing$, in other words, $x \notin \operatorname{cl}\left(\bigcup_{A\in F} A\right)$. On rewriting, $\psi_{A_0} \in \lim_{[X,\$]} \mathcal{F}$ if and only if

$$\bigcap_{F\in\mathcal{F}} \operatorname{cl}\left(\bigcup_{A\in F} A\right) \subset A_0.$$

□

Open sets for a topology $\alpha(X,\$)$, are generated by a subbase

$$[\mathcal{A},\{1\}] = \mathcal{A}_c = \{B \in \mathcal{O}_X : X \setminus B \in \mathcal{A}\}, \tag{4.1}$$

where $\mathcal{A} \in \alpha$. Actually, it is a base if α contains finite intersections of its elements, and it is a collection of all open sets (of $\alpha(X,\$)$) if α is stable by unions. Therefore

Proposition 4.2. $A_0 \in \lim_{\alpha(X,\$)} \mathcal{F}$ *if and only if* $A_0 \in \mathcal{A}_c$ *then* $\mathcal{A}_c \in \mathcal{F}$ *for every* $\mathcal{A} \in \alpha(X)$*.*

It was proved in [9] that $\mathcal{D}$ is an open set for $[X,\$]$ if and only if $\mathcal{D}_c$ is an openly isotone compact family on X. Therefore, it follows from (4.1) that the Isbell hyperspace topology $\kappa(X,\$)$ is equal to the topologization $T[X,\$]$ of $[X,\$]$, that is, to the *upper Kuratowski topology.* Nevertheless in general $T[X,Z]$ is not equal to $\kappa(X,Z)$ [13].

If $\alpha = \{\mathcal{O}_X(D) : D \in \mathcal{D}\}$ then

Corollary 4.3. $A_0 \in \lim_{\mathcal{D}(X,\$)} \mathcal{F}$ *if and only if for every* $D \in \mathcal{D}$ *such that* $A_0 \cap D = \varnothing$ *there is* $F \in \mathcal{F}$ *such that* $A \cap D = \varnothing$ *for each* $A \in F$*.*

Example 4.1 (pointwise topology)**.** *If* $\mathcal{D}$ *is the family of all finite subsets of* X*, then we get on* $C(X,\$)$ *the* upper set-theoretic convergence*, that is,* $B \in \lim \mathfrak{F}$ *whenever* $\bigcap_{\mathcal{F}\in\mathfrak{F}} \bigcup_{F\in\mathcal{F}} F \subset B$*. We denote this space by* $C_p(X,\$)$*.*

Example 4.2 (cocompact topology)**.** *(also called* upper Wijsman *topology) If* $\mathcal{K}$ *is the family of compact subsets of* X*, then* $\mathcal{A} \in \mathcal{N}_{\mathcal{K}}(B)$ *if for every* $K \in \mathcal{K}$ *disjoint from* B*, one has* $\{A \in C(X,\$) : A \cap K = \varnothing\} \subset \mathcal{A}$*. We denote it* $C_k(X,\$)$*.*

Example 4.3 (upper Vietoris topology)**.** *If* $\mathcal{Z} = C(X,\$)$ *is the family of closed subsets of* X*, then* $\mathcal{A} \in \mathcal{N}_z(B)$ *if for every* $O \in \mathcal{O}_X$ *such that* $B \subset O$*, one has* $\{A \in C(X,\$) : A \subset O\} \subset \mathcal{A}$*. This is a very strong topology, as indicates the Choquet theorem* [1][6]*.*

If $\mathcal{F}$ is a filter on $C(X,Z)$ then for every closed subset C of Z the filter $\mathcal{F}^{-1}(C)$ is generated by $\{\bigcup_{f\in F}\{f^{-1}(C)\}: F\in\mathcal{F}\}$ on $C(X,\$)$. It is proved in [10] that

Theorem 4.4. *$f_0 \in \lim_{[X,Z]}\mathcal{F}$ if and only if $f_0^{-1}(C) \in \lim_{[X,\$]}\mathcal{F}^{-1}(C)$ for each closed subset C of Z.*

Theorem 4.5. *$f_0 \in \lim_{\alpha(X,Z)}\mathcal{F}$ if and only if $f_0^{-1}(C) \in \lim_{\alpha(X,\$)}\mathcal{F}^{-1}(C)$ for each subset C of Z.*

5. Closed-open topology

Although the upper Vietoris topology $C_z(X,\$)$ has been known for a long time (see, e.g., [2]), the corresponding convergence of functions (the closed-open topology) $C_z(X,\mathbb{R})$ seems to appear here for the first time.

It is well-known and straightforward that if X is a regular topological space then the upper Vietoris topology is stronger than the upper Kuratowski convergence. Choquet observed in [1] that the upper Vietoris topology is very strong, and for that reason he deemed it uninteresting. He noticed that if a countably based filter $\mathcal{F}$ in a metrizable space X converges to A in the upper Vietoris topology [3], then there is a compact subset K of A such that the filter generated by $\{F\setminus A : F\in\mathcal{F}\}$ converges to K in the upper Vietoris topology (see also [3],[11],[16]). Nowadays some weaker conditions on the space and the filter have similar consequences [6].

The considerations above indicate that, more generally, closed-open topology is very strong. I observe

Proposition 5.1. *If the underlying topology is regular, then the closed-open topology is stronger than the continuous convergence.*

Proof. Let $f_0 \in \lim_{z(X,Z)}\mathcal{F}$ and let $x\in X$. Then for every open neighborhood O of $f(x_0)$ and an open neighborhood V of x such that $f_0(V)\subset O$ there exists a closed neighborhood W of x such that $f_0(W)\subset O$. It follows that $[W,O]\in\mathcal{F}$. □

Denote by $\mathcal{N}_\iota(y)$ the principal ultrafilter of y. If $\mathcal{F}$ is a filter on $C(X,Z)$ then $\mathcal{F}(x)$ is the filter generated by $\{\{f(x): f\in F\}: F\in\mathcal{F}\}$. Finally if a filter $\mathcal{F}$ on $C(X,Z)$ converges to f_0 in $z(X,\mathbb{R})$, then define the *active set* of $\mathcal{F}$ by

$$\operatorname{act}(\mathcal{F}) := \{x\in X : \mathcal{F}(x)\neq \mathcal{N}_\iota(f_0(x))\}.$$

Recall that a (Hausdorff) topological X space is *hemicompact* if there exists a sequence $(K_n)_n$ of compact subsets of X such that each compact subset of X is included in one of the elements of this sequence; it is a *k-space* if a set is closed provided that its intersection with each compact set K is closed in K. There exist hemicompact spaces, which are not k-spaces; on the other hand, each locally compact space of countable weight is a hemicompact k-space [12, p. 165].

Theorem 5.2. *Let X be a hemicompact k-space and let $\mathcal{F}$ be a countably based filter on $C(X,\mathbb{R})$. Then $f_0\in\lim_{z(X,\mathbb{R})}\mathcal{F}$ if and only if $f_0\in\lim_{k(X,\mathbb{R})}\mathcal{F}$ and there is a compact subset K of X such that $f_0(\operatorname{act}(\mathcal{F})\setminus K)$ is finite, where $\operatorname{act}(\mathcal{F})$ is the active set of $\mathcal{F}$, and $\mathcal{F}$ converges to a constant function z uniformly on $f_0^{-1}(z)$ for each $z\in f_0(\operatorname{act}(\mathcal{F})\setminus K)$.*

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[3]Actually, the set A was an arbitrary (not necessarily closed) subset of X.

Proof. As, in a Hausdorff space, each compact set is closed, $z(X,\mathbb{R})$ is stronger than $k(X,\mathbb{R})$. Assume that $f_0(\mathrm{act}(\mathcal{F})\setminus K)$ is infinite for each compact set K. Thus if $(K_n)_n$ is as in the definition of hemicompactness, then there is a sequence $(x_n)_n$ such that $x_n \notin K_n$ and $\{f_0(x_n) : n \in \mathbb{N}\}$ are all distinct. The set $C := \{x_n : n \in \mathbb{N}\}$ is closed, because it intersection with each compact set is finite, but it is not compact. Consequently, there exist $(\eta_k)_k$ such that $\{B(f_0(x_k),\eta_k) : k \in \mathbb{N}\}$ is pairwise disjoint. As $\mathcal{F}$ is generated by a decreasing sequence $(F_n)_n$, there exists a sequence $(\varepsilon_k)_k$ and a subsequence $(n_k)_k$ and $f_k \in F_{n_k}$ such that $f_k(x_k) \in B(f_0(x_k),\eta_k) \setminus B(f_0(x_k),\varepsilon_k)$. Therefore $f_0 \in [C, \bigcup_{k\in\mathbb{N}} B(f_0(x_k),\varepsilon_k)] \notin \mathcal{F}$. As $f_0^{-1}(z)$ is closed (for each z), if $\mathcal{F}$ converges to f_0 in $z(X,\mathbb{R})$, then the convergence on $f_0^{-1}(z)$ is uniform. Conversely, if $f_0 \in \lim_{k(X,\mathbb{R})} \mathcal{F}$ and the condition holds, then there are exist finitely many closed sets, say, $C_0, C_1, \ldots, C_m$ such that $X = \bigcup_{l=1}^{m} C_l$, and the set C_0 is compact, and $\mathcal{F}$ converges uniformly to a constant function on C_l for each $1 \le l \le m$. □

Corollary 5.3. *If X is a hemicompact k-space, and if $f = \lim_{z(X,\mathbb{R})}(f_n)$, then f is bounded on $\{x \in X : \forall_n \, f_n(x) \neq f(x)\}$.*

Corollary 5.4. *Let X be a k-space in which a sequence $(K_n)_n$ of compact sets is cofinal for compact sets, and $X \setminus K_n$ is connected for each n. If $f = \lim_{z(X,\mathbb{R})}(f_n)$ and the active set of (f_n) is X, then there is a compact subset K of X such that f is constant on $X \setminus K$.*

Proof. By Theorem 5.2, there is n such that f takes finitely many values on $X \setminus K_n$, say $\{r_1, r_2, \cdots, r_m\}$. Therefore the sets $f^{-1}(r_j) \setminus K_n$ are clopen in a (connected) set $X \setminus K_n$ for $1 \le j \le m$. Hence $m = 1$. □

6. Convergence of infima and of sets of minimizers

In our case, Theorems 4.4 and 4.5 take a particular form. A function $f : V \to \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ is *lower semicontinuous* whenever it is continuous with respect to the *lower topology* of $\bar{\mathbb{R}}$ (for which $\{(r,\infty] : -\infty \le r \le \infty\}$ is a base). Therefore $C(V,\bar{\mathbb{R}})$ is the space of maps that are continuous from X to $\bar{\mathbb{R}}$ endowed with the lower topology, in other words, *lower semicontinuous functions.*

A subset A of V is closed if and only if its *indicator function* ψ_A

$$\psi_A(x) = \begin{cases} -\infty \text{ if } x \in A \\ \infty \text{ if } x \notin A \end{cases} \tag{6.1}$$

is continuous for the lower topology of $\{-\infty, \infty\}$. Actually this topology is homeomorphic to the Sierpiński topology.

The well-known fact, that a function is lower semicontinuous if and only if its every lower level is closed, links up with Theorem 4.4 as follows: let $\mathcal{F}$ be a filter on $C(X,\bar{\mathbb{R}})$ and $f_0 \in C(X,\bar{\mathbb{R}})$. We denote by $\{\mathcal{F} \le r\}$ the filter (on the set of closed subsets of X) generated by $\{\bigcup_{f\in F}\{f \le r\} : F \in \mathcal{F}\}$. Then

Proposition 6.1. *If $\bar{\mathbb{R}}$ is endowed with the lower topology, then $f_0 \in \lim_{[X,\bar{\mathbb{R}}]} \mathcal{F}$ if and only if $\{f_0 \le r\} \in \lim_{[X,\$]}\{\mathcal{F} \le r\}$ for each $r \in \mathbb{R}$.*

Similarly, Theorem 4.5 specializes to

Proposition 6.2. *If $\bar{\mathbb{R}}$ is endowed with the lower topology, then $f_0 \in \lim_{\alpha(X,\bar{\mathbb{R}})} \mathcal{F}$ if and only if $\{f_0 \le r\} \in \lim_{\alpha(X,\$)}\{\mathcal{F} \le r\}$ for each $r \in \mathbb{R}$.*

The lower convergence of the infima of functions can be interpreted in terms of a set-open topology. Indeed, if we consider the family $o(X)$ consisting of a single set (the whole space X), and $\bar{\mathbb{R}}$ is endowed with the lower topology, then

Proposition 6.3. *If $\bar{\mathbb{R}}$ is endowed with the lower topology, and $f_0 \in \lim_{o(X,\bar{\mathbb{R}})} \mathcal{F}$, then $\inf_X f_0 \leq \sup_{F\in\mathcal{F}} \inf_{f\in F} \inf_X f$.*

Proof. Suppose the former and let $r < \inf_X f_0$. Then $f_0 \in [X, (r, \infty]]$ hence, by assumption, $[X, (r, \infty]] \in \mathcal{F}$, thus there is $F \in \mathcal{F}$ such that $\inf f \geq r$ for each $f \in F$. □

The topology $o(X, \bar{\mathbb{R}})$ is coarser than the (lower) closed-open topology $z(X, \bar{\mathbb{R}})$, because X is a closed set. Therefore

Corollary 6.4. *If $\bar{\mathbb{R}}$ is endowed with the lower topology, and $f_0 \in \lim_{z(X,\bar{\mathbb{R}})} \mathcal{F}$, then $\inf_X f_0 \leq \sup_{F\in\mathcal{F}} \inf_{f\in F} \inf_X f$.*

It is well-known that the continuous (lower) convergence does not imply the lower convergence of the corresponding infima, and that some supplementary conditions related to compactness are needed to assure it (e.g., [4]). In some sense, closed-open topology encompasses such conditions. Of course, all the topologies and convergences on $C(X, \bar{\mathbb{R}})$ (for $\bar{\mathbb{R}}$ with the lower topology) that weaker than the continuous convergence, they do not imply the lower convergence of infima.

On the other hand, the lower convergence of the suprema of functions follows from the lower pointwise convergence of the corresponding functions. Indeed,

Proposition 6.5. *If $\bar{\mathbb{R}}$ is endowed with the lower topology, and $f_0 \in \lim_{p(X,\bar{\mathbb{R}})} \mathcal{F}$, then $\sup_X f_0 \leq \sup_{F\in\mathcal{F}} \inf_{f\in F} \sup_X f$.*

Proof. If $r < \sup_X f_0$ then there is $x \in X$ such that $r < f_0(x)$, thus by the (lower) pointwise convergence, there is $F \in \mathcal{F}$ such that $r < f(x)$ for each $f \in F$, that is, $r \leq \sup_{F\in\mathcal{F}} \inf_{f\in F} \sup_X f$. □

As the pointwise convergence is the weakest among all the topologies and convergences considered in this paper, each of them implies the lower continuity of suprema.

If we endow the extended real line $\bar{\mathbb{R}}$ with the *upper topology* (the sets $\{[-\infty, r) : -\infty \leq r \leq \infty\}$ constitute a base), then a mirror proposition holds.

Proposition 6.6. *If $\bar{\mathbb{R}}$ is endowed with the upper topology, and $f_0 \in \lim_{p(X,\bar{\mathbb{R}})} \mathcal{F}$, then $\inf_X f_0 \leq \inf_{F\in\mathcal{F}} \sup_{f\in F} \inf_X f$.*

Corollary 6.7. *Let $\bar{\mathbb{R}}$ be endowed with the upper topology. If $\alpha(X)$ is a collection of families with $X^{<\aleph_0} \subset \alpha(X)$ and $f_0 \in \lim_{\alpha(X,\bar{\mathbb{R}})} \mathcal{F}$, then $\inf_X f_0 \leq \inf_{F\in\mathcal{F}} \sup_{f\in F} \inf_X f$.*

For simplicity sake, I shall prove the following auxiliary fact. Denote $g_0 := f_0 + r_{f_0}, g := f + r_f, \mathcal{G}$ a filter generated by $\{\{f + r_f : f \in F\} : F \in \mathcal{F}\}$, where r_f is a real number depending on the function f.

Lemma 6.8. *Let $\bar{\mathbb{R}}$ be endowed with the lower topology. If $f_0 \in \lim_{\alpha(X,\bar{\mathbb{R}})} \mathcal{F}$ and $r_{f_0} \leq \sup_{F\in\mathcal{F}} \inf_{f\in F} r_f$, then $g_0 \in \lim_{\alpha(X,\bar{\mathbb{R}})} \mathcal{G}$.*

Proof. If $f_0 + r_{f_0} \in [\mathcal{A}, (s, \infty]]$, then there is $A_0 \in \mathcal{A}$ such that $s < f_0(x) + r_{f_0}$ for each $x \in A_0$. On the other hand, there exist s_0 and s_1 such that $s = s_0 + s_1$ and $s_0 < f_0(x)$ for each $x \in A_0$ and $s_1 < r_{f_0}$. By assumption, there is $F \in \mathcal{F}$ such that $F \subset [\mathcal{A}, (s_0, \infty]]$ and $s_1 < r_f$ for every $f \in F$. Therefore, for every $f \in F$, there is $A_f \in \mathcal{A}$ such that $s_0 \leq \inf_{A_f} f$, thus $s < \inf_{A_f} f + r_f$ for each $f \in F$. Consequently $f + r_f \in [A_f, (s, \infty]]$ for every $f \in F$, that is, $[\mathcal{A}, (s, \infty]] \in \mathcal{G}$. □

We notice that the set of minimizers of a function $f : X \to \bar{\mathbb{R}}$ can be represented as

$$\mathrm{Min}_X f = \{x : f(x) \leq \inf_X f\}.$$

We will use this representation in establishing a general convergence result for the sets of minimizers. Denote by $\mathrm{Min}\,\mathcal{F}$ the filter generated by $\{\bigcup_{f \in F} \mathrm{Min}\, f : F \in \mathcal{F}\}$.

Theorem 6.9. *If α is a collection of families that fulfills $X^{<\aleph_0} \subset \alpha(X)$, and if $f_0 \in \lim_{\alpha(X,\mathbb{R})} \mathcal{F}$ and $\inf_X f_0 > -\infty$, then $\mathrm{Min}\, f_0 \in \lim_{\alpha(X,\$)} \mathrm{Min}\,\mathcal{F}$.*

Proof. As the considered functions f do not take infinite values, $\inf_X f < \infty$ for each f, and since $-\inf_X f_0 < \infty$, there is $F \in \mathcal{F}$ such that $-\inf_X f < \infty$ for each $f \in F$. Therefore $\mathrm{Min}_X f = \{x : f(x) - \inf_X f \leq 0\}$ for all such f. As $f_0 - \inf_X f_0 \in \lim_{\alpha(X,\mathbb{R})}(f - \inf f)\mathcal{F}$ it follows that the 0-lower levels converge in $\alpha(X, \$)$. □

Another formalism describing the situation above is based on Γ-convergence (c.f. [5]).

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