ELIMINATION OF COVERS IN COMPLETENES

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Abstract. It is shown that nonadherent filters can totally eliminate covers from topological arguments, which enhances the unity of convergence approach. In particular, cocomplete collections of nonadherent filters replace complete collections of covers. Arhangel'skii-Frolík characterization of Čech complete spaces and its generalizations by Frolík are extended and refined. Hereditary completeness is dually characterized (in terms of pavements of the upper Kuratowski convergence). As a corollary a characterization by Dolecki and Mynard of the pretopologicity of the upper Kuratowski convergence (which generalizes to arbitrary convergences the characterization of Hofmann and Lawson) is recovered.

1. Introduction

The elimination mentioned in the title is founded on the fact that a family $\mathcal{P}$ is a cover of a subset A of a convergence space if and only if the adherence of $\mathcal{P}_c = \{P^c : P \in \mathcal{P}\}$ is disjoint from A. Therefore covers can be entirely eliminated from definitions and arguments by the dual concept of non-adherent families of sets. As numerous variants of compactness and of completeness can be defined with the aid of ideal (that is, closed for finite unions and subsets) covers, they can be also expressed in terms of non-adherent filters. The latter approach has been widespread in relation with miscellaneous notions of compactness, but it seems to have been absent from the study of completeness.

In this paper I use the concept of complete collection $\mathbb{P}$ of covers (such that, each filter that contains an element of every cover from $\mathbb{P}$ is adherent), which is similar to that of Frolík [6], and a dual notion of collection of cocomplete non-adherent filters (that is, of families of complements of covers from a complete collection). This dual point of view can often simplify arguments; one can judge it by what follows. In case of completeness of the type of Čech and Frolík, the dual approach enabled me to improve some results of the latter in several ways. On one hand, I prove (without any regularity assumption), that a dense κ-complete subset of a Hausdorff weakly diagonal convergence is G_κ (an intersection of κ many open sets)

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and on the other, every G_κ-subset of a regular κ-complete convergence is κ-complete (without any diagonality, a fortiori, topologicity, assumption). My results generalize also the classical characterization of Čech-completeness by Arhangel'skii and Frolík (this corresponds to the case $\kappa = \aleph_0$). Diagonality (which is a property somewhat weaker than topologicity) and regularity are two antithetic properties, one implying the other in the presence of compactness and Hausdorffness.

(Open) hereditary κ-completeness of an epitopology is equivalent to the existence of a pavement of cardinality κ at every element of the upper Kuratowski convergence. As a corollary, I recover a result of [4], which generalizes that of [7], that an epitopology is topologically core-compact if and only if the upper Kuratowski convergence is a pretopology.

2. Convergences

A *convergence space* is a set X endowed with a relation lim between filters on X and elements of X such that $\mathcal{F} \subset \mathcal{G}$ implies $\lim \mathcal{F} \subset \lim \mathcal{G}$, and $x \in \lim\{x\}^{\uparrow}$ for every $x \in X$ (where $\{x\}^{\uparrow}$ is the principal ultrafilter determined by x). If X, Y are convergence spaces, then a map $f : X \to Y$ is *continuous* if $f(\lim_X \mathcal{F}) \subset \lim_Y f(\mathcal{F})$ for every filter $\mathcal{F}$ on X.[1] This naturally entails the definitions of order (*finer, coarser*), initial and final convergences, quotient, product, and so on. A convergence is a *pseudotopology* if $\lim \mathcal{F} = \bigcap_{\mathcal{U} \in \beta\mathcal{F}} \operatorname{adh} \mathcal{U}$ for every filter $\mathcal{F}$, where $\beta\mathcal{F}$ denotes the set of all ultrafilters that are finer than $\mathcal{F}$. A convergence is *Hausdorff* if the limit of each filter is at most a singleton. A subset O of a convergence space is *open* if $O \cap \lim \mathcal{F} \neq \varnothing$ implies $O \in \mathcal{F}$; *closed*, if it is the complement of an open subset.

The *adherence* $\operatorname{adh} \mathcal{H}$ of a family $\mathcal{H}$ of subsets of a convergence space is the union of the limits of the filters $\mathcal{F}$ that *mesh* with $\mathcal{H}$ (in symbols, $\mathcal{F} \# \mathcal{H}$), that is, such that $F \cap H \neq \varnothing$ for every $F \in \mathcal{F}$ and $H \in \mathcal{H}$. In other words,

$$\operatorname{adh} \mathcal{H} = \bigcup_{\mathcal{F} \# \mathcal{H}} \lim \mathcal{F}.$$

In particular, the *adherence* $\operatorname{adh} H$ of a subset H of convergence is defined as the adherence of the principal filter of H. I denote by $\operatorname{adh}^{\natural} \mathcal{H} = \{\operatorname{adh} H : H \in \mathcal{H}\}$. Recall that if $\mathcal{H}$ is a family of subsets of a set X, then $\mathcal{H}_c = \{X \backslash H : H \in \mathcal{H}\}$. The *inherence* $\operatorname{inh} \mathcal{P}$ of a family $\mathcal{P}$ of subsets of X is defined by $\operatorname{inh} \mathcal{P} = (\operatorname{adh} \mathcal{P}_c)^c$ and $\operatorname{inh} P = \operatorname{inh} \mathcal{P}$ whenever $\{P\} \subset \mathcal{P} \subset \{P\}^{\uparrow}$. As usual, $\operatorname{inh}^{\natural} \mathcal{P} = \{\operatorname{inh} P : P \in \mathcal{P}\}$.

If $\mathcal{G}(y)$ is a family of subsets of X for every $y \in Y$, and $\mathcal{A}$ is a family of subsets of Y, then

$$\mathcal{G}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \bigcap_{y \in A} \mathcal{G}(y)$$

[1] $f(\mathcal{F})$ stands for the filter generated by $\{f(F) : F \in \mathcal{F}\}$.

is the *contour of* $\mathcal{G}(\cdot)$ *along* $\mathcal{A}$. In particular, if $\mathcal{A} = \{A\}$ or $\mathcal{A} = \{A\}^{\uparrow}$, then we abridge $\mathcal{G}(A) = \bigcap_{y\in A}\mathcal{G}(y)$. A set V is a *vicinity* of a set A with respect to a convergence θ if $\operatorname{adh}_\theta V^c \cap A = \varnothing$. The set of vicinities of A with respect to θ is denoted by $\mathcal{V}_\theta(A)$; in particular, $\mathcal{V}_\theta(x)$ stands for the set of vicinities of the singleton $\{x\}$. If $\mathcal{A}$ is a family of subsets of X, then $\mathcal{V}_\theta(\mathcal{A}) = \bigcup_{A\in\mathcal{A}}\mathcal{V}_\theta(A)$ is the contour of $\mathcal{V}_\theta$ along $\mathcal{A}$. In the sequel I shall use the following equivalence (see [2, Corollary 2.2]):

$$\mathcal{H}\#\mathcal{V}_\theta(\mathcal{A}) \Leftrightarrow \operatorname{adh}^\natural_\theta \mathcal{H}\#\mathcal{A}. \tag{2.1}$$

A convergence ξ on X is *diagonal* (respectively, *weakly diagonal*) if $x_0 \in \lim_\xi \mathcal{F}$ and if $x \in \lim_\xi \mathcal{G}(x)$ for every $x \in X$, then $x_0 \in \lim_\xi \mathcal{G}(\mathcal{F})$ (respectively, $x_0 \in \operatorname{adh}\mathcal{G}(F)$ for every $F \in \mathcal{F}$). Of course, each topology is diagonal, and every diagonal convergence is weakly diagonal. E. Lowen-Colebunders proved in [8] that the adherence of every filter is closed if and only if the convergence is weakly diagonal.

A convergence is *regular* if

$$\lim \mathcal{F} \subset \lim(\operatorname{adh}^\natural \mathcal{F})$$

for every filter $\mathcal{F}$. A convergence ξ on a set X is θ*-regular* (where θ is another convergence on X) whenever $\lim_\xi \mathcal{F} \subset \lim_\xi(\operatorname{adh}^\natural_\theta \mathcal{F})$ for every filter $\mathcal{F}$. If a convergence ξ is θ-regular, then for every family $\mathcal{H}$,

$$\operatorname{adh}_\xi \mathcal{V}_\theta(\mathcal{H}) \subset \operatorname{adh}_\xi \mathcal{H}.$$

If moreover ξ is a pseudotopology, then the converse also holds.

A family $\mathcal{A}$ of subsets of a convergence space is *compactoid*, respectively, *compact* (see, for example, [1]) if every filter $\mathcal{G}$ that meshes with $\mathcal{A}$ has non-empty adherence, respectively, $\operatorname{adh}\mathcal{G}$ meshes $\mathcal{A}$. In particular, a filter is compactoid if every finer ultrafilter is convergent. A subset of a convergence space is *compactoid* (*compact*) if the principal filter of the underlying set is compactoid (respectively, compact); *locally compactoid* (*locally compact*) if every convergent filter contains a compactoid (respectively, compact set).

3. Elimination of covers and pseudocovers

A family $\mathcal{P}$ of subsets of a convergence space X is a *cover* of a subset A of X if every filter convergent to an element of A has a common element with $\mathcal{P}$.[2] If we specialize this notion of cover for topologies, which are such convergences in which a filter $\mathcal{F}$ converges to an element x if and only if $O \in \mathcal{F}$ for every open set O that contains x, then we get the condition $\bigcup_{P\in\mathcal{P}} \operatorname{int} P \supset A$. Clearly for families $\mathcal{P}$ of open sets, $\mathcal{P}$ is a cover of A in the present sense if and only if it is in the classical one. Therefore the new notion can be seen as an extension of the classical one to arbitrary (not necessarily open) families of sets. Moreover, various concepts related to compactness

[2] A family is a cover of a convergence (space X) if it is a cover of the improper subset X of X.

and to completeness, which are usually introduced in the language of open covers, remain unaltered if we express them in terms of arbitrary covers.[3]

Non-adherent families are complementary with respect to covers.

Theorem 3.1. *A family $\mathcal{P}$ is a cover of A if and only if*

$$\operatorname{adh} \mathcal{P}_c \cap A = \varnothing. \tag{3.1}$$

Proof. By definition, (3.1) means that if a filter $\mathcal{F}$ converges to an element of A then $\mathcal{F}$ does not mesh with $\mathcal{P}_c$, that is, there exist $F \in \mathcal{F}$ and $P \in \mathcal{P}$ such that $F \cap P^c = \varnothing$, equivalently $F \subset P$, that is, $\mathcal{F} \cap \mathcal{P} \neq \varnothing$, which means that $\mathcal{P}$ is a cover of A. ∎

Notice that $\{X\}$ is a cover of every convergence on X. Of course, $\{X\}_c = \{\varnothing\}$, and clearly $\operatorname{adh}\{\varnothing\} = \varnothing$.

If $\mathcal{P}$ is a family of sets, then $\mathcal{P}^{\cup\downarrow}$ denotes the (possibly degenerate) ideal generated by $\mathcal{P}$, that is, $S \in \mathcal{P}^{\cup\downarrow}$ whenever there exists $\mathcal{R} \in [\mathcal{P}]^{<\omega}$ such that $S \subset \bigcup \mathcal{R}$. In turns out that in the definitions of miscellaneous variants of compactness and of completeness, one can use either all covers or the ideal covers without altering the meaning.[4]

A family $\mathcal{P}$ of subsets of a convergence space is a *pseudocover* of a subset A of a convergence space X if every ultrafilter convergent to an element of A contains an element of $\mathcal{P}$. It is immediate that a family $\mathcal{P}$ is a pseudocover of A if and only if for every filter $\mathcal{F}$ that converges to an element of A there is a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\bigcup \mathcal{P}_0 \in \mathcal{F}$. Of course, every cover is a pseudocover, and each additive pseudocover is a cover. If P is a subset of a topological space X such that $\operatorname{cl} P \cap \operatorname{cl} P^c \neq \varnothing$, then $\{P, P^c\}$ is a pseudocover, but not a cover, of X, because the neighborhood filter of $x \in \operatorname{cl} P \cap \operatorname{cl} P^c$ contains neither P nor P^c. An open pseudocover in a topological space is a cover. Indeed, let $\mathcal{P}$ be an open pseudocover of A in a topological space, that is, for each $x \in A$ there is a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $x \in \operatorname{int}(\bigcup \mathcal{P}_0)$, but since the elements of $\mathcal{P}_0$ are open, there is $P \in \mathcal{P}_0$ such that $x \in P$. It is straightforward that

Proposition 3.2. *$\mathcal{P}$ is a pseudocover of A if and only* $\operatorname{adh} \mathcal{H} \cap A = \varnothing$ *for every filter $\mathcal{H} \geq \mathcal{P}_c$.*

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[3]For example, consider the property: for every cover of A there exists a finite subcover of A. Of course, this property implies compactness of A: for every open cover of A there exists a finite subcover of A. Conversely, if $\mathcal{P}$ is a cover of a compact set A, then $\{\operatorname{int} P : P \in \mathcal{P}\}$ is an open cover of A, hence there is a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $\{\operatorname{int} P : P \in \mathcal{P}_0\}$ is a cover of A, thus $\mathcal{P}_0$ is a (finite) cover of A.

[4]For example, if A is compact, then obviously for every ideal cover $\mathcal{R}$ of A, there is a finite subfamily $\mathcal{R}_0$ of $\mathcal{R}$ that is a cover of A. But $\bigcup \mathcal{R}_0$ is also a cover of A, and since $\mathcal{R}$ is an ideal, $\bigcup \mathcal{R}_0 \in \mathcal{R}$, so that there is an element R of $\mathcal{R}$ such that $\{R\}$ is a cover of A. Conversely, if the last property holds and $\mathcal{P}$ is an open cover of A, then $\mathcal{P}^{\vee\downarrow}$ is also a cover of A, thus there is $R \in \mathcal{P}^{\vee\downarrow}$ such that $\{R\}$ is a cover of A. This means that there is a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $R \subset \bigcup \mathcal{P}_0$, and since the space is topological this implies that $A \subset \bigcup \mathcal{P}_0$. Therefore if $\mathcal{F}$ converges to $x \in A$, then there is $P \in \mathcal{P}_0$ such that $x \in P$ and $P \in \mathcal{F}$ because P is open.

4. Complete collections

If $\mathbb{P}$ is a collection of families of subsets (of a given set), then a filter $\mathcal{F}$ is $\mathbb{P}$-*Cauchy* if $\mathcal{F} \cap \mathcal{P} \neq \varnothing$ for every $\mathcal{P} \in \mathbb{P}$. A collection $\mathbb{P}$ of families of subsets of a convergence space is *complete* if every $\mathbb{P}$-Cauchy filter has non empty adherence.

Let ξ, θ be convergences on X. A collection $\mathbb{P}$ is called θ-*openly* ξ-*complete* if every θ-open $\mathbb{P}$-Cauchy filter is ξ-adherent. Every ξ-complete collection is θ-openly ξ-complete for every θ. If $\xi = \theta$ is fixed, then we say *openly complete.* I will postpone further comparison of the two completeness properties to subsequent sections. In [6] Frolík uses the term *complete* for what I call here *openly complete.*

Every Cauchy filter with respect to a complete collection is compactoid. Indeed, if $\mathcal{F}$ is $\mathbb{P}$-Cauchy then each $\mathcal{G} \geq \mathcal{F}$ is $\mathbb{P}$-Cauchy, hence if $\mathbb{P}$ is complete, then $\operatorname{adh} \mathcal{G} \neq \varnothing$, that is, $\mathcal{F}$ is compactoid.

As we shall see later, every convergence admits a complete collection of covers. What makes the difference is the cardinality that such a collection can have.

A family $\mathcal{P}$ (of subsets of a convergence space) is called *complete* if the collection $\{\mathcal{P}\}$ is complete.

If $\mathbb{P}$ is complete, then for every choice $P_{\mathcal{P}} \in \mathcal{P}$ with $\mathcal{P} \in \mathbb{P}$, the set $\bigcap_{\mathcal{P} \in \mathbb{P}} P_{\mathcal{P}}$ is compactoid (possibly empty). In fact, every filter on $\bigcap_{\mathcal{P} \in \mathbb{P}} P_{\mathcal{P}}$ is $\mathbb{P}$-Cauchy. In particular, if $\mathcal{P}$ is a complete family and $P \in \mathcal{P}$ then P is compactoid.

Let $\mathbb{P}, \mathbb{R}$ be collections of families of sets. Then $\mathbb{R}$ is a *refinement* of $\mathbb{P}$ if for every $\mathcal{P} \in \mathbb{P}$ there is $\mathcal{R} \in \mathbb{R}$ such that $\mathcal{R}$ is a refinement of $\mathcal{P}$.

Proposition 4.1. *A refinement of a complete collection is complete.*

Proof. Let $\mathbb{R}$ be a refinement of a complete collection $\mathbb{P}$. Let $\mathcal{H}$ be $\mathbb{R}$-Cauchy and let $\mathcal{P} \in \mathbb{P}$. Then there exists $\mathcal{R} \in \mathbb{R}$ such that $\mathcal{R}$ is a refinement of $\mathcal{P}$. As $\mathcal{H}$ is $\mathbb{R}$-Cauchy, $\mathcal{H} \cap \mathcal{R} \neq \varnothing$ hence $\mathcal{H} \cap \mathcal{P} \neq \varnothing$. Therefore $\mathcal{H}$ is $\mathbb{P}$-Cauchy, and adherent by the completeness of $\mathbb{P}$. ∎

Recall that $\mathcal{P}^{\cup\downarrow}$ denotes the (possibly degenerate) ideal generated by $\mathcal{P}$.

Proposition 4.2. *A collection $\mathbb{P}$ of covers is complete if and only the collection $\mathbb{P}_{\sim} = \{\mathcal{P}^{\cup\downarrow} : \mathcal{P} \in \mathbb{P}\}$ is complete.*

Proof. Let $\mathbb{P}$ be complete and let $\mathcal{F}$ be a filter such that $\mathcal{F} \cap \mathcal{P}^{\cup\downarrow} \neq \varnothing$ for each $\mathcal{P} \in \mathbb{P}$. If $\mathcal{U} \in \beta(\mathcal{F})$ then $\mathcal{U} \cap \mathcal{P} \neq \varnothing$ and thus $\varnothing \neq \lim \mathcal{U} \subset \operatorname{adh} \mathcal{F}$ which means that $\mathbb{P}_{\sim}$ is complete. Conversely, if $\mathbb{P}_{\sim}$ is complete and $\mathcal{F} \cap \mathcal{P} \neq \varnothing$, hence $\mathcal{F} \cap \mathcal{P}^{\cup\downarrow} \neq \varnothing$, for each $\mathcal{P} \in \mathbb{P}$ and thus $\operatorname{adh} \mathcal{F} \neq \varnothing$. ∎

Let $\mathcal{P}, \mathcal{R}$ be families of subsets of a convergence space; $\mathcal{R}$ is a *strong refinement* of $\mathcal{P}$ if $\operatorname{adh}^{\natural} \mathcal{R}$ is a refinement of $\mathcal{P}$. A collection $\mathbb{P}$ (of families of subsets) of a convergence space is said to be *regular* if for every $\mathcal{P} \in \mathbb{P}$ there is $\mathcal{R} \in \mathbb{P}$ that is a strong refinement of $\mathcal{P}$. A family $\mathcal{P}$ of subsets of a convergence space is called *regular* if the collection $\{\mathcal{P}\}$ is regular. For example, an ideal generated by a family consisting of closed sets is regular.

Lemma 4.3. *If $\mathbb{P}$ is a complete regular collection, and $\mathcal{H}$ is $\mathbb{P}$-Cauchy, then $\operatorname{adh}^\natural \mathcal{H}$ is also $\mathbb{P}$-Cauchy.*

Proof. Let $\mathcal{F}$ be $\mathbb{P}$-Cauchy, where $\mathbb{P}$ is such a collection. Then for every $\mathcal{P} \in \mathbb{P}$ there exists $\mathcal{R}_\mathcal{P} \in \mathbb{P}$ that refines $\mathcal{P}$. As $\mathbb{P}$ is regular, for every $\mathcal{P} \in \mathbb{P}$ there is $R_\mathcal{P} \in \mathcal{F} \cap \mathcal{R}_\mathcal{P}$ and $P \in \mathcal{P}$ such that $\operatorname{adh} R_\mathcal{P} \subset P$. Therefore $\operatorname{adh}^\natural \mathcal{F} \cap \mathcal{P} \neq \varnothing$ for every $\mathcal{P} \in \mathbb{P}$ and thus is $\mathbb{P}$-Cauchy. ∎

If a convergence is regular, then each cover admits a strong refinement that is a cover. More generally,

Proposition 4.4. *If X is a subset of a regular convergence space Y and $\mathcal{P}$ is a cover of X, then there is a strong refinement of $\mathcal{P}$, which is a cover of X.*

Proof. If $x \in X \cap \lim_Y \mathcal{F}$, then $x \in \lim_Y(\operatorname{adh}_Y^\natural \mathcal{F})$ by the regularity of Y, thus there is $F_\mathcal{F} \in \mathcal{F}$ and $P_\mathcal{F} \in \mathcal{P}$ such that $\operatorname{adh}_Y F_\mathcal{F} \subset P_\mathcal{F}$ because $\mathcal{P}$ is a cover of X. The family of $F_\mathcal{F}$, where $\mathcal{F}$ are all the filters containing X and convergent in Y, is a strong refinement of $\mathcal{P}$ that covers X. ∎

5. Convergence of Cauchy filters

A collection $\mathbb{P}$ of families of sets is *narrow* if for every choice $P_\mathcal{P} \in \mathcal{P}$ the set $\bigcap_{\mathcal{P}\in\mathbb{P}} P_\mathcal{P}$ is at most a singleton.

Lemma 5.1. *If $\mathbb{P}$ is a narrow regular collection, then the adherence of each $\mathbb{P}$-Cauchy filter is at most a singleton.*

Proof. By the regularity of $\mathbb{P}$, for every $\mathcal{P} \in \mathbb{P}$ there is $\mathcal{R}_\mathcal{P} \in \mathbb{P}$ such that $\operatorname{adh}^\natural \mathcal{R}_\mathcal{P}$ is a refinement of $\mathcal{P}$. If $\mathcal{H}$ is $\mathbb{P}$-Cauchy, then for each $\mathcal{P} \in \mathbb{P}$ there is $H_\mathcal{P} \in \mathcal{H} \cap \mathcal{R}_\mathcal{P}$. Therefore $\operatorname{adh} \mathcal{H} \subset \bigcap_{\mathcal{P}\in\mathbb{P}} \operatorname{adh} H_\mathcal{P} \subset \bigcap_{\mathcal{P}\in\mathbb{P}} P_\mathcal{P}$ is a singleton by the narrowness of $\mathbb{P}$. ∎

Proposition 5.2. *A convergence that admits a narrow regular collection of covers, is Hausdorff.*

Proof. Let $\mathcal{F}$ be a convergent filter, and $\mathbb{P}$ be a narrow regular collection of covers. Since each element of $\mathbb{P}$ is a cover, $\mathcal{F}$ is $\mathbb{P}$-Cauchy, and since $\mathbb{P}$ is narrow and regular, $\operatorname{adh} \mathcal{F}$ is at most a singleton by Lemma 5.1. As $\varnothing \neq \lim \mathcal{F} \subset \operatorname{adh} \mathcal{F}$, the proof is complete. ∎

Example 5.3. *Let X be a metrizable space, d be a compatible metric on X and let*

$$B_d(x,r) = \{w \in X : d(w,x) < r\},$$

and $\mathcal{P}_n = \{B_d(x, \frac{1}{n}) : x \in X, n < \omega\}$. Then $\{\mathcal{P}_n : n < \omega\}$ is a narrow, regular sequence of covers of X.

Proposition 5.4. *A pseudotopology that admits a complete narrow regular collection of covers, is regular.*

Proof. If $\mathbb{P}$ is such a collection and $x \in \lim \mathcal{F}$, then $\mathcal{F}$ is $\mathbb{P}$-Cauchy, because $\mathbb{P}$ consists of covers. As $\mathbb{P}$ is narrow and regular, by Proposition 5.2, $\lim \mathcal{F} = \{x\}$. By Lemma 4.3, $\mathrm{adh}^{\natural} \mathcal{F}$ is $\mathbb{P}$-Cauchy, thus by completeness, $\mathrm{adh}^{\natural} \mathcal{F}$ is compactoid, and by Lemma 5.1, $\{x\} = \lim \mathcal{F} \subset \mathrm{adh}(\mathrm{adh}^{\natural} \mathcal{F}) = \{x\}$. Therefore $\varnothing \neq \mathrm{adh}\, \mathcal{U} = \lim \mathcal{U} \subset \{x\}$ for every ultrafilter $\mathcal{U}$ finer than $\mathrm{adh}^{\natural} \mathcal{F}$. Hence by pseudotopologicity, $\{x\} = \lim(\mathrm{adh}^{\natural} \mathcal{F})$. ∎

Theorem 5.5. *If $\mathbb{P}$ is a complete, narrow, regular collection of covers of a pseudotopology, then every $\mathbb{P}$-Cauchy filter is convergent.*

Proof. Let $\mathcal{H}$ be a $\mathbb{P}$-Cauchy filter. Then by Lemma 5.1 there is such an element x of the convergence space that $\mathrm{adh}\, \mathcal{H} = \{x\}$. If $\mathcal{U}$ is an ultrafilter finer than $\mathcal{H}$, then $\mathcal{U}$ is also $\mathbb{P}$-Cauchy, hence by completeness, $\varnothing \neq \mathrm{adh}\, \mathcal{U} \subset \mathrm{adh}\, \mathcal{H} = \{x\}$. By pseudotopologicity,

$$\{x\} = \bigcap_{\mathcal{U} \in \beta \mathcal{H}} \mathrm{adh}\, \mathcal{U} \subset \lim \mathcal{H} \subset \mathrm{adh}\, \mathcal{H} = \{x\}.$$

∎

Corollary 5.6. *If a collection $\mathbb{P}$ of regular covers of a pseudotopological space is complete and narrow, then every $\mathbb{P}$-Cauchy filter converges.*

6. Cocomplete collections of filters

By virtue of Theorem 3.1 a family is a cover of a convergence space if and only if the family of complements has empty adherence. This duality between covers and non adherent families enables us to eliminate covers altogether, in particular, in case of completeness. As by Proposition 4.2, completeness can be investigated by using only ideal covers, the only families of complements in a study of completeness are filters. Notice that the collection 2^X of all subsets of X is an (improper) ideal cover of every convergence on X. As $(2^X)_c = 2^X$, the degenerate filter 2^X is non-adherent for every convergence on X.

A collection $\mathbb{G}$ of filters on a convergence space is said to be *cocomplete* if every filter $\mathcal{H}$ that does not mesh any $\mathcal{G} \in \mathbb{G}$ is compactoid (equivalently, adherent). Now a filter is compactoid if and only if each finer ultrafilter converges. Hence $\mathbb{G}$ is cocomplete if and only if for every non-convergent ultrafilter $\mathcal{U}$ there exists $\mathcal{G} \in \mathbb{G}$ such that $\mathcal{U} \geq \mathcal{G}$. Therefore,

Proposition 6.1. *A collection $\mathbb{G}$ of non-adherent filters is cocomplete if and only if convergent ultrafilters are precisely those meshing no element of $\mathbb{G}$.*

Proposition 6.2. *A collection $\mathbb{G}$ of filters is cocomplete if and only if $\mathbb{G}_*$ is complete.*

Proof. Let $\mathbb{G}$ be cocomplete and let $\mathcal{F}$ be a filter such that $\mathcal{F} \cap \mathcal{G}_c \neq \varnothing$ for every $\mathcal{G} \in \mathbb{G}$. This means that there is $G \in \mathcal{G}$ such that $G^c \in \mathcal{F}$, that is, $\mathcal{F}$ and $\mathcal{G}$ do not mesh. Therefore $\mathcal{F}$ is compactoid by the cocompleteness of $\mathbb{G}$, hence $\operatorname{adh}\mathcal{F} \neq \varnothing$. Conversely, assume that $\mathbb{G}_*$ is complete and $\mathcal{F}$ is a filter that meshes with no $\mathcal{G} \in \mathbb{G}$, equivalently $\mathcal{F} \cap \mathcal{G}_c \neq \varnothing$ for every $\mathcal{G} \in \mathbb{G}$. Thus $\mathcal{U} \cap \mathcal{G}_c \neq \varnothing$ for each $\mathcal{U} \in \beta(\mathcal{F})$ and every $\mathcal{G} \in \mathbb{G}$. The completeness of $\mathbb{G}_*$ implies that $\operatorname{adh}\mathcal{U} \neq \varnothing$, so that $\mathcal{F}$ is compactoid. ∎

Therefore there is a duality between complete collections (of ideals) and cocomplete collections (of filters), and between collections of covers and collections of non-adherent families. We may add to that another observation: $\mathbb{R}$ is a refinement of $\mathbb{P}$ if and only if for every $\mathcal{F} \in \mathbb{P}_*$ there exists $\mathcal{G} \in \mathbb{R}_*$ such that $\mathcal{G} \leq \mathcal{F}$, that is, whenever the collection $\mathbb{R}_*$ is *coarser* than the collection $\mathbb{P}_*$.

Proposition 6.3. *Suppose that a convergence is θ-regular. If $\mathbb{F}$ is a cocomplete collection of filters, then $\{\operatorname{adh}_\theta^\natural \mathcal{F} : \mathcal{F} \in \mathbb{F}\}$ is also cocomplete.*

Proof. Let $\mathcal{U}$ be an ultrafilter that does not mesh $\operatorname{adh}_\theta^\natural \mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}$, equivalently $\mathcal{V}_\theta(\mathcal{U})$ does not mesh $\mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}$, hence $\operatorname{adh}\mathcal{V}_\theta(\mathcal{U}) \neq \varnothing$ by the cocompleteness of $\mathbb{F}$. By θ-regularity, $\operatorname{adh}\mathcal{V}_\theta(\mathcal{U}) \subset \operatorname{adh}\mathcal{U}$ showing the cocompleteness of $\{\operatorname{adh}_\theta^\natural \mathcal{F} : \mathcal{F} \in \mathbb{F}\}$. ∎

By Proposition 6.2, if a collection $\mathbb{P}$ in a θ-regular space is complete, then $\{\operatorname{inh}_\theta^\natural \mathcal{P} : \mathcal{P} \in \mathbb{P}\}$ is complete.

A collection of filters is called *θ-closedly ξ-cocomplete* if $\mathbb{F}_*$ is θ-openly ξ-complete. If $\mathbb{F}$ is a collection, then

$$\operatorname{cl}_\theta \mathbb{F} = \{\operatorname{cl}_\theta^\natural \mathcal{F} : \mathcal{F} \in \mathbb{F}\}.$$

Proposition 6.4. *If $\operatorname{cl}_\theta \mathbb{F}$ is cocomplete, then $\mathbb{F}$ is θ-closedly cocomplete.*

Proof. If $\mathcal{H} = \mathcal{O}_\theta(\mathcal{H})$ does not mesh $\mathcal{F}$ for each $\mathcal{F} \in \mathbb{F}$, then $\mathcal{H}$ does not mesh $\operatorname{cl}_\theta^\natural \mathcal{F}$ for each $\mathcal{F} \in \mathbb{F}$, hence $\operatorname{adh}\mathcal{H} \neq \varnothing$. ∎

Theorem 6.5. *In a regular topology open completeness coincides with completeness.*

Proof. Let $\mathbb{F}$ be a closedly cocomplete collection of non-adherent filters. We shall see that $\operatorname{cl}\mathbb{F}$ is a cocomplete collection of non-adherent filters. If $\mathcal{H}$ does not mesh $\operatorname{cl}^\natural \mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}$, equivalently $\mathcal{O}(\mathcal{H})$ does not mesh $\mathcal{F}$ for every $\mathcal{F} \in \mathbb{F}$ and thus $\operatorname{adh}\mathcal{O}(\mathcal{H}) \neq \varnothing$. By regularity, $\operatorname{adh}\mathcal{O}(\mathcal{H}) \subset \operatorname{adh}\mathcal{H}$, hence $\operatorname{cl}\mathbb{F}$ is a cocomplete collection. By topologicity $\operatorname{adh}\mathcal{F} = \operatorname{adh}\operatorname{cl}^\natural \mathcal{F}$ for every filter $\mathcal{F}$, thus all the elements of $\operatorname{cl}\mathbb{F}$ are not adherent. ∎

7. Completeness number

Let κ be a cardinal. A convergence space is *κ-complete* if there is a complete collection, of cardinality κ, of covers. In other words, a convergence is κ-complete whenever there exists a cocomplete collection of cardinality κ of non-adherent filters.

Proposition 7.1. *If X_α is a κ_α-complete convergence for $\alpha < \lambda$, then $\prod_{\alpha<\lambda} X_\alpha$ is κ-complete, where $\kappa = \sum_{\alpha<\lambda} \kappa_\alpha$.*

Proof. If $\mathbb{F}_\alpha$ is a cocomplete collection of non-adherent filters on X_α, then let $\mathbb{F}$ consist of all those filters on $X = \prod_{\alpha<\kappa} X_\alpha$ that are generated by

$$[\mathcal{F}_\beta] = \{\prod_{\alpha<\lambda} F_\alpha : F_\beta \in \mathcal{F}_\beta, \alpha \neq \beta \Rightarrow F_\alpha = X_\alpha\},$$

with $\mathcal{F}_\beta \in \mathbb{F}_\beta$. The cardinality of $\mathbb{F}$ is $\sum_{\alpha<\lambda} \kappa_\alpha$. Now it is enough to observe that a filter $\mathcal{H}$ on X meshes with $[\mathcal{F}_\beta]$ if and only the projection $p_\beta(\mathcal{H})$ on X_β meshes with $\mathcal{F}_\beta$. Therefore if an ultrafilter $\mathcal{U}$ is finer than an element of $\mathbb{F}$, then there is β such that its projection on X_β is finer then an element of $\mathbb{F}_\beta$, hence $\mathcal{U}$ does not converge, because $\mathbb{F}_\beta$ consists of non-adherent filters; on the other hand, if an ultrafilter $\mathcal{U}$ meshes with no element of $\mathbb{F}$, then for every $\beta < \lambda$, its projection on X_β meshes with no element of $\mathbb{F}_\beta$, hence converges in X_β by the cocompleteness of $\mathbb{F}_\beta$, so that $\mathcal{U}$ converges. ∎

A convergence is called *openly κ-complete* if there exists an openly complete collection of covers of cardinality κ (equivalently, a closedly cocomplete collection of cardinality κ of non-adherent filters). As there are not more complete collections of covers than openly complete collections of covers, every κ-complete convergence is openly κ-complete. As the projection of an openly based filter from a product convergence space onto every component space is openly based, an adaptation of the proof above yields a generalization of [6, Theorem 2.10] which was established for products of completely regular topological spaces.

Proposition 7.2. *If X_α is a openly κ_α-complete convergence for $\alpha < \lambda$, then $\prod_{\alpha<\lambda} X_\alpha$ is openly κ-complete, where $\kappa = \sum_{\alpha<\lambda} \kappa_\alpha$.*

The least cardinal κ such that a convergence is κ-complete is called the *completeness number.* This is also the least cardinal for which there exists a cocomplete collection (of that cardinality) of non-adherent filters. Analogously one can define the *open completeness number* (of course, it is always less than or equal to the completeness number).

Every convergence is κ-complete for some cardinal κ. Indeed, each convergence on a set X is $2^{2^{\mathrm{card}(X)}}$-complete. In fact, the collection of all non-adherent ultrafilters on X is a cocomplete collection and its cardinality is not greater than that of all ultrafilters on X. We infer that the completeness number is well defined.

Proposition 7.3. *The completeness number is finite (equivalently, less than or equal to* 1*) if and only if the convergence is locally compactoid.*

Proof. If there is a complete cover, then its elements are compactoid, so that every convergent filter contains a compactoid set. If now there is a finite complete cover, say, $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n\}$, then the family

$$\{\bigcap_{k=1}^{n} P_k : (P_1, P_2, \ldots, P_n) \in \prod_{k=1}^{n} \mathcal{P}_k\}$$

is also a complete cover. If a convergence is locally compactoid, then the family of all compactoid sets is a complete cover. ∎

If $\mathcal{K}$ stands for the family of compactoid sets (of a fixed convergence), then $\mathcal{K}_c$ is the filter of *cocompactoid sets* (the complements of compactoid sets) that we shall call the *cocompactoid filter*. The cocompactoid filter is degenerate if and only if the convergence is compact. Indeed, this happens if and only if the whole space (the complement of the empty set) is compactoid, equivalently, compact.

The cocompactoid filter is cocomplete in every convergence, because an ultrafilter that does not mesh with it, contains a compactoid set, hence converges.

Proposition 7.4. *A convergence is locally compactoid if and only the cocompactoid filter is non-adherent.*

Proof. If $\mathcal{K}_c$ is non-adherent, and an ultrafilter $\mathcal{U}$ converges, then $\mathcal{U}$ does not mesh with $\mathcal{K}_c$, hence $\mathcal{U}$ contains a compactoid set, so that the convergence is locally compactoid. Conversely, if a convergence is locally compactoid, then every convergent ultrafilter contains a compactoid set, hence does not mesh with $\mathcal{K}_c$, which means that $\mathcal{K}_c$ is non-adherent. ∎

I say that a convergence is *Čech-complete* if its completeness number is $\aleph_0$. Originally Čech-completeness was defined as a property of Tychonoff topological spaces in different terms (see the next section); the property used here to extend the definition to convergence spaces, was a characterization (Arhangel'skii-Frolík theorem [5, Theorem 3.9.2]).

8. Subspaces

In this section I shall discuss the relation between the completeness number and the property of being homeomorphic to an intersection of a family, of certain cardinality, of open subsets.

A locally compact dense subset of a Hausdorff topological space is open [5, Theorem 3.3.9], and each open subset of a locally compact space is locally compact [5, Theorem 3.3.8]. In Engelking's terminology, a *locally compact* space is T_1, hence by [5, Theorem 3.3.1] Tychonoff.

By definition, a topological space is *Čech-complete* if it is Tychonoff, and is a G_δ subset (that is, a countable intersection of open subsets) of its Stone-Čech compactification [5]. A theorem of Frolík and Arhangel'skii [5, Theorem 3.9.2] says that a Tychonoff space is Čech-complete if and only if it admits a countable complete collection of (open) covers, (that is, according to our terminology, if it is $\aleph_0$-complete).

In other words, in the class of Tychonoff spaces, 1-complete spaces are open subsets of their extensions, and $\aleph_0$-complete spaces are G_δ-subsets of their extensions. In [6] Frolík generalizes the two facts mentioned above, showing a relationship between open κ-completeness and G_κ property for an arbitrary cardinal κ.

A subset X of a convergence space Y is called a G_κ*-subset* (of Y) if it is the intersection of κ vicinities of X in Y; it is a *topologically* G_κ*-subset* if it is the intersection of κ open sets. In view of this definition, topologically $G_{\aleph_0}$-sets are classically called G_δ-sets. Of course, each topologically G_κ-subset is a G_κ-subset.

By definition, a subset A of a convergence space X is *dense* if $\operatorname{adh} A = X$.

Theorem 8.1. *A dense* κ*-complete subset of a Hausdorff weakly diagonal convergence is a topologically* G_κ*-subset.*

Proof. Let X be a dense κ-complete subset of Y, where Y is a Hausdorff weakly diagonal convergence. Let $\mathbb{G}$ be a cocomplete collection of non-adherent filters on X. Then

$$Y \backslash X = \bigcup_{\mathcal{G} \in \mathbb{G}} \operatorname{adh}_Y \mathcal{G}. \tag{8.1}$$

Indeed, $X \cap \operatorname{adh}_Y \mathcal{G} = \varnothing$ for every $\mathcal{G} \in \mathbb{G}$, because $\mathcal{G}$ is non-adherent in X, thus $\supset$ holds in (8.1). Conversely, if $y \in Y \backslash X$, then because X is dense in Y and Y is Hausdorff, there is an ultrafilter $\mathcal{U}$ on X such that $\{y\} = \lim_Y \mathcal{U} = \operatorname{adh}_Y \mathcal{U}$ and thus $\mathcal{U}$ is non-adherent in X. By the completeness of $\mathbb{G}$, there is $\mathcal{G} \in \mathbb{G}$ such that $\mathcal{U} \# \mathcal{G}$, thus $y \in \operatorname{adh}_Y \mathcal{G}$, and hence $\subset$ holds in (8.1). Because Y is weakly diagonal, by [8, Theorem 1.3] $\operatorname{adh}_Y \mathcal{G}$ is closed for every filter $\mathcal{G}$, and thus X is a topologically G_κ-set. ∎

A subset A of a convergence X is called *openly dense* if for every $x \in X$ there exists an openly based filter on A that converges to x. Each openly dense set is dense, and the converse holds in topological spaces.

Theorem 8.2. *An openly dense openly* κ*-complete subset of a Hausdorff weakly diagonal convergence is* G_κ*.*

Proof. As before if $\mathbb{G}$ is an openly cocomplete collection of non-adherent filters, then $\supset$ holds in (8.1). Conversely, if $y \in Y \backslash X$, then because X is openly dense in Y and Y is Hausdorff, there is an ultrafilter $\mathcal{U}$ on X such that $\{y\} = \lim_Y \mathcal{O}_X(\mathcal{U}) = \operatorname{adh}_Y \mathcal{O}_X(\mathcal{U})$. Therefore by the open completeness of $\mathbb{G}$, there is $\mathcal{G} \in \mathbb{G}$ such that $\mathcal{O}_X(\mathcal{U}) \# \mathcal{G}$, and thus $y \in \operatorname{adh}_Y \mathcal{G}$, hence $\subset$ holds in (8.1). ∎

Theorem 8.3. *Every* G_κ*-subset of a regular* κ*-complete convergence space is* κ*-complete.*

Proof. Let $\{\mathcal{H}_\alpha : \alpha < \kappa\}$ be a cocomplete collection of non-adherent filters of a regular convergence space Y, and let $X = \bigcap_{\alpha<\kappa} V_\alpha$, where $V_\alpha \in \mathcal{V}_Y(X)$ for each $\alpha < \kappa$. In other words, $\operatorname{adh}_Y V_\alpha^c \cap X = \varnothing$ and $\operatorname{adh}_Y \mathcal{H}_\alpha = \varnothing$ for every $\alpha < \kappa$. Hence $\operatorname{adh}_Y(\mathcal{H}_\alpha \wedge V_\alpha^c) \cap X = \varnothing$, and if we set $\mathcal{Z}_\alpha = \mathcal{V}_Y(\mathcal{H}_\alpha \wedge V_\alpha^c)$, then $\operatorname{adh}_Y \mathcal{Z}_\alpha \cap X = \varnothing$ by the regularity of Y. The collection $\{\mathcal{Z}_\alpha : \alpha < \kappa\}$ is cocomplete on X. Indeed, let $\mathcal{U}$ be an ultrafilter on X that does not mesh with $\mathcal{Z}_\alpha$ for each $\alpha < \kappa$. In particular, $\mathcal{U}$ does not mesh with $\mathcal{H}_\alpha$ for $\alpha < \kappa$. Because $\{\mathcal{H}_\alpha : \alpha < \kappa\}$ is cocomplete, $\lim_Y \mathcal{U} \neq \varnothing$. On the other hand, $\mathcal{U}$

does not mesh with $\mathcal{V}_Y(V_\alpha^c)$ for $\alpha < \kappa$, that is, for every $\alpha < \kappa$ there is $U_\alpha \in \mathcal{U}$ such that $\mathrm{adh}_Y U_\alpha \subset V_\alpha$. We can conclude that

$$y \in \lim_Y \mathcal{U} \subset \bigcap_{\alpha<\kappa} \mathrm{adh}_Y U_\alpha \subset \bigcap_{\alpha<\kappa} V_\alpha = X,$$

that is, $y \in \mathrm{adh}_X \mathcal{U}$, which proves that X is κ-complete. ∎

It follows from the two theorems above that in Hausdorff regular weakly diagonal κ-complete convergence spaces, each G_κ set is openly κ-complete. The two theorems of this section imply in case $\kappa = \aleph_0$ the Arhangel'skii-Frolík theorem [5].

Corollary 8.4. *A space is Čech-complete if and only if it is a G_δ subspace of its every compactification.*

Theorem 8.3 slightly improves [6, Theorem 2.4] of Z. Frolík who proved that every openly κ-complete Hausdorff topological space is a G_κ-subset of every Hausdorff extension (every κ-complete collection is openly κ-complete).

Theorem 8.3 generalizes, from Hausdorff topologies to arbitrary convergences, [6, Theorem 2.5], which says that every openly G_κ-subset of a regular openly κ-complete topology is openly κ-complete. In fact in case of regular topologies open completeness and completeness coincide by Theorem 6.5.

Frolík proved that the assumption of regularity in Theorem 8.3 must not be dropped. Actually he showed that neither a closed nor an open subset of a G_κ-space need be a G_κ-space (a Hausdorff topological space a *G_κ-space* if it is a G_κ-subset of every Hausdorff topology in which it is densely homeomorphically embedded).

By Theorem 8.1, every κ-complete Hausdorff topology is a G_κ-space. It is straightforward that a closed subset of a κ-complete (not necessarily Hausdorff) convergence space is κ-complete.

9. Completion

Every convergence can be easily completed, that is, for a given cardinal κ, it can be *extended* to (that is, densely isomorphically embedded in) a κ-complete convergence. In particular, every convergence can be extended to a locally compactoid convergence. The existence of an extension that fulfils some separation axioms, like Hausdorffness or regularity, is another, often tougher, problem. Moreover, in general an extension of a topological space need not be topological.

If X is a convergence space and $\mathbb{F}$ is a collection of non-adherent filters on X, then we define on the disjoint union $X \cup \mathbb{F}$ a convergence, called the *simple extension* of X *over* $\mathbb{F}$ and denoted by $X \wedge \mathbb{F}$. An element $\mathcal{F}$ of $\mathbb{F}$ can be considered either as a filter on X or as an element of $X \cup \mathbb{F}$. Therefore, in order to distinguish these two usages, I shall write $\mathcal{F}^\flat$ in the second case. The convergence $X \wedge \mathbb{F}$ is defined as follows: if $x \in X$, then $x \in \lim_{X\wedge\mathbb{F}} \mathcal{G}$ if and only if $x \in \lim_X \mathcal{G}$, and if $\mathcal{F} \in \mathbb{F}$ and $\mathcal{G}$ is a filter on X, then $\mathcal{F}^\flat \in \lim_{X\wedge\mathbb{F}} \mathcal{G}$ whenever $\mathcal{F} \leq \mathcal{G}$. Notice that $\mathbb{F}$ is a discrete closed subset of $X \cup \mathbb{F}$, because the only convergent filters on $\mathbb{F}$ are principal ultrafilters, and that X is dense in $X \cup \mathbb{F}$.

It is straightforward that the simple extension of a pseudotopology (respectively, a pretopology) over any collection of non-adherent filters is a pseudotopology (respectively, a pretopology). However, the simple extension of a topology over $\mathbb{F}$ is a topology if and only if the elements of $\mathbb{F}$ are openly based filters. The simple extension of a convergence over $\mathbb{F}$ is Hausdorff if and only if the convergence is Hausdorff and $\mathbb{F}$ is a collection of non-adherent pairwise disjoint filters (that is, if $\mathcal{F}_0, \mathcal{F}_1 \in \mathbb{F}$ then there exist $F_0 \in \mathcal{F}_0$ and $F_1 \in \mathcal{F}_1$ such that $F_0 \cap F_1 = \varnothing$). Simple extensions are typically non-regular; the simple extension of a regular convergence space X over $\mathbb{F}$ is regular whenever every element $\mathcal{F}$ of $\mathbb{F}$ is regular (that is, $\mathcal{F} = \operatorname{adh}^{\natural} \mathcal{F}$) and for every filter $\mathcal{G}$ on $\mathbb{F}$ the contour $\mathcal{F}(\mathcal{G})$ is non adherent and disjoint from each $\mathcal{F} \in \mathbb{F}$ (where $\mathcal{F}(\cdot)$ associates with $\mathcal{F}^{\flat}$ the filter $\mathcal{F}$ for each $\mathcal{F} \in \mathbb{F}$). Regular extensions will be investigated in a future paper.

Theorem 9.1. *If $\lambda > \kappa > 0$ and X is a λ-complete convergence space, then there exists a κ-complete convergence space Y such that X is a dense subconvergence of Y.*

Proof. Let $\mathbb{F}$ be a complete collection of non-adherent filters on X of cardinality λ. Choose a subcollection $\mathbb{F}_0$ of $\mathbb{F}$ of cardinality κ, and let Y be the simple extension of X over $\mathbb{F} \backslash \mathbb{F}_0$. We infer from this definition that the only convergent filters on $\mathbb{F} \backslash \mathbb{F}_0$ are the principal ultrafilters (and they converge to their defining points). The constructed convergence is $(1+\kappa)$-complete, because the collection $\mathbb{F}_0 \cup \{\mathcal{Y}\}$, where $\mathcal{Y}$ is the cofinite filter of $\mathbb{F} \backslash \mathbb{F}_0$ is complete. In fact, the elements of this collection are not adherent. On the other hand, if $\mathcal{W}$ is a (free) ultrafilter on Y that does not mesh with any element of $\mathbb{F}_0 \cup \{\mathcal{Y}\}$, then necessarily $X \in \mathcal{W}$, hence either $\mathcal{W}$ meshes with no element of $\mathbb{F} \backslash \mathbb{F}_0$, and thus is convergent by the completeness of $\mathbb{F}$ on X, or meshes with some $\mathcal{F} \in \mathbb{F} \backslash \mathbb{F}_0$, hence converges to $\mathcal{F}^{\flat}$. If κ is infinite, then $1+\kappa = \kappa$; if $0 < \kappa < \aleph_0$, then κ-, $(1+\kappa)$- and 1-completeness coincide. ∎

If in Theorem 9.1 $\kappa = 1$ and λ is infinite, then we can simplify the proof above by putting $\mathbb{F}_0 = \mathbb{F}$. The resulting extension is then 1-complete, hence locally compactoid (no separation axiom is required in the definition of local compactoidness). On the other hand, each locally compactoid convergence can be compactified (no separation axiom is involved) by declaring that the cocompactoid filter converges to a single added point (like in the Alexandroff compactification).

10. Duality

As we shall see, a pavement is a dual concept with respect to that of a cocomplete collection of non-adherent filters. A convergence space X is *κ-paved* if for every element x of X there exists a collection of filters $\mathbb{V}(x)$ of cardinality κ such that $x \in \lim \mathcal{V}$ for every $\mathcal{V} \in \mathbb{V}(x)$ and if an ultrafilter

$\mathcal{U}$ converges to x, then there exists $\mathcal{V} \in \mathbb{V}(x)$ such that $\mathcal{U} \geq \mathcal{V}$. Notice that a convergence is a *pretopology* if and only if its is 1-paved, because a pavement of X at $x \in X$ consists of a single filter $\mathcal{V}(x)$ if and only if $\mathcal{V}(x)$ is the coarsest filter that converges to x. The existence of such a filter at every point characterizes pretopologies among all convergences.

As usual, the evaluation map $\langle \cdot, \cdot \rangle : X \times Z^X \to Z$ is defined by $\langle x, f \rangle = f(x)$. If ξ and σ are convergences, respectively, on X and Z, then $C(\xi, \sigma)$ denotes the set of maps from X to Z that are continuous (from ξ to σ). The *σ-dual* $[\xi, \sigma]$ of ξ (also called the *continuous convergence*) is the coarsest convergence on for which the *evaluation* map (restricted to $X \times C(\xi, \sigma)$) is jointly continuous. In the case of the *Sierpiński topology* $\$ = \{\varnothing, \{1\}, \{0, 1\}\}$ on $\{0, 1\}$, the set $C(\xi, \$)$ can be identified with the set of ξ-closed subsets by $\langle x, A \rangle = 0$ if $x \in A$ and $\langle x, A \rangle = 1$ if $x \notin A$. Indeed, $\langle \cdot, A \rangle$ is continuous from ξ to $\$$ if and only if A is ξ-closed. It is known that the *Sierpiński dual* (that is, $\$$-dual) $[\xi, \$]$ is homeomorphic with the *upper Kuratowski convergence* (also called *hyperconvergence*) with respect to ξ.

I shall now resume those few facts from [4] that will be needed in the sequel. For a filter $\mathcal{G}$ on $C(\xi, \$)$ the *reduced filter* is defined by

$$\mathrm{r}(\mathcal{G}) \approx \{ \bigcup_{A \in G} A : G \in \mathcal{G} \}.$$

A reduced filter can be degenerate (in the case of the principal ultrafilter of the element $\varnothing$ of $C(\xi, \$)$). It is straightforward that

$$A_0 \in \lim_{[\xi,\$]} \mathcal{G} \Leftrightarrow \mathrm{adh}_\xi \, \mathrm{r}(\mathcal{G}) \subset A_0.$$

A subset G of $C(\xi, \$)$ is *saturated* if $\mathrm{cl}_\xi B = B \subset \bigcup_{A \in G} A$ implies $B \in G$. A filter on $C(\xi, \$)$ is *saturated* if it admits a base of saturated sets. If $H \subset X$ then $\mathrm{e}_\xi H = \{B = \mathrm{cl}_\xi B : B \subset H\}$. A filter $\mathcal{G}$ is saturated if and only there exists a filter $\mathcal{H}$ on X such that $\mathcal{G} = \mathrm{e}_\xi^\natural \mathcal{H}$.

The restriction $A\xi$ of $[[\xi, \$], \$]$ to X (considered as a subset of $C(C(\xi, \$), \$)$) is coarser than ξ and is called the *epitopologization of* ξ. Because $[A\xi, \$] = [\xi, \$]$ for every convergence ξ, in the study of hyperconvergences one can assume, without affecting the generality, that the underlying convergences are *epitopologies*, that is, equal to their epitopologizations. In what follows I shall use the fact that every epitopology is *star-regular*. To explain what this means, it is handy to introduce two topologies $\xi^\bullet$ and ξ^* associated with a convergence ξ. The formula $\mathrm{cl}_{\xi^\bullet} x = \mathrm{cl}_\xi \{x\}$ defines a relation $\{(x, w) \in X \times X : w \in \mathrm{cl}_{\xi^\bullet} x\}$. As usual, $\mathrm{cl}_{\xi^\bullet} A = \bigcup_{x \in A} \mathrm{cl}_{\xi^\bullet} x$ is the image of $A \subset X$ by the relation. It turns out that so defined closure is topological; the corresponding topology $\xi^\bullet$ (the *point topology* of ξ) has a peculiar property: every union of closed sets is closed. Therefore, the point-closed sets are the open sets of another topology, called the *star topology* (of ξ) that one denotes by ξ^*. It is clear that cl_{ξ^*} is the inverse relation of $\mathrm{cl}_{\xi^\bullet}$. If ξ is T_1 (that is, if all the singletons are closed for ξ) then the star and the point topologies of ξ become the discrete topology. A filter $\mathcal{F}$ on a convergence space τ is *regular* if $\mathrm{adh}_\tau^\natural \mathcal{F} = \mathcal{F}$; it is called *inherent* if $\mathrm{inh}_\tau^\natural \mathcal{F} = \mathcal{F}$. Of course, a filter is star-inherent if and only if it is point-regular, and vice versa. The map $\mathrm{e}_\xi^\natural$ is a bijection between point-regular and saturated filters, the inverse being r. A convergence is an epitopology if and only if it is a star-regular pseudotopology with closed limits.

Theorem 10.1. *An epitopology is open-hereditarily κ-complete if and only if its \$-dual is κ-paved.*

Proof. Let ξ be an epitopology on X and let A be a closed subset of X. If $\mathbb{F}$ is a cocomplete collection of non-adherent filters on $X\backslash A$, that is, $\mathrm{adh}_\xi \mathcal{F} \subset A$ for every $\mathcal{F} \in \mathbb{F}$, and if an ultrafilter $\mathcal{U}$ on $X\backslash A$ is such that $\lim_\xi \mathcal{U} = \mathrm{adh}_\xi \mathcal{U} \subset A$, then $\mathcal{U} \geq \mathcal{F}$ for some $\mathcal{F} \in \mathbb{F}$. As ξ is star-regular, and because the star topology and the point topology are complementary,

$$\mathrm{adh}_\xi \mathcal{G} = \mathrm{adh}_\xi \mathcal{V}_{\xi^*}(\mathcal{G}) = \mathrm{adh}_\xi(\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{G}) \tag{10.1}$$

for every filter $\mathcal{G}$. Therefore $\{\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F} : \mathcal{F} \in \mathbb{F}\}$ is a collection of filters (on X) the adherences of which are included in A and such that if an ultrafilter $\mathcal{U}$ on $X\backslash A$ has the limit included in A, then there is $\mathcal{F} \in \mathbb{F}$ such that

$$\mathcal{U} \geq \mathrm{cl}_{\xi^\bullet}^\natural \mathcal{U} \geq \mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F}. \tag{10.2}$$

I claim that the collection $\{\mathrm{e}_\xi^\natural(\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F} \wedge A) : \mathcal{F} \in \mathbb{F}\}$ is a pavement of $[\xi, \$]$ at A. Indeed, since r is the inverse of $\mathrm{e}_\xi^\natural$ restricted to point-regular filters, $\mathrm{r}\,\mathrm{e}_\xi^\natural(\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F}) = \mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F}$ and as $\mathrm{adh}_\xi(\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F}) \subset A$, we infer that $A \in \lim_{[\xi,\$]} \mathrm{e}_\xi^\natural(\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F} \wedge A)$ for every $\mathcal{F} \in \mathbb{F}$. On the other hand, if $\mathcal{W}$ is an ultrafilter on $C(\xi, \$)$ that converges to A in $[\xi, \$]$, then $\mathrm{adh}_\xi \mathrm{r}(\mathcal{W}) \subset A$, and there is an ultrafilter $\mathcal{U}$ on X such that $\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{U} = \mathrm{r}(\mathcal{W})$ so that, by (10.1), $\lim_\xi \mathcal{U} \subset A$. If $A \in \mathcal{U}$, then $\mathrm{e}_\xi A \in \mathrm{e}_\xi^\natural \mathrm{r}(\mathcal{W}) \leq \mathcal{W}$, and thus $\mathcal{W} \geq \mathrm{e}_\xi^\natural(\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F} \wedge A)$ for every $\mathcal{F} \in \mathbb{F}$; if $A \notin \mathcal{U}$, hence by the cocompleteness of $\{\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F} : \mathcal{F} \in \mathbb{F}\}$ in $X\backslash A$ there is $\mathcal{F} \in \mathbb{F}$ such that $\mathcal{U} \geq \mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F}$, that is, $\mathrm{r}(\mathcal{W}) \geq \mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F}$, thus $\mathcal{W} \geq \mathrm{e}_\xi^\natural(\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{F} \wedge A)$.

Conversely if A is a closed subset of X, and $\mathbb{G}$ is a pavement of $[\xi, \$]$ at A, then $\{\mathrm{r}(\mathcal{G}) \vee A^c : \mathcal{G} \in \mathbb{G}\}$ is a cocomplete collection of non-adherent filters on $X\backslash A$. Indeed, every $\mathcal{G}$ in $\mathbb{G}$ converges to A in $[\xi, \$]$, hence $\mathrm{adh}_\xi \mathrm{r}(\mathcal{G}) \subset A$. Let $\mathcal{U}$ be an ultrafilter on $X\backslash A$ such that $\lim_\xi \mathcal{U} = \mathrm{adh}_\xi \mathcal{U} \subset A$, hence by star-regularity, $\mathrm{adh}_\xi \mathrm{cl}_{\xi^\bullet}^\natural \mathcal{U} \subset A$. There exists a maximal point-regular filter $\mathcal{H}$ such that

$$\mathrm{cl}_{\xi^\bullet}^\natural \mathcal{U} \leq \mathcal{H} \leq \mathcal{U},$$

because the set of point-regular filters $\mathcal{H}$ fulfilling the condition above is non empty, and the supremum of every chain of such filters is point-regular. Clearly, $\mathrm{adh}_\xi \mathcal{H} \subset A$, hence $A \in \lim_{[\xi,\$]} \mathrm{e}_\xi^\natural(\mathcal{H})$ because $\mathrm{r}\,\mathrm{e}_\xi^\natural(\mathcal{H}) = \mathcal{H}$; moreover, $\mathrm{e}_\xi^\natural(\mathcal{H})$ is maximal within the saturated filters. If $\mathcal{W}$ is an ultrafilter finer than $e^{\natural}_{\xi}(\mathcal{H})$, then $A \in \lim_{[\xi,\$]} \mathcal{W}$ and because $\mathbb{G}$ is a pavement at A, there is $\mathcal{G} \in \mathbb{G}$ such that $\mathcal{W} \geq \mathcal{G}$. Then $e^{\natural}_{\xi}(\mathcal{H}) = e^{\natural}_{\xi} \mathrm{r}(\mathcal{W}) \geq e^{\natural}_{\xi} \mathrm{r}(\mathcal{G})$ and thus $\mathcal{H} = \mathrm{r}\, e^{\natural}_{\xi}(\mathcal{H}) \geq \mathrm{r}\, e^{\natural}_{\xi} \mathrm{r}(\mathcal{G}) = \mathrm{r}(\mathcal{G})$. ∎

Hofmann and Lawson proved in [7] that the upper Kuratowski convergence of a topology is topological if and only if the underlying topology is core-compact. Mynard and the present author extended in [3],[4, Theorem 16.4] this characterization to general convergence spaces (if the underlying convergence is topological, and the upper Kuratowski convergence is pretopological, then it is topological). A convergence is called *topologically core-compact* if for every open set O and for every filter $\mathcal{F}$ that converges to an element of O, there exists $F \in \mathcal{F}$ which is compactoid in O, that is, such that every ultrafilter on F converges to an element of O. It is clear that a convergence is topologically core-compact if and only if it is open-hereditarily 1-complete. Hence, we recover

Corollary 10.2. [4, Theorem 16.4] *An epitopology is topologically core-compact if and only if its Sierpiński dual is a pretopology.*

The result above is restricted to epitopologies in order to simplify the formulation; in full generality, the first condition concerns the epitopologization of an arbitrary convergence.

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