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THE CLARKE'S TANGENT CONE AND LIMITS OF TANGENT CONES

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Abstract : It is shown that the Clarke's strict tangent cone at a point a *to a subset* X *of a Banach space* E *contains the limit inferior of the contingent cones at neighbouring points* $x \in X$ *as* $x \to a$, $x \in X$ *. The reverse inclusion is shown to be false by means of a counterexample.*

In this paper we study the relationships between the strict tangent cone (or Clarke's tangent cone) $T_X(a)$ at $a \in X$ to a closed subset X of a Banach space E and the limit inferior of the contingent cones (or usual tangent cones) $K_X(x)$ to X at neighbouring points $x \in X$ as $x \to a$, $x \in X$. It is a direct sequel of [10] in which the equality

$$(1) \qquad T_X(a) = \liminf_{\substack{x \to a \\ x \in A}} K_X(x)$$

was shown for E finite dimensional ([10], corol.3.4 and 3.5) along with some extensions in the infinite dimensional case under some restrictions on E or X. Here we provide an example showing that (1) is not true in the infinite dimensional case, thus contradicting a statement in [9]. However we show that the inclusion

$$(2) \qquad T_X(a) \supset \liminf_{\substack{x \to a \\ x \in A}} K_X(x)$$

is always valid ; this was shown in [11] with a completely different proof. This can also be shown using the method of proof of [9] provided that E is reflexive and X is weakly closed. The present proof relies on the drop theorem [7] which was qualified by J. Danes a "useful theorem in nonlinear analysis", a statement we are glad to verify. The drop theorem can be shown to be an easy consequence of the Brézis-Browder's general ordering principle [2] either through the use of nonlinear semigroups as in [2] or more directly via a geometric inequality dealing with drops as shown in the appendix. It is also connected with the problem of normal solvability ([3], [4], [13] for instance) whose links with the use of tangent cones are pointed out in [8]. Thus we get a striking circle of results in nonlinear analysis.

Let us observe that inclusion (2) does not seem to be very promising for generalized subdifferential calculus. There, one is lead to look for a concept of normal cone as small as possible in order that the necessary conditions written for instance under the form

$$(3) \qquad (-\partial f(a)) \cap N_X(a) \neq \emptyset ,$$

where $N_X(a) = T_X(a)^o$ is the negative polar cone of $T_X(a)$, should not be too trivial and useless. In view of inclusion (2), replacing $N_X(a)$ by $(\liminf_{x\to a,\, x\in X} K_X(a))^o$ would make things worse since this latter cone is still larger than $T_X(a)^o = N_X(a)$, which itself may be too big to offer significative results discriminating for instance local minima from local maxima.

§.1 - A GENERAL INCLUSION

In the sequel we adopt the notations and the terminology of [14]. We denote by $B(a,r)$ the closed ball with center a and radius r in the Banach space E and by $\mathcal{N}(a)$ the family of neighbourhoods of a. Let X be a closed subset of E.

Definition 1

The contingent cone (or usual tangent cone) to X at $a \in X$ is the set $K_X(a) = \limsup_{t\to 0_+} t^{-1}(X-a)$:

$$v \in K_X(a) \text{ iff } \forall V \in \mathcal{N}(v) \ \forall \varepsilon > 0 \quad (a+]0,\varepsilon[\, V) \cap X \neq \emptyset .$$

The Clarke's tangent cone (or strict tangent cone) to X at $a \in X$ is the set $T_X(a) = \liminf_{x\to a,\, x\in X,\ t\to 0_+} t^{-1}(X-a)$:

$$v \in T_X(a) \text{ iff } \forall V \in \mathcal{N}(v) \quad \exists N \in \mathcal{N}(a), \exists \lambda > 0 \text{ such that}$$

$$(x + tV) \cap X \neq \emptyset \quad \forall x \in X \cap N , \quad \forall t \in]0,\lambda[.$$

Let us recall that for a multifunction $F : Z \rightrightarrows E$ from a subset Z of a topological space W into E and for a point w in the closure of Z one has

$$\liminf_{z\to w\,,\; z\in Z} F(z) = \bigcap_{\varepsilon>0} \bigcup_{N\in\mathcal{N}(x)} \bigcap_{z\in Z\cap N} [F(z) + B(0,\varepsilon)]$$

while

$$\limsup_{z\to w\,,\; z\in Z} F(z) = \bigcap_{\varepsilon>0} \bigcap_{N\in\mathcal{N}(w)} \bigcup_{z\in Z\cap N} [F(z) + B(0,\varepsilon)] \quad .$$

The following result shows that the Clarke's tangent cone always contains the "stable part" of the neighbourhing contingent cones.

Theorem 1

For any closed subset X of a Banach space E and any $a \in X$ one has

$$\liminf_{\substack{x\to a\\ x\in X}} K_X(x) \subset T_X(a) \quad .$$

The proof of this result relies on the drop theorem [7] we recall now after introducing a notation. Given a convex subset B of E and $y \in E$, the drop with vertex y and body B is $D(y,B) = y + [0,1](B-y)$, the convex hull of $\{y\}$ and B.

Proposition 1 (the drop theorem [7]), [15], [16] and appendix)

Let X be a closed subset of a Banach space E, and let B be a closed ball in E with $B \cap X = \emptyset$. Then for any $y \in X$ there exists $x \in X \cap D(y,B)$ with

$$D(x,B) \cap X = \{x\} \quad .$$

We shall use the following simple geometric fact about drops.

Lemma 1

For any $x \in D(y,B) \setminus B$, where B is an arbitrary convex set, there exists $s > 0$ such that $s(B-y) \subset B-x$, hence $B-y \subset (0,+\infty)(B-x)$.

<u>Proof</u> : As $x \in D(y,B)$ we can find $s \in [0,1]$ and $z \in B$ such that $x = sy + (1-s)z$. Since $x \notin B$ we have $s \neq 0$. Then, for each $b \in B$ we have

$$s(b-y) = sb + (1-s)z - x \in B - x$$

as B is convex. The last assertion follows immediately. □

Lemma 2

Let $v \in E \setminus T_X(a)$. Then there exists $V \in \mathcal{N}(v)$ such that for each $N \in \mathcal{N}(a)$ there exists $t > 0$ and $x,y \in X \cap N$ such that $x \notin y + tV$ and $D(x,y+tV) \cap X = \{x\}$.

<u>Proof</u> : By definition, the assumption $v \notin T_X(a)$ can be translated as :

$$\exists V \in \mathcal{N}(v) \text{ such that } \forall U \in \mathcal{N}(a) \;\; \forall \lambda > 0 \;\; \exists y \in U \cap X \, , \, \exists t \in (0,\lambda) :$$
$$(y + tV) \cap X = \emptyset \;\; .$$

Without loss of generality we may suppose V is a closed ball with center v and radius $r > 0$. Given $N \in \mathcal{N}(a)$ we choose $\delta > 0$ with $B(a,2\delta) \subset N$ and we take $U = B(a,\delta)$, $\lambda = \delta(\|v\| + r)^{-1}$ above. Then, setting $B=y+tV$, we have $D(y,B) \subset N$ since for each $d \in D(y,B)$ we have

$$\|a-d\| \leq \|a-y\| + t\,(\|v\| + r) \leq 2\delta \;\; .$$

Then the drop theorem gives $x \in D(y,B) \cap X$ with $D(x,B) \cap X = \{x\}$; as $B \cap X = \emptyset$ we can not have $x \in B$. □

<u>Proof of the theorem</u> Keeping the notations of lemma 2 and using lemma 1 we have

$$V = t^{-1}(B-y) \subset t^{-1}s^{-1}(B-x)$$

hence $[0,st]V \subset [0,1]\;(B-x)$.

Using lemma 2 we see that $x + (0,1]\;(B-x) \subset D(x,B) \setminus \{x\}$ does not intersect X , so that

$$(x +]0,st]V) \cap X = \emptyset$$

and we get $\text{int } V \subset E \setminus K_X(x)$. As N is arbitrary we can not have $v \in \liminf_{x \to a,\, x \in X} K_X(x)$. □

Corollary 1

Let $v : X \to E$ be a continuous mapping (continuous vector field in the classical terminology). Then the following assertions are equivalent :

(a) for each $x \in X$ $v(x)$ belongs to $K_X(x)$;

(b) for each $x \in X$ $v(x)$ belongs to $T_X(x)$.

Proof : Clearly (b) implies (a) as $T_X(x) \subset K_X(x)$. Now if (a) is satisfied, for each $a \in X$ we have :

$$v(a) = \lim_{\substack{x \to a \\ x \in X}} v(x) \in \liminf_{\substack{x \to a \\ x \in X}} K_X(x) \subset T_X(a)$$

and (b) is satisfied. □

We just used the fact that for a multifunction $F : X \rightrightarrows E$ one has $u \in \liminf_{\substack{x \to a \\ x \in X}} F(x)$ iff there exists a mapping $v : X \to E$ continuous at a with $v(a) = u$ and such that $v(x) \in F(x)$ for each x in some neighborhood of a in X . This fact and the preceding corollary may suggest that $\liminf_{\substack{x \to a \\ x \in X}} T_X(x)$ and $\liminf_{\substack{x \to a \\ x \in X}} K_X(x)$ coincide. This is not true, even in the finite dimensional case as the following example taken from [14] shows.

Exemple 1 : Let $E = \mathbb{R}^3$, $X = \{x \in E : (x_3 - x_1 x_2)(x_3 + x_1 x_2) = 0\}$, $a = 0$. Then, as E is finite dimensional ,

$$\liminf_{\substack{x \to 0 \\ x \in X}} K_X(x) = T_X(0) = \mathbb{R}^2 \times \{0\}.$$

For $x = (x_1, 0, 0)$ with $x_1 > 0$ one has $T_X(x) = \mathbb{R} \times \{0\} \times \{0\}$ and for $x = (0, x_2, 0)$ with $x_2 > 0$ one has $T_X(x) = \{0\} \times \mathbb{R} \times \{0\}$. Thus

$$\liminf_{\substack{x \to 0 \\ x \in X}} T_X(0) = \{0\} . \quad \square$$

§.2 - COUNTEREXAMPLE.

Our aim now consists in showing that the inclusion

$$\liminf_{\substack{x \to a \\ x \in X}} K_X(x) \subset T_X(a)$$

may be strict. Let us observe that for finding a counterexample the requirement that X is a closed subset of E can be dropped. In fact if we have an arbitrary subset X of E, a point a in the closure C of X and a vector $v \in T_X(a) \setminus \liminf_{\substack{x \to a \\ x \in X}} K_X(x)$, then we also have $v \in T_C(a)$ as $T_C(a) = T_X(a)$ and $v \notin \liminf_{\substack{x \to 0 \\ x \in C}} K_C(x)$ as $K_C(x) = K_X(x)$ for $x \in C$ and as the limit over $x \in C$ is smaller than the limit over $x \in X$. This simple observation will dispense us with tedious inspections.

Let E be a separable Hilbert space with orthonormal basis $(e_n)_{n \geq 0}$. Let N be the set of positive integers and let S be the set of non null sequences $s = (s_n)_{n \in N}$ with values 0 or 1 and with finitely many non null terms. We denote by $m(s)$ (resp. $p(s)$) the rank of the first (resp. the last) non null term of s and we set $x(s) = y(s)e_o + z(s)$ with

$$y(s) = \sum_{n \in N} 3^{-n} s_n$$

$$z(s) = \sum_{n \in N} 3^{-m(s)} 3^{-n} s_n e_n \quad .$$

Let $X = \{x(s) : s \in S\}$ and let us show that $K_X(x) = \{0\}$ for each $x = x(s)$ in X. It suffices to prove that $K_X(x)$ has no element with norm one, or that we are lead to a contradiction if we suppose we can find a sequence $(s^k)_{k \in N}$ in S such that $(x^k) := (x(s^k))$ converges to x while $(r_k^{-1}(x^k - x))$ has a limit, where $r_k = \|x^k - x\|$. As $y(s) \in (0,1)$ we can find $h \in N$ with

$$3^{-h} \leq y(s) < 3^{-h+1} \quad ,$$

so that, for k large enough we have

$$\frac{3}{2}^{-h} < y(s^k) < 3^{-h+1} \quad ,$$

and as $y(s^k) \in [3^{-m(s^k)} \;,\; \frac{3}{2}\; 3^{-m(s^k)})$ we get

$$h - 1 < m(s^k) < h + 1$$

or $m(s^k) = h$ for k large enough.

Now, for each $n \in N$ we have $\lim_{k\to+\infty} x_n^k = x_n$, hence $\lim_{k\to+\infty} 3^{-h} s_n^k = 3^{-m} s_n$, with $m = m(s)$, so that $m = h$ and $s_n^k = s_n$ for $k \geq k_n$ large enough. Let us write $\lim_{k\to+\infty} r_k^{-1}(x^k - x) = te_o + w$ with $t \in \mathbb{R}_+$, $w \perp e_o$. Then, for each $n \in N$, we have $(w|e_n) = \lim_{k\to+\infty} r_k^{-1}(x^k - x|e_n) = 0$, hence $w = 0$. As $\| r_k^{-1}(x^k - x) \| = 1$, we have $t = 1$ or $\lim_{k\to+\infty} r_k^{-1}(y(s^k) - y(s)) = 1$ so that

$$\lim_{k\to\infty} (y(s^k) - y(s))^{-1} \| z(s^k) - z(s) \| = 0 \quad .$$

However, if $n(k)$ is the rank of the first non null term of $s^k - s$ we have

$$\| z(s^k) - z(s) \| \geq 3^{-m} 3^{-n(k)}$$

$$|y(s^k) - y(s)| \leq \sum_{j \geq n(k)} 3^{-j} = \frac{3}{2}\, 3^{-n(k)} \qquad ,$$

a contradiction with our assertion about the limit of the quotient. Thus $K_X(x) = \{0\}$.

Let us show now that $e_o \in T_X(0)$ by showing that

$$\lim_{\substack{x\to 0\\ x\in X}} \; \liminf_{t\to 0_+} \; t^{-1} d_X(x + te_o) = 0$$

and using [11] theorem 1. It suffices to prove that for each $\varepsilon > 0$ we have

$$\liminf_{t\to 0_+} \; t^{-1} d_X(x + te_o) \leq \varepsilon$$

provided that $x \in X$ satisfies $\| x \| \leq \varepsilon$. Now if $x := x(s)$ and $n > p(s)$ we have

$$(1/3^{-n}) d_X(x + 3^{-n} e_o) \leq 3^n \| (x + 3^{-n} e_o) - (x + 3^{-n} e_o + 3^{-m(s)} 3^{-n} e_n) \|$$

$$\leq 3^{-m(s)} \leq \| x \| \leq \varepsilon$$

and our assertion is proved.

<u>Remark</u> : The same example shows that the dual inclusion (in which P° is the negative polar of the cone P)

$$\limsup_{\substack{x \to a \\ x \in X}} K_X(x)^\circ \subset N_X(a) := (T_X(a))^\circ$$

is not true. In fact, taking s^k to be the sequence with $m(s^k) = k = p(s^k)$ and $x^k = x(s^k)$, we see that $(x^k) \to 0$, $K_X(x^k)^\circ = E$ for each k while $N_X(0) \subset (\mathbb{R}_+ e_o)^\circ$ so that $e_o \notin N_X(0)$, $e_o \in \limsup_{\substack{x \to 0 \\ x \in X}} K_X(x)^\circ$. □

APPENDIX : A PROOF OF THE DROP THEOREM

Let us derive the drop theorem from the following general ordering principle and a geometric inequality given in lemma A_2.

Proposition A_1 ([2])

Let X be a Hausdorff topological space endowed with an order relation $\geq$ such that for each $x \in X$ the set $A(x) = \{y \in X : y \geq x\}$ is closed and such that any increasing sequence has a limit. Suppose there exists a strictly decreasing map $f : X \to \mathbb{R}_+$ (i.e. $y \geq x$, $y \neq x \Longrightarrow f(y) < f(x)$). Then, for each $y \in X$ there exists $x \in A(y)$ such that x is maximal in $(X, \geq)$.

Given B and X as in proposition 1 , we defined an order relation $\geq$ on X by

$$(y \geq x) \Longleftrightarrow (y \in D(x,B)) \Longleftrightarrow (D(y,B) \subset D(x,B))$$

Then any maximal element x in $(X, \geq)$ is such that $D(x,B) \cap X = \{x\}$. Moreover $A(y) = D(y,B) \cap X$, a closed subset of X .

Lemma A_2 [16]

If the ball B has radius r and center c then for each x,y in X with $y \geq x$ one has

$$\|x-y\| \leq (\|x-c\| - \|y-c\|)(\|x-c\| + r)(\|x-c\|-r)^{-1}$$

Proof : As $y \in D(x,B)$ there exists $t \in [0,1]$ and $b \in B$ with $y = tb + (1-t)x$. Then

$$\|y-x\| = t\|b-x\| \leq t(\|b-c\| + \|c-x\|) \leq t(\|x-c\| + r)$$

and

$$\|y-c\| \leq t\|b-c\| + (1-t)\|x-c\| \leq tr + (1-t)\|x-c\|$$

or, equivalently,

$$t(\|x-c\| - r) \leq \|x-c\| - \|y-c\| .$$

Plugging this estimate of t into our first inequality gives the result. □

The preceding inequality shows that if $y \geq x$ and $y \neq x$ we necessarily have $\|x-c\| - \|y-c\| > 0$, so that the mapping f given by

$f(x) = \|x-c\|$ is strictly decreasing. Finally, if (x_n) is an increasing sequence of X, then $(\|x_n - c\|)$ is decreasing and has limit $s = \inf_{n\geq 0} \|x_n - c\|$ in $\mathbb{R}_+$. Lemma A_1 shows that (x_n) is Cauchy, hence has a limit. □

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