A SUBSEQUENTIAL FILTER THAT CANNOT BE EXTENDED TO A COUNTABLE HAUSDORFF SEQUENTIAL SPACE

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Abstract. Salvador Garcia-Ferreira asked if every subsequential filter on a countable set can be extended to a countable Hausdorff sequential space. A counter-example is constructed.

During his talk at ICTA in Kyoto (3-7 December 2007) professor Salvador Garcia-Ferreira asked if every atomic subsequential topology (on a countable set) can be extended to a countable Hausdorff sequential space. A topological space is *atomic*[1] if it admits at most one non-isolated point. A filter is *subsequential* if it is homeomorphic of a neighborhood filter of a subsequential topology.

Theorem. *There exists an atomic Hausdorff subsequential topology of rank* 2 *that cannot be extended to a Hausdorff sequential topology on a countable set.*

Proof. For a maximal almost disjoint family $\mathcal{A}$ of subsets of ω, consider the one-point compactification of the *Isbell-Mrówka* topology on the disjoint union $X := \{\infty\}\cup\mathcal{A}\cup\omega$. This is a Hausdorff sequential topology of order 2. Hence its restriction τ_0 to $\{\infty\}\cup\omega$ is subsequential and atomic. Denote by $\mathcal{F}$ the restriction to ω of the neighborhood filter of ∞.

Suppose that B is a countable set such that there exists a Hausdorff sequential topology τ on the disjoint union $Y := \{\infty\} \cup B \cup \omega$ so that the restriction of τ to $\{\infty\}\cup\omega$ is equal to τ_0. Let $\mathcal{N}_{\tau_0}(y)$ denote the trace on ω of the neighborhood filter $\mathcal{N}_\tau(y)$.

Because τ is Hausdorff, for every $y \in B$ the filters $\mathcal{N}_{\tau_0}(y)$ and $\mathcal{F}$ do not mesh. Therefore there exists a countable subfamily $\mathcal{A}_0$ of $\mathcal{A}$ such that for each $y \in Y$ there is a finite subfamily $\mathcal{B}_y$ of $\mathcal{A}_0$ such that $\bigcup_{A\in\mathcal{B}_y} A \in \mathcal{N}_\tau(y)$. Let $\mathcal{A}_1$ be another countably infinite subfamily of $\mathcal{A}$ so that $\mathcal{A}_0$ and $\mathcal{A}_1$ have no common element. Thus there are two disjoint subsets E_0 and E_1 of ω such that $A \setminus E_0$ is finite for each $A \in \mathcal{A}_0$ and $A \setminus E_1$ is finite for each $A \in \mathcal{A}_1$.

If $\infty \in \mathrm{cl}_{\tau_0} H \subset \mathrm{cl}_\tau H$ then there exists a sequential cascade T and a multisequence $f : T \to Y$ such that $f(\max T) \subset H$ and $\infty \in \lim_\tau f$, thus $\infty \in \lim_{\tau_0} \int f$, where $\int f$ is the contour of f (see e.g., [1]). In other words, each filter on ω that converges to ∞ in τ, contains E_0. This implies that $E_0 \in \mathcal{F}$. On the other hand, $E_1 \in \mathcal{F}^{\#}$, which is a contradiction with $E_0 \cap E_1 = \varnothing$. □

References

[1] S. Dolecki. Multisequences. *Quaestiones Mathematicae*, **29**:239–277, 2006.

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[1]called also *prime*.

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