AMAZING OBLIVION OF PEANO'S CONTRIBUTIONS TO MATHEMATICS

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Commemorating the 150th Birthday of Giuseppe Peano (1858-1932)

ABSTRACT. At the end of 19th century Peano discerned many mathematical concepts in a perfect form that remained such till today. The formal language of logic that he developed, enabled him to perceive mathematics with great precision and depth. Actually he built mathematics axiomatically based exclusively on logical and set-theoretic primitive terms and properties, which was a revolutionary turning point in the development of mathematics.

Ask a mathematician about Peano's achievements and you would probably hear about Peano's continuous curve that maps the unit interval onto a square and about Peano's axioms of natural numbers. One might have heard of Peano series and Peano remainder.

Fig. 1. Giuseppe Peano (1858-1932)

It is unlikely that he/she would mention *"Zermelo" axiom of choice, "Borel-Lebesgue" theorem, "Fréchet" derivative, "Bouligand" tangent cone, "Grönwall" inequality, "Banach" operator norm, "Kuratowski" upper and lower limits of sequences of sets, "Choquet" filter grill* or *"Mamikon" sweeping-tangent theorem*, in spite of the fact that Peano anticipated these notions or proved these theorems well before, and often in a more accomplished and general form than those who granted them their names.

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It is plausible that you won't be told that he was at the origin of many mathematical symbols (like $\in, \cup, \cap, \subset, \exists$, N, R, Q), of reduction of all mathematical objects and properties to sets, of axiomatic approach to Euclidean space (with vectors and scalar product), of the theory of linear systems of differential equations (with matrix exponential and resolvents), of modern necessary optimality conditions, of derivation of measures, of definition of surface area and of many others.

On the faculty at the University of Turin since 1880, Peano reached the summit of fame at the turn of the century, when he took part in the International Congress of Philosophy and the International Congress of Mathematicians in Paris in 1900. Bertrand Russell, who participated in this congress, reported that Peano was always more precise than anyone else in discussions and that he invariably got the better of any argument upon which he embarked.

We should have in mind that at the time when Peano appeared on the mathematical scene, mathematical discourse was in general vague and approximative ([1]). Although Cauchy was praised for having given solid bases to mathematics, he was not exempt from errors that would be today qualified as elementary. Nor had new rigor of Weierstraß stopped vagueness and imprecision.

Peano's rigorous refoundation of mathematics was not in the main streams of mathematical activity and the simplicity and ease, with which Peano grasped the essence of things, contrasted with commonly involved and tedious pace. His achievements were often unnoticed, because they were quite in advance for his epoch. In a letter to Camille Jordan of 6th November 1894, Peano writes

> [...] I have been teaching at the university for fourteen years and I am not yet appointed full professor, contrary to others who are younger by age and by seniority; because here my work is little known and not much appreciated ([2]).

We shall try to recall what is more or less forgotten about the importance of this great scientist. In doing so, we will exploit many facts gathered in the papers commemorating the 150th anniversary of the birth of Peano by Dolecki, Greco [7], [8], Greco, Pagani [16], Greco, Mazzucchi, Pagani [14], [15] and Bigolin, Greco [1].

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[1] For example, historians of mathematics agree that the first rigorous proof that a function is constant provided that its derivative is null, was given by H. A. Schwarz in 1870. Much later, in 1946, J. H. Pearce, a reviewer of a textbook "The Theory of Functions of Real Variables" by L. M. Graves, stresses that the Rolle's Theorem has a *correct* proof, "a comparative rarity in books of this kind".

[2] Translation of an excerpt of a letter written in French: [...] il y a quatorze ans que je professe à l'Université, et je ne suis pas encore nommé ordinaire, à différence d'autres plus jeunes d'âge et d'enseignement ; car mes travaux sont ici peu connus et peu estimés.

Contents

1. Youthful achievements

Peano graduated in 1880 and became an assistant of Genocchi in 1881/82, starting soon to write down a calculus course that his master taught at that time. Peano was charged with exercises to that course, but he soon took it over entirely and continued till 1884, because Genocchi fell ill.

Peano utilized notes made by students at Genocchi's lessons and compared them point by point with all the principal calculus texts. Consequently he made many additions and some changes in these lessons, so that when Genocchi saw the result, that is, *Calcolo differenziale e integrale* [11, (1884)], he disclaimed his authorship, stating that "everything [in the book] was due to that outstanding young man Dr. Giuseppe Peano". In a celebrated *Encyclopädie der Mathematischen Wissenschaften* [53, (1899)],[60, (1899)] *Calcolo differenziale e integrale* and another Peano's book *Lezioni di analisi infinitesimale* [44, (1893)] are cited among most influential treaties of infinitesimal calculus together those of Euler (1748) and Cauchy (1821).

Making the mentioned comparisons with the existing literature, Peano realized that numerous mathematical definitions were flawed, many proofs were defective and multiple theorems had overabundant hypotheses. Critical analysis of those defects urged him to rectify them.

Most of major achievements of Peano were realized or prefigured before he turned thirty. They are collected in the already mentioned *Calcolo differenziale e integrale*, in *Applicazioni geometriche* [35, (1887)] and in *Calcolo geometrico* [36, (1888)]. Hence among youthful accomplishments we will mention those carried out a couple of years from graduation.

1.1. **Surface area.** In 1882 Peano at the age of 24 discovers that the definition of surface area, presented by Serret in his *Cours d'Analyse* [56, (1868)], is incorrect. According to Serret, the area of a surface should be given by the limit of the areas of inscribed polyhedral surfaces. Peano observes that in the case of cylindrical surface, it is possible to choose a sequence of inscribed polyhedral surfaces fulfilling Serret's condition, so that the sum of the areas converges to infinity. Here is the construction that was discovered independently by Schwarz and Peano. A Venetian lantern of height H (divided into n equal intervals) and of radius R (circles corresponding to the ends of the intervals are divide into m equal segments). The lantern is hence composed of $2mn$ triangles.

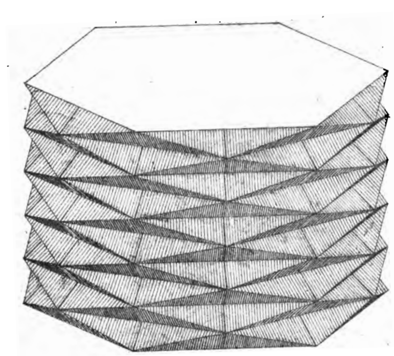

Fig. 2. A Venetian lantern taken from a course of C. Hermite [20, (1883) p. 36].

Divide the cylinder vertically into intervals of equal length $\frac{H}{n}$, on each of $n+1$ circles corresponding to the ends of these intervals take m equally distributed points, so that the points of each circle are midway of those of the circle above. Consider the triangles, the vertices of which are two consecutive points on one circle and midway point on the circle above or on the circle below. Then the area of the lantern is ([3])

$$V(n,m) := 2mnR\sin\frac{\pi}{m}\sqrt{4R^2\sin^4\frac{\pi}{2m}+\frac{H^2}{n^2}}. \tag{1}$$

The limit depends on the proportion of n and m, for example, if $n = m$, then $\lim_{m\to\infty} V(m,m) = 2\pi RH$, but if, for instance, $n = m^3$, then $\lim_{m\to\infty} V(m^3,m) = \infty$.

Moreover, Peano identifies a principal error of the Serret's method, that is, that a variable plane passing through three non-collinear points of a surface S, does not necessarily tend to the tangent plane of S at a point x, when these three points tend to x.

Genocchi moderates enthusiasm of the young mathematician, telling him that a similar counterexample has already been discovered by H. A. Schwarz two years earlier. But when Genocchi invites Schwarz to propose an alternative correct definition, Schwarz declines and stresses several difficulties.

In *Applicazioni geometriche* [35, (1887)] Peano overcomes all the difficulties of defining that area and gives the following algorithm: fix a plane L and for an arbitrary finite partition of the surface S, move arbitrarily but rigidly each element of the partition and project it orthogonally on L. Then take the sum of so obtained plane areas. This sum depends on the partition and on the positions of its elements after the transport. The supremum of so obtained sums over all the partitions and all the positions defines the area of S.

This definition does not lead to contradictions as that of Serret. In fact it coincides with the Lagrange formula for the area in case of Cartesian surfaces, that is, given by C^1-functions f,

$$\iint_D \sqrt{1+\|\nabla f(x,y)\|^2}\,\mathrm{d}x\,\mathrm{d}y. \tag{2}$$

1.2. **Concept of plane measure.** In [31, (1883)] Peano is the first to prove that a positive function f of one variable is *integrable* if and only if the positive hypograph of f is *measurable*. If this is the case, Peano shows that the integral of f is equal to the area of the positive hypograph of f.

In the same paper, he presents concepts of *external* and *internal area* and, what is most considerable, that of *measurability* ([4]) for planar sets, ten years before the work of Jordan [25, (1893)]. In introducing the inner and

[3]The area of the lateral surface of the cylinder of height H and radius R, is $2\pi RH$. The length of the bases of these isosceles triangles is

$$2R\sin\frac{\pi}{m}$$

and the altitude is

$$\sqrt{R^2(1-\cos\frac{\pi}{m})^2+\frac{H^2}{n^2}}.$$

As $1-\cos\frac{\pi}{m} = 2\sin^2\frac{\pi}{2m}$, the area of the polygon formed by these triangles is equal to (1).

[4]Peano does not use the term *measurability.*

outer area of planar sets as well as in defining surface area, Peano is also influenced by Archimedes's approach on calculus of area, length and volume of convex figures.

Till then the concept of measure was commonly used, but was not defined. Only later appears a concept of *Inhalt* (content) in the works by Stolz [57, (1884)], Cantor [3, (1884)], Harnack [18, (1885)], corresponding to external measure.

Peano considers finite unions of polygons that cover a given planar set A and finite unions of polygons that are included in A. Denote by $\mathbb{P}$ the collection of finite families of polygons. The infimum over $\mathcal{P} \in \mathbb{P}$ of

$$\sum\nolimits_{P\in\mathcal{P}} \operatorname{area}(P) \tag{3}$$

such that $\bigcup_{P\in\mathcal{P}} P \supset A$, defines the *external area* of A, and the supremum over $\mathcal{P} \in \mathbb{P}$ of (3) such that $\bigcup_{P\in\mathcal{P}} P \subset A$, defines the *internal area* of A. If these two values coincide, A is said to be *measurable*, and the common value is called the *area* of A.

Mind that the polygons of a given family $\mathcal{P} \in \mathbb{P}$ considered in these definition can overlap! This is a deliberate choice of Peano, which enables one to immediately infer that the measure of isometrically invariant. The corresponding construction of Jordan, using grills of rectangles, requires a proof of such invariance.

1.3. **Dirichlet function.** Dirichlet was probably the first to conceive functions as arbitrary assignments which need not to be expressed analytically, that is, by algebraic operations, elementary functions and their limits. To show the extent of his new concept, he gave in 1829, as an example, the celebrated *Dirichlet function*, which is the characteristic function of irrational numbers $\chi_{\mathbb{R}\setminus\mathbb{Q}}$ ([5]).

In [11, (1884)] Peano shows that, surprisingly, the Dirichlet function is analytically expressible as double limit of rational functions

$$\chi_{\mathbb{R}\setminus\mathbb{Q}}(x) = \lim\nolimits_{m\to\infty} \varphi\left(\sin\left(m!\pi x\right)\right),$$

where

$$\varphi(x) := \lim\nolimits_{t\to 0} \frac{x^2}{x^2+t^2} = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Indeed, if x is rational, then $\sin(m!\pi x) = 0$, hence $\varphi(\sin(m!\pi x)) = 0$ for m greater than the denominator of the fraction of integers that represents x; if x is irrational, then $m!x$ is irrational and thus $\sin(m!\pi x) \neq 0$ so that $\varphi(\sin(m!\pi x)) = 1$ for each m.

Peano adopts Dirichlet's definition of function in [11, (1884)]; in [48, (1908)] he defines *functions* and, more generally, *realtions* as subsets of Cartesian product. In [49, (1911)], commenting freshly published *Principia Mathematica*, where relations are primary notions, Peano reiterates

[5] that assigns 0 to the rational numbers and 1 to the irrational numbers.

his preference to consider *set* as a primitive notion and defines functions as particular relations, as it is commonly done today.

1.4. **Dispute about the mean value theorem.** In nineteenth century *Nouvelles Annales de Mathématiques* published letters and short notes, offering a forum to mathematical community. In [32, (1884)] Peano observes that the proof of the *mean value theorem* given by Jordan in his *Cours d'Analyse* [23, (1882)] is faulty. Mind that, at that time, Peano was a young assistant, while Jordan was a famous professor almost twice as old as Peano.

Theorem 1 (mean value theorem)**.** *If a real function f is continuous in $[a, b]$ and differentiable in (a, b), then there is $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

It is impressive that an easy basic fact, which is nowadays taught in freshmen calculus courses, constituted a difficulty for a great mathematician like Jordan.

In his proof, Jordan divides the interval to subintervals $a = a_0 < a_1 < \ldots < a_{n-1} = b$, of diameter tending to 0, and claims that

$$\frac{f(a_r) - f(a_{r-1})}{a_r - a_{r-1}} - f'(a_{r-1}) \tag{4}$$

tend also to 0. Peano's example $f(x) := x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f[x] := 0$ for $x = 0$,

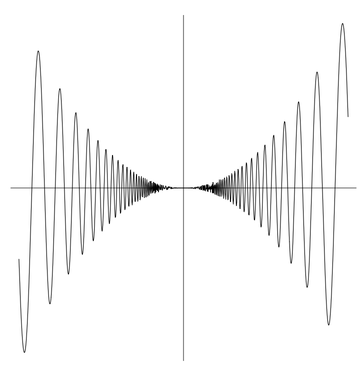

Fig. 3. $f(x) := x^2 \sin \frac{1}{x}$ if $x \neq 0$.

with appropriately chosen a_r and a_{r-1} tending to 0, shows that (4) does not hold in general.

Peano indicates that Jordan's claim is true under the continuous differentiability of f and adds that the mean value theorem can be easily proved without that assumption.

Jordan replies that Peano's objections are founded, and that, [Jordan] implicitly assumed that

$$\frac{f(x+h) - f(x)}{h} \to_h f'(x) \text{ uniformly as } h \text{ tends to } 0 \tag{5}$$

in the interval $[a, b]$, and asks Peano to furnish a proof of his claim of futility of continuity of derivative, as he does not know a satisfactory one ([6]). In [33, (1884)] Peano remarked that a correct proof of Theorem 1 was due to Bonnet and that property (5) amounts to the cotinuous differentiability of f.

This dispute originates Peano's study of *strict derivatives* of functions and measures (Sections 7 and 11).

1.5. **Some of Peano's counter-examples.** Peano encounters numerous inaccuracies and errors in mathematical literature and provides, with astonishing ease, a long list of counter-examples. He remains perhaps the champion of counter-examples in the mathematical world. Of course, it is natural that errors happen to (almost) everyone and papers of numerous great mathematicians contain, sometimes fecund, errors. Peano rigor was however quite exceptional; Bertrand Russell comments in *The Principles of Mathematics* that Peano had a rare immunity from error. We list below some of Peano's counter-examples from *Calcolo differenziale e integrale* of 1884, to sundry statements of Cauchy, Lagrange, Serret, Bertrand, Todhunter, Sturm, Hermite, Schlömilch and others.

A. *The order of partial derivation cannot be altered in general*: if

$$f(x,y) := \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, \text{ if } x^2+y^2>0, \\ 0, \text{ if } x=y=0, \end{cases}$$

then $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ ([7]).

B. *Existence of partial derivatives is not sufficient for the mean value theorem in two (and more) variables:* Peano shows that for

$$f(x,y) := \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, \text{ if } x^2+y^2>0, \\ 0, \text{ if } x=y=0, \end{cases}$$

[6]At this point, Ph. Gilbert of Louvain intervenes in the exchange, saying that the request of professor Jordan was done with archness, because the mean value theorem without the continuity of derivative is false. The example he proposes to support his claim is (of course!) wrong.

In his answer [33, (1884)], Peano gives the (today standard) proof of Bonnet, using the theorems of Weierstraß and Rolle, and mentions that satisfactory proofs can be found in the books of Serret, Dini, Harnack and Pasch.

[7]Peano mentions other, more complicated counter-examples, for instance, that of Dini and of Schwarz (1873) (quoted below):

$$f(x,y) := \begin{cases} x^2 \arctan \frac{y}{x} - y^2 \arctan \frac{x}{y}, \text{ if } xy \neq 0, \\ 0, \text{ if } xy = 0. \end{cases}$$

the mean value formula does not hold ([8]).

C. *On the formula of de l'Hôpital:* $f(0) = 0 = g(0)$ and $\lim_{x\to 0} \dfrac{f(x)}{g(x)}$ exists without the existence of $\lim_{x\to 0} \dfrac{f'(x)}{g'(x)}$. Peano proposes a counter-example, using f as defined in Figure 3 and $g(x) := x$. Here the derivatives exist, but f' is discontinuous at 0. In another example:

$$f(x) := x^2 \int_0^x \sin \frac{1}{t^4} \frac{dt}{t} \text{ and } g(x) := x^2,$$

f' is everywhere continuous, but $\lim_{x\to 0} \dfrac{f'(x)}{g'(x)}$ does not exist.

D. A function can attain, on each straight line passing through $(0,0)$, a local minimum at $(0,0)$, without attaining its local minimum at $(0,0)$ ([9]). Consider

$$f(x,y) := (y - x^2)(y - 2x^2).$$

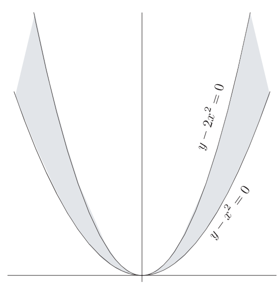

Fig. 4. Zero sub-level of f.

The value $f(x)$ is positive if $y \geq 2x^2$ and if $y \leq x^2$; it is negative if $x^2 \leq y \leq 2x^2$. Therefore $(0,0)$ is neither local minimum nor local maximum. Each straight line L passing through $(0,0)$ remains in the positivity area of f on an open interval around $(0,0)$, that is, f attains a local on L.

[8]Indeed, both partial derivatives are null at $(0,0)$, otherwise

$$f_x(x,y) = \frac{y^3}{(x^2+y^2)^{3/2}} \text{ and } f_y(x,y) = \frac{x^3}{(x^2+y^2)^{3/2}}.$$

For every real number t,

$$f(t,t) = \frac{|t|}{\sqrt{2}}, \quad f_x(t,t) = f_y(t,t) = \frac{(\mathrm{sgn}(t))^3}{2\sqrt{2}} \in \{-\frac{1}{2\sqrt{2}}, 0, \frac{1}{2\sqrt{2}}\}$$

For $x_0 := y_0 := -1$ and $h := k := 3$ and $\theta \in \mathbb{R}$,

$$f(x_0+h, y_0+k) - f(x_0,y_0) = \frac{|x_0+h| - |x_0|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

and hence the mean value formula does not hold.

[9]Peano constructs this counter-example of a related statement of Serret: if $df(x,y) = 0$, and $d^4 f(x,y)(h,k) > 0$ for each (h,k) such that $d^2 f(x,y)(h,k) = d^3 f(x,y)(h,k) = 0$, then (x,y) is a local minimum.

2. Logic and set theory

The formal language of logic that Peano developed, enabled him to perceive mathematics with great precision and depth. Actually he built mathematics axiomatically based exclusively on logical and set-theoretic primitive terms and properties, which was a revolutionary turning point in the development of mathematics. For Peano, logic is the common part of all theories.

It should be emphasized that the formal language conceived and used by Peano was not a kind of shorthand adapted for a mathematical discourse, but a collection of ideographic symbols and syntactic rules with unambigous set-semantics, which produced precise mathematical propositions, as well as inferential rules that ensure the correctness of arguments.

For Peano, semantics is inherent to syntax, a mathematical point of view as opposed to that of logicians.

$\cdot 2 \quad g,h\,\varepsilon\, \mathrm{qFq} \ .\supset\!:\ f\,\varepsilon\, \mathrm{qFq} \,.\, \mathrm{D}f = [(gx\times fx+hx)|x, \mathrm{q}] \ .=.$
$f = \{\mathrm{e}\!\upharpoonright\!\mathrm{S}(g;\,0,x)[f0+\mathrm{S}\{[\mathrm{e}\!\upharpoonright\!-\mathrm{S}(g;\,0,x)]\times hx|x;\,0,x\}]|x, \mathrm{q}\}$

{ Leibniz A.Erud. a.1694 Math.S. t.5 p.313 }

Dato duo functione g et h, æquatione, identitate in numero reale x, conditione in functione f:

$$\mathrm{D}fx = gx\times fx+hx$$

vocare in f « æquatione differentiale lineare ». Si gx es constante, et hx es nullo, resulta æquatione considerato in p.323.

Calculo de functione f: $\quad x\varepsilon\mathrm{q} \ .\supset_x .\ \mathrm{D}fx = gx\, fx - hx$
Transporta: $\quad x\varepsilon\mathrm{q} \ .\supset_x .\ \mathrm{D}fx - gx\, fx = hx$
Multiplica per $\mathrm{e}\!\upharpoonright\!-\mathrm{S}(g;0,x)$:
$x\varepsilon\mathrm{q} \ .\supset_x .\ \mathrm{D}[\mathrm{e}\!\upharpoonright\!-\mathrm{S}(g;0,x)\times fx|x,\mathrm{q}]x = \mathrm{e}\!\upharpoonright\!-\mathrm{S}(g;\,0,x)\times hx$
Integra ab 0 ad x:
$x\varepsilon\mathrm{q} \ .\supset_x .\ \mathrm{e}\!\upharpoonright\!-\mathrm{S}(g;0,x)\times fx - f0 = \mathrm{S}[\mathrm{e}\!\upharpoonright\!-\mathrm{S}(g;0,x)\times hx|x;0,x]$
Trahe fx:
$x\varepsilon\mathrm{q} \ .\supset_x .\ fx = \mathrm{e}\!\upharpoonright\!\mathrm{S}(g;0,x)\{f0+\mathrm{S}[\mathrm{e}\!\upharpoonright\!-\mathrm{S}(g,0,x)\times hx|x;0,x]\}$
Unde nos deduce $f =$ secundo membro, ubi varia x in campo de numeros reale.

Fig. 5. Formal statement and proof of Leibniz theorem from Peano's *Formulario Mathematico* [48, (1908) p. 431] (in latino sine flexione and Peano's symbolic language).

In the figure above the following Leibniz theorem from 1694 is stated and proved: *If g, h are functions from $\mathbb{R}$ to $\mathbb{R}$, then a function f from $\mathbb{R}$ to $\mathbb{R}$ verifies the equality $f' = g\,f + h$ if and only if*

$$f(x) = e^{\int_0^x g}\left(f(0) + \int_0^x e^{-\int_0^u g} h(u)\mathrm{d}u\right)$$

for every $x \in \mathbb{R}$.

2.1. **Reduction of mathematics to sets.** Till nineteenth century, there was a great variety of mathematical objects: numbers, lines, surfaces, figures, all considered as entities. The language of mathematics was constituted of a mixture of symbols and of common language. Because of semantic ambiguity of natural languages, mathematical facts expressed with their aid may not be univocal.

With Dedekind and Cantor sets become mathematical objects, while Peano reduces all the objects and properties to sets. Relations become subsets of Cartesian products, functions are particular relations and operations are expressed by means of functions. All this constitutes a conceptual revolution.

It follows that two objects x and y are equal if and only if $x \in X$ is equivalent to $y \in X$ for every set X. In this way, Peano implements the principle of Aristotle ([10]), Saint Thomas ([11]) and Leibniz ([12]); see [50, (1915)] and [50, (1916)].

Peano understands urgent necessity of inequivocal formal language to refound mathematics on solid bases. Starting from 1889, he formalizes a significative part of mathematics of his times.

Mathematical objects are accessible through symbols. Peano introduces symbols he needs for his formalism of mathematics. Among them

$$\in, \cup, \cap, \subset, \exists,$$

that today have become universal. He denotes the sets of natural numbers with N, of rational numbers with R (for *rational*) of real numbers with q (for *quantity*), of numerical finite-dimensional Euclidean space with q_n and so on. He forms all mathematical expressions using two primitive propositions $x \in y$ and $x = y$. Therefore he keeps the distinction between $\in$ and $\subset$, hence between an element x and the corresponding singleton $\{x\}$. A relevant subject of research activity of Peano and his School concerned *definitions* in Mathematics, a subject that received and till now receives more attention by philosophers than by mathematicians.

Peano uses formal expressions to announce mathematical facts and formal inferential transformations to prove them. Peano's symbolic propositions are not stenographic, but organic, with precise univocal semantic values. Thanks to this absolute precision of his formalism, Peano could easily detect errors and see necessity of hypotheses or axioms. For Peano, mathematical facts are precisely those that can be expressed in terms of set-theoretic and logical symbols; therefore in [47, (1906)] Peano rejects the paradox of Richard [55, (1905)], as pertaining to linguistics and not to mathematics.

2.2. **Axiom of choice.** Peano realized that the *principle of infinite arbitrary choices* was not guaranteed by the axioms traditionally used in mathematics, when he elaborated a proof of existence of solutions to systems of ordinary differential equations under the sole hypothesis of continuity

[10] Nam quaecumque de uno praedicatur, ea etiam de altero praedicari debent.

[11] Quaecumque sunt idem, ita se habent, quod quidquid praedicatur de uno, praedicatur et de alio.

[12] Eadem sunt quorum unum in alterius locum substitui potest, salva veritate.

[39, (1890)], which we will discuss later. Only the *principle of determined choices* was allowed. For his specific problem of selecting elements from closed bounded subsets of Euclidean space, he based his choice on the lexicographic order of Euclidean space.

After the rediscovery of the *axiom of choice* by Zermelo in [61, (1904)], the pertinence of this axiom was discussed by mathematical community, among whom Russell and Poincaré had their say, but Peano's contribution was forgotten. A promise of 1924 of Zariski to reestablish Peano's priority was not kept [8, p. 321].

3. Arithmetic

Peano proposes *six axioms* to define *natural numbers*. We resume them as follows: the primitive notions $\mathbb{N}_0$, 0 and an operation σ fulfill the following axioms:

P0. $\mathbb{N}_0$ is a set,
P1. $0 \in \mathbb{N}_0$,
P2. $\sigma(n) \in \mathbb{N}_0$ for every $n \in \mathbb{N}_0$,
P3. if S is a set, $0 \in S$ and $\sigma(S) \subset S$, then $\mathbb{N}_0 \subset S$,
P4. σ is injective,
P5. $\sigma(n) \neq 0$ for every $n \in \mathbb{N}_0$.

[. . .] nos sume tres idea N_0, 0, + ut idea primitivo, per que nos defini omni symbolo de Arithmetica.

Nos determina valore de symbolo non definito N_0, 0, + per systema de propositio primitivo sequente.

✻ 1. Pp

·0 $N_0 \ \varepsilon$ Cls

·1 $0 \ \varepsilon N_0$

·2 $a\varepsilon N_0 .\supset. a+ \ \varepsilon N_0$

·3 $s\varepsilon \mathrm{Cls} . 0\varepsilon s : a\varepsilon s .\supset_a . a+ \ \varepsilon s :\supset. N_0 \supset s$ Induct

·4 $a,b\varepsilon N_0 . a+ = b+ .\supset. a=b$

·5 $a\varepsilon N_0 .\supset. a+ -= 0$

Fig. 6. Here are Peano's six axioms of arithmetic as they appear in *Formulario Matematico* [48, (1908) p. 27] (Peano's comments are in *latino sine flexione*).

It follows from the axioms of Peano that $\mathbb{N}_0$ is infinite in the sense of Peirce and Dedekind, that is, there exists a map $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ that is injective but not surjective and that $\mathbb{N}_0$ is a minimal infinite set, because of the induction principle P3. In the introduction to *Arithmetices principia, nova methodo exposita* [38, (1889)] Peano writes ([13])

> Questions pertaining to the foundations of mathematics, although treated by many these days, still lack a satisfactory solution. The difficulty arises principally from the ambiguity of ordinary language. For this reason it is of the greatest concern to consider attentively the words we use. [...]
>
> I have indicated by signs all the ideas which occur in the fundamentals of arithmetic, so that every proposition is stated with just these signs. The signs pertain either to logic or to arithmetic.

Following *Lehrbuch der Arithmetic* of Grassmann [12, (1861)], Peano extends by induction the operation σ to those of addition and multiplication. He is then in a position to extend arithmetic to integers, rationals and reals.

Some claim that Peano is beholden to Dedekind for his foundation of arithmetic. This is however not the case, because Peano proceeds axiomatically, proving, by the way, the independence of his axioms, while Dedekind proves everything, even improvable, like the existence of infinite set. Peano uses a completely formal and coherent language, while Dedekind is often vague (he does not distinguish *membership* from *inclusion*).

4. Peano's filling curve

In 1914 Hausdorff wrote in *Grundzüge der Mengenlehre* [19] of Peano's filling curve that this is one of the most remarkable facts of set theory, the discovery of which we owe to G. Peano ([14]). Nowadays this fact is rather qualified as topological. We present it in a separate section preceding that of other topological achievements of Peano.

Invited by Felix Klein to publish in Mathematische Annalen, Peano sent a paper [40, (1890)], in which he proves the existence of a continuous map from the interval $[0,1]$ onto the square $[0,1] \times [0,1]$.

Fig. 7. The figure representing the second approximations of Peano's curve as it appears in Formulario Matematico [48, (1908) p. 240].

In order to construct such a map, he uses the ternary representation of each element t of $[0,1]$ and transforms it into ternary representations of

[13] Translation from Latin is taken from [27].

[14] *Das ist eine der merkwürdigsten Tatsachen der Mengenlehere, deren Entdeckung wir G. Peano verdanken.*

$x(t) \in [0,1]$ and $y(t) \in [0,1]$, that is, of an element of $[0,1] \times [0,1]$. Because the sought map need to be continuous, Peano's construction is necessarily more sophisticated than that of Cantor that established a bijection between $[0,1]$ and $[0,1] \times [0,1]$.

More precisely, Peano uses ternary representations of the elements $t = 0, a_1 a_2 \ldots$ of $[0,1]$ to define ternary representations of

$$x(t) = 0, b_1 b_2 \ldots \text{ and } y(t) = 0, c_1 c_2 \ldots,$$

in such a way that successive subdivisions of $[0,1]$ into $3^2, 9^2, 27^2, \ldots$ segments are mapped onto $3^2, 9^2, 27^2, \ldots$ squares that subdivide $[0,1] \times [0,1]$ so that if two segments are adjacent then the corresponding squares are also adjacent. This rule implies the continuity of the constructed curve.

Let us make this explicit for the first subdivision of $[0,1]$: the first two fractionary digits of $t = 0, a_1 a_2, \ldots$ correspond to the left end of an interval of this subdivision:

$0,00$	$0,01$	$0,02$	$0,10$	$0,11$	$0,12$	$0,20$	$0,21$	$0,22$
0	1	2	3	4	5	6	7	8

.

Accordingly the n-th interval is mapped onto the n-th square, as in the following table.

$0,2$	2	3	8
$0,1$	1	4	7
$0,0$	0	5	6
	$0,0$	$0,1$	$0,2$

Peano defines an involution $\boldsymbol{k} : \{0,1,2\} \to \{0,1,2\}$ by

$$\boldsymbol{k}(0) = 2,\ \boldsymbol{k}(1) = 1,\ \boldsymbol{k}(2) = 0.$$

Consequently, $\boldsymbol{k}^n(a) = a$ if n is even and $\boldsymbol{k}^n(a) = \boldsymbol{k}(a)$ if n is odd. Observe that the first subdivision above corresponds to

$$b_1 = a_1 \text{ and } c_1 = \boldsymbol{k}^{a_1}(a_2).$$

In full generality, the map is defined by

$$b_n = \boldsymbol{k}^{a_2 + a_4 + \cdots + a_{2n-2}}(a_{2n-1}) \text{ and } c_n = \boldsymbol{k}^{a_1 + a_3 + \cdots + a_{2n-1}}(a_{2n}).$$

The vertices of the three polygonal lines inscribed in the Peano's curve in the figure below, are calculated at $0, 1$ and, respectively, at

(first polygonal)	$0, a_1 a_2 \overline{1}$,
(second polygonal)	$0, a_1 a_2 a_3 a_4 \overline{1}$,
(third polygonal)	$0, a_1 a_2 a_3 a_4 a_5 a_6 \overline{1}$.

for $a_1, a_2, a_3, a_4, a_5, a_6 \in \{0, 1, 2\}$, where $\bar{1}$ stands for the periodic 1.

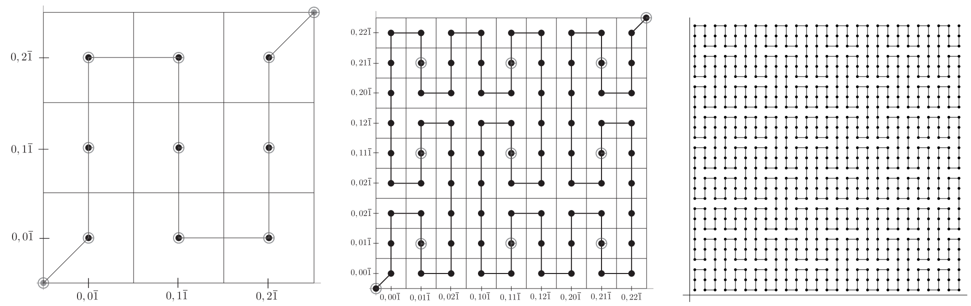

Fig. 8. The first three approximations of Peano's curve corresponding to first digits of the ternary representations.

Peano concedes himself that he conceived the filling curve as a counterexample to commonly diffused ideas of curve, for instance, that the area of a curve is null ([15]). He observes that his curve is (and even its components are) nowhere differentiable. This fact is obvious, because the image of each segment of the n-th subdivision of $[0, 1]$ by Peano's curve is equal to the corresponding square of the n-th subdivision of $[0, 1] \times [0, 1]$.

Peano's original construction was not illustrated by any figure. Solicited by Klein, David Hilbert published a note [21, (1891)] on Peano's curve (see the figure below), presenting a variant based on binary representations. He described a Cauchy sequence (for the uniform convergence) of curves, hence convergent to a continuous map, the image of which is dense by construction. On the other hand, it is also closed (hence surjective), as the image of a compact set by continuous map.

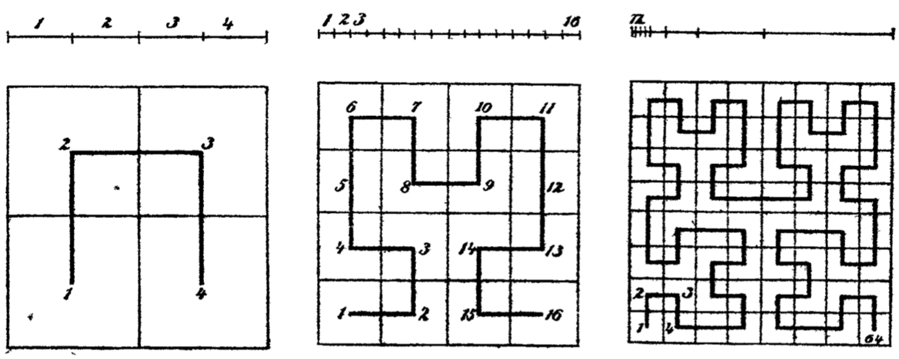

Fig.9. The figure representing the first three approximations of Hilbert's, as it appears in [21, (1891)]].

[15]Peano had occasional epistolary exchange with Jordan: two letters from Peano to Jordan (from 1884 and 1894) are known, while no letter from Jordan to Peano has been found. In spite of their familiarity, in 1894 in *L'intermédiare des mathématiciens* [26] Jordan asks if there exists a curve of undetermined area. Peano replies in [45, (1896)] that if one joins the ends of his curve with those of a rectifiable curve lying outside the square, then the difference between the outer and inner measures of the set inside the curve is equal to the area of the square.

5. Topology

5.1. **Interior and closure.** The notions of interior, exterior and boundary points of subsets of Euclidean space existed informally in mathematical literature before 1887, but were precisely defined for the first time in *Applicazioni Geometriche* [35, (1887)], where x is said to be an *interior point* of a subset A of Euclidean space X there is $r > 0$ such that $B(x, r) \subset A$; an x is called an *exterior point* of A if it is an interior point of $X \setminus A$

An x is a *boundary point* ([16]) if it is neither exterior nor interior. Subsequently, Peano defines the *interior* $\operatorname{int} A$ of A as the set of interior points, and the *closure* of A by

$$\operatorname{cl} A := \{x \in X : \operatorname{dist}(x, A) = 0\},$$

and relates it to the notion of *closed* set of Cantor, that is, $\operatorname{cl} A$ is the least closed set that includes A.

These fundamental topological concepts reappeared several years later in the second edition of *Cours d'Analyse* [25, (1893)] of Jordan.

It is significant that Peano introduced interior and exterior points in connection with internal and external measures (see Section 11).

5.2. **Distributive and antidistributive families.** Miscellaneous distributive properties were studied in *Applicazioni geometriche* [35, (1887)]. A *distributive* family $\mathcal{H}$ of subsets of X ($\mathcal{H} \in \mathbb{D}$, for short) is defined as a family fulfilling

$$H_0 \cup H_1 \in \mathcal{H} \Longleftrightarrow H_0 \in \mathcal{H} \text{ or } H_1 \in \mathcal{H}. \tag{6}$$

Among examples of distributive families given by Peano are the family of infinite sets and that of unbounded subsets of Euclidean space. He calls a family $\mathcal{A}$ (of subsets of X) *antidistributive* ($\mathcal{A} \in \mathbb{I}$, for short) if

$$A_0 \cup A_1 \in \mathcal{A} \Longleftrightarrow A_0 \in \mathcal{A} \text{ and } A_1 \in \mathcal{A}. \tag{7}$$

By the way, such families are nowadays called *ideals*. For instance, the family of finite sets and that of bounded subsets of Euclidean space are antidistributive.

We call a family $\mathcal{F}$ of subsets of a set X is called a *filter* ($\mathcal{F} \in \mathbb{F}$, for short) (see H. Cartan [4, (1937)]) if

$$F_0 \in \mathcal{F} \text{ and } F_1 \in \mathcal{F} \Longleftrightarrow F_0 \cap F_1 \in \mathcal{F}. \tag{8}$$

We include here neither the usual non-degeneracy condition: $\varnothing \notin \mathcal{F}$ (the only filter $\mathcal{F}$ on X fulfilling $\varnothing \in \mathcal{F}$ is the power set 2^X of X) nor the nonemptiness: $\mathcal{F} \neq \varnothing$.

[16] In Peano terminology, *limit point*.

In order to relate the properties (6),(7) and (8), we consider three unitary operations $2^X \to 2^X$, namely, for $\mathcal{A} \in 2^X$,

(grill) $$\mathcal{A}^{\#} := \{H \subset X : \underset{A\in\mathcal{A}}{\forall} H \cap A \neq \varnothing\},$$

(complementation) $$\mathcal{A}^{c} := \{H \subset X : H \notin \mathcal{A}\},$$

(complementary) $$\mathcal{A}^{\flat} := \{X \setminus A : A \in \mathcal{A}\}.$$

These operations are involutions and

$$\#(\mathbb{F}) = \mathbb{D},\ c(\mathbb{D}) = \mathbb{I} \text{ and } \flat(\mathbb{I}) = \mathbb{F}.$$

The *grill* was introduced by Choquet [6, (1947)], who notices that $\mathcal{H}$ is the grill of a filter if and only if (6) holds. Therefore Choquet rediscovers distributive property sixty years after they were introduced by Peano.

5.3. **Compactness.** It is stupefying that in the eighties of the nineteenth century Peano routinely used as a matter of fact two dual properties of abstract compactness. Peano attributes one of them to Cantor [3, (1884)]. The definition of "compactness" by Heine (1872) came earlier, while those of Borel (1895), Lebesgue (1902) Vietoris (1921) and Alexandrov and Urysohn (1923) were posterior to Cantor and Peano.

Theorem 2. *Let S be a bounded non-empty set of Euclidean space X. If $\mathcal{H}$ is a distributive family of subsets of X and $S \in \mathcal{H}$, then there exists a point $\bar{x} \in \operatorname{cl} S$, such that any neighborhood of $\bar{x}$ belongs to $\mathcal{H}$.*

Peano cites Cantor [3, (1884) p. 454] for Theorem 2 and its proof ([17]) and restates it in terms of antidistributive families:

Theorem 3. *Let S be a bounded non-empty set of Euclidean space X. If $\mathcal{A}$ is an antidistributive family of subsets of X and for each $x \in \operatorname{cl} S$ there is a neighborhood of x belonging to $\mathcal{A}$, then $S \in \mathcal{A}$.*

A restatement in terms of filters yields readily

Theorem 4. *Let S be a bounded non-empty set of Euclidean space X. If $\mathcal{F}$ is a filter X and $S \in \mathcal{F}^{\#}$, then there exists a point $\bar{x} \in \operatorname{cl} S$, such that $\bar{x} \in \operatorname{adh} \mathcal{F}$.*

Recall that $\mathcal{N}(x)$ denotes the *neighborhood filter* of x and the adherence of $\mathcal{F}$ can be defined by

$$\operatorname{adh} \mathcal{F} := \left\{x \in X : \mathcal{F} \subset \mathcal{N}(x)^{\#}\right\}.$$

A subset S of a topological space X is said to be *relatively compact* if for each family $\mathcal{R}$ of open sets is a cover such that $\bigcup_{R\in\mathcal{R}} R = X$, there is a finite subfamily $\mathcal{R}_0$ of $\mathcal{R}$ such that $S \subset \bigcup_{R\in\mathcal{R}_0} R$. A contraposition of the definition produces the well known fact that S is relatively compact if and only if $S \cap \operatorname{adh} \mathcal{F}$ for each filter $\mathcal{F}$ such that $S \in \mathcal{F}^{\#}$.

[17] In his proof, Peano, as Cantor, considers successive partitions of S of diameter tending to 0 He mentions that this method was used by Cauchy.

As an immediate consequence of any of three equivalent theorems above, we get

Theorem 5 (Borel-Lebesgue)**.** *Each bounded subset of Euclidean space is relatively compact.*

Consequently, the Borel-Lebesgue theorem was known to Peano in two dual versions (Theorems 2 and 3).

Peano quotes as an immediate corollary of Theorem 2, the *Weierstraß theorem* saying that

Corollary 6. *A continuous real-valued function on a closed bounded set attains its minimum and maximum.*

It is enough to consider, in the case of maximum, the family $\mathcal{H}$ of subsets of S such that $H \in \mathcal{H}$ whenever $\sup f(H) = \sup f(S)$ and to notice that $\mathcal{H}$ is distributive. Although the framework remains that of Euclidean space, the method is valid for a continuous function on a compact subset of a topological space.

Applying the corollary above in metric space, the distance from $X \setminus O$ (which is continuous) attains its minimum on a compact set F, hence Peano gets

Corollary 7. *If F is a compact set and O is an open set such that $F \subset O$, then there exists $r > 0$ such that $B(F, r) \subset O$.*

5.4. **Lower and upper limits of variable set.** Generalizing the notions of limit of straight lines, planes, circles and spheres (that depend on parameter) considered as sets, he defines two limits of variable figures (in particular, curves and surfaces).

A *variable figure* (or *set*) is a family, indexed by the reals, of subsets A_λ of an affine Euclidean space X. Peano defines in *Applicazioni Geometriche* [35, (1887)] the *lower limit of a variable figure* by

$$\mathrm{Li}_{\lambda\to+\infty} A_\lambda := \{y \in X : \lim_{\lambda\to+\infty} \mathrm{d}(y, A_\lambda) = 0\}.$$

In the last two editions of *Formulario Mathematico* [46, (1903)], [48, (1908)] he defines also the *upper limit of a variable figure*:

$$\mathrm{Ls}_{\lambda\to+\infty} A_\lambda := \{y \in X : \liminf_{\lambda\to+\infty} \mathrm{d}(y, A_\lambda) = 0\},$$

that he also expresses as

$$\mathrm{Ls}_{\lambda\to\infty} A_\lambda = \bigcap\nolimits_{n\in\mathbb{N}} \mathrm{cl} \bigcup\nolimits_{\lambda\geq n} A_\lambda.$$

These notions will serve Peano in the definitions of lower and upper tangent cones (see Section 8) and, as Peano himself stresses, in the theory of differential equations (see Section 10).

6. Vector spaces

6.1. **Affine and vector spaces.** Peano maintains firmly the distinction between points and vectors and so on. He applies the geometric calculus of Grassmann and refounds axiomatically affine spaces and Euclidean geometry, based on the primitive notions of *point*, *vector* (i.e., difference of points) and *scalar product*.

In *Calcolo geometrico* [36, (1888)] he provides a modern definition of vector space structured by *addition* and *multiplication by scalars*, which fulfill

$$\begin{aligned}
&\text{(comutativity)} && a+b=b+a,\\
&\text{(associativity)} && a+(b+c)=(a+b)+c,\ m(na)=(mn)a,\\
&\text{(distributivity)} && m(a+b)=ma+mb,\ (m+n)a=ma+na,\\
&\text{(normalization)} && 1a=a,\ 0a=\mathbf{0},
\end{aligned}$$

for every vectors a, b, c, and scalars m, n. That concept was implicit in the work of Grassmann [13, (1862)] and based on the notions of *sum*, *difference* and *multiplication by scalars*.

6.2. **Norms.** In [37, (1888)] Peano defines the *Euclidean norm* in numerical *Euclidean* space $\mathbb{R}^n$ for arbirary natural n, realizing an abstraction from orthogonal coordinates. He recognizes its equivalence with the l_∞-norm.

Subsequently he defines the norm of linear maps F between Euclidean spaces by

$$\|F\| := \max_{x\neq 0} \frac{\|Fx\|}{\|x\|},$$

which constitutes the first occurrence of the *Banach operator norm*. Furthermore he shows its basic properties and its compatibility with the linear operator algebra, for example,

$$\|GF\| \leq \|G\| \, \|F\| \, .$$

He confronts it with the Euclidean norm in the corresponding space of matrices and relates it to the eigenvalues of $F^T F$, where F^T is the transposed operator of F. He also gives the *Liouville formula*:

$$\det(e^A) = e^{\operatorname{tr} A}.$$

7. Differentiability

The definition of *derivative* at a point x of a real-valued function defined on a subset of Euclidean space appears already in *Applicazioni geometriche* [35, (1987)] and is generalized in *Formulario Mathematico* [48, (1908)] to function valued in Euclidean space: a function $f : A \to \mathbb{R}^n$, where A is a subset of $\mathbb{R}^m$ and x is an accumulation point of A, is said to be *differentiable* *at* x if there exists a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{A\ni y\to x} \frac{f(y) - f(x) - L(y-x)}{\|y-x\|} = 0. \tag{9}$$

Before Peano, differentiability of functions of several variables was understood as the existence of *total differential* and in practice was assured by the continuity of partial derivatives, that is, by strict differentiability (see below). It was Thomae who pointed out in [59, (1875)] that differentiability was not equivalent to partial differentiability. Peano's definition frees the derivative of particular coordinate system, thus makes it possible to pass from one coordinate system to another.

It should be stressed that Peano's definition appeared in a rigorous modern form (9), that is used nowadays in contrast to the standard language of mathematical definition in that epoch, was usually informal and often vague. Even if, in giving this definition, Peano refers to the concepts of Grassmann [13, (1862)] and of Jacobi [22, (1841)], those however were more rudimentary (radial derivative and Jacobian matrix). As A is not the whole of Euclidean space, in general the linear operator L in (9) is not unique; if it is, it is called the *derivative* of f at x and is denoted by $Df(x)$.

$Df(x)$ is called nowadays the *Fréchet derivative* of f at x, although Fréchet gave its informal (geometric) definition only in [9] in 1911. Fréchet was apparently unaware of Peano's definition, because one month later [10] he published another note, acknowledging contributions of Stolz (1893), of Pierpoint (1905) and of W. H. Young (1910), but not that of Peano.

In [41, (1892)] Peano introduces *strict differentiability* at x of $f : A \to \mathbb{R}$, with $A \subset \mathbb{R}$ interval and $x \in A$, that is, if

$$\lim_{A\ni y,z\to x} \frac{f(y) - f(z)}{y-z} = f'(x). \tag{10}$$

He noticed that strictly differentiability amounts to continuos continuos diferentiability.

In [36, (1888)] Peano gives the following *mean value theorem* for vector-valued functions f of one variable: if f has an $(n+1)$-derivative $f^{(n+1)}$ on $[t, t+h]$, then there exists an element $k \in \operatorname{cl}\operatorname{conv} f^{(n+1)}([t, t+h])$ such that

$$f(t+h) = f(t) + hf'(t) + \cdots + \frac{h^n}{n!} f^{(n)}(t) + \frac{h^{n+1}}{(n+1)!} k, \tag{11}$$

where "conv" stands for the *convex hull.* Here is another surprise, because the concept of convex hull has been usually attributed to Minkowski [29, (1896)].

In [42, (1892)] Peano says that a polynomial function $a_0 + a_1(x - x_0) + \ldots a_n(x - x_0)^n$ is said to be a *development* of f of *rank* n with respect to powers of $x - x_0$ if

$$\lim_{x\to x_0} \frac{f(x) - (a_0 + a_1(x-x_0) + \ldots a_n(x-x_0)^n)}{(x-x_0)^n} = 0. \tag{12}$$

([18]) This equality can be rewritten in such a way that the n-th coefficient is given by

$$a_n = \lim_{x\to x_0} \frac{f(x) - (a_0 + a_1(x-x_0) + \dots a_{n-1}(x-x_0)^{n-1})}{(x-x_0)^n},$$

which leads to the *Peano generalized derivative of order* n, that is, $a_n n!$ If $f^{(n)}(x_0)$ exists, then $a_n n! = f^{(n)}(x_0)$, but even a discontinuous function can have a development. For example,

$$f_n(x) := \begin{cases} x^{n+1}\theta\left(\frac{1}{x}\right), \text{ if } x \neq 0, \\ 0, \text{ if } x = 0, \end{cases}$$

where $\theta(t)$ is the fractional part of t, has a development of rank n, and

$$f_\infty(x) := \begin{cases} \exp(-\frac{1}{x^2})\theta\left(\frac{1}{x}\right) \text{ if } x \neq 0, \\ 0, \text{ if } x = 0, \end{cases},$$

has developments of arbitrary rank; they both have discontinuities in each neighborhood of 0. Indeed, $\lim_{x\to 0} \dfrac{f_n(x)}{x^k} = 0$ for $0 \leq k \leq n$, because $0 \leq \theta\left(\frac{1}{x}\right) < 1$, so that $a_0 = a_1 = \ldots = a_n = 0$, so that (12) holds with $x_0 = 0$. Similarly for f_∞. On the other hand, $t \longmapsto \theta\left(\frac{1}{t}\right)$ is discontinuous at every $t \in \{\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\}$.

If f is n times differentiable and $f^{(n+1)}(t)$ exists, then there exists $\xi \in [t, t+h]$ such that the *Peano remainder* of rank $n+1$ of the *Taylor formula* is

$$\Big(\frac{f^{(n)}(\xi) - f^{(n)}(t)}{\xi - t} - f^{(n+1)}(t)\Big)\frac{h^{n+1}}{(n+1)!}.$$

8. Tangency

The notion of tangent to a circle can be found already in Euclid's work and to a curve in *Géométrie* of Descartes (1637). Till the time of Peano, several definitions of tangent set to arbitrary figures ([19]) were formulated, for example,

(α) *a tangent plane to a surface S at a point p is a plane that contains the tangent straight line at p of every curve traced on the surface S and passing through p;*

(β) *a tangent plane to S at p is a plane that contains the tangents at p to those curves on S that has a tangent straight line and pass through p.*

But these definitions failed to produce non-controversial results in some cases. In *Applicazioni Geometriche* [35, (1887)] Peano gives a metric definition of tangent straight line and of tangent plane and, finally, introduces a

[18] Peano gives rules for the developments of sums, products and compositions.

[19] That is, subsets of Euclidean space.

unifying notion, that of *affine tangent cone*:

$$\operatorname{tang}(A,x) := x + \operatorname{Li}_{\lambda\to+\infty}\lambda(A-x). \tag{13}$$

Later, in *Formulario Mathematico* [48, (1908)], he introduces another type of tangent cone, namely

$$\operatorname{Tang}(A,x) := x + \operatorname*{Ls}_{\lambda\to+\infty}\lambda(A-x) \tag{14}$$

To distinguish the two notions above, we shall call the first *lower affine tangent cone* and the second *upper affine tangent cone*.

As usual, after abstract investigation of a notion, Peano considers significant special cases; he calculates the upper affine tangent cone in several basic figures (closed ball, curves and surfaces parametrized in a regular way).

9. Optimality conditions

A well-known necessary conditions of maximality of a function at a point, is formulated in terms of derivative of the function and of tangent cone of the constraint at that point. Consider a real-valued function $f : X \to \mathbb{R}$, where X is a Euclidean affine space, and a subset A of X.

Regula (of optimality) *If f is differentiable at $x \in A$ and $f(x) = \max\{f(y) : y \in A\}$, then*

$$\langle Df(x), y-x\rangle \leq 0 \textit{ for every } y \in \operatorname{Tang}(A,x). \tag{15}$$

Here $Df(x) : X \to \mathbb{R}$ is the derivative of the f at x and $\operatorname{Tang}(A,x)$ is the upper tangent cone of A at x.

This is exactly a today formulation of necessary optimality conditions. But Regula was known to Peano already in [35, (1887)] and it appeared in the form (15) in [48, (1908)].

Peano applies his regula to numerous examples of minimization problems, in particular, to those of minimizing the sum of distances of a point from one or several fixed points or figures.

10. Differential equations

10.1. **Linear systems of differential equations.** In [37, (1888)] Peano introduces, what is now called *Peano series,* to represent the solution of a general linear system of differential equations. He transforms such a system into an equation

$$\frac{dx}{dt} = Ax, \tag{16}$$

where $A(t) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator continuously depending on t. Starting with a constant $x_0 \in \mathbb{R}^n$, he defines

$$x_1 := \int Ax_0\,dt,\ x_2 =: \int Ax_1\,dt, \ldots,\ x_{n+1} =: \int Ax_n\,dt, \ldots \tag{17}$$

(integrating from t_0 to t), shows that there is M such that $|x_n(t)| \leq M^n \|x_0\| (t-t_0)^n / n!$ that for each n, so that the *Peano series*

$$x := x_0 + x_1 + x_2 + \ldots$$

uniformly converges as well as its derivative $Ax_1 + Ax_2 + Ax_3 + \ldots$ and, clearly, is a solution of (16) such that $x(t_0) = x_0$. Then, on using (17), he defines

$$R_{t_0}^t := (I + \int_{t_0}^t A\,ds + \int_{t_0}^t A\,ds \int_{t_0}^t A\,ds + \ldots),$$

where $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity, which, of course, is the *resolvent* operator of (16), and thus the solution to (16) with the initial condition $x(t_0) = x_0$, is given by $x(t) = R(t_0, t)\, x_0$. For A constant, he represents the resolvent by the exponential of A given by

$$e^A := I + A + \tfrac{1}{2}A^2 + \ldots + \tfrac{1}{n!}A^n + \ldots,$$

so that

$$x(t) = e^{A(t-t_0)}x_0.$$

Finally, he gives a solution of a non-homogeneous equation $\frac{dx}{dt} = Ax + b$ in the form

$$x(t) = R_{t_0}^t x_0 + R_{t_0}^t \int_{t_0}^t R_s^{t_0}\, b(s)\, ds.$$

In this short paper, Peano precedes this luminous theory by the theory of linear operators, their matrix representations, their norms and the convergent series of operators, in particular, the exponentials of operators.

Peano's view is unprecedented in that epoch. As observed Garret Birkhoff in [2], these were foreshadows of the modern theory of Banach spaces and algebras.

10.2. **Nonlinear differential equations.** In [34, (1886)] Peano shows that the equation

$$\frac{dx}{dt} = f(t,x), \quad x(t_0) = x_0, \tag{18}$$

where $t_0 \leq t \leq t_1$ and $f : [t_0, t_1] \times \mathbb{R} \to \mathbb{R}$, has a solution provided that f is continuous and that this solution is unique if, moreover, f fulfills the *Lipschitz condition* with respect to x, that is, there exists a constant $c > 0$ such that

$$|f(t, x_0) - f(t, x_1)| \leq c\,|x_0 - x_1| \tag{19}$$

for every t, x_0 and x_1.

This is the first result of uniqueness of solutions of differential equations in the literature. To prove it, Peano uses an argument that amounts to the following *Grönwall inequality*: if c a real number and $\frac{du}{dt}(t) \leq c\,u(t)$ for each $t \geq t_0$, then

$$u(t) \leq u(t_0)\, e^{c\,(t-t_0)} \tag{20}$$

for $t \geq t_0$. Moreover, Peano uses (20) also to prove continuous dependence of solution on initial value. Of course, a statement of this inequality as a fact of its own interest by Grönwall [17, (1919)] has greater merit than its implicit use.

Later on, Klein asks Peano to generalize this theorem to systems of differential equation in view of publication in Mathematische Annalen. Peano replies that passing from a scalar equation to a system of equations considerably complicates the quest, but a few years later presents a paper [39, (1890)], in which he solves the problem, which can be stated in the same terms, the only difference being that now $f : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ for a natural $n \geq 1$. By the way, he gives examples of non-uniqueness in the absence of the Lipschitz condition (19), for instance,

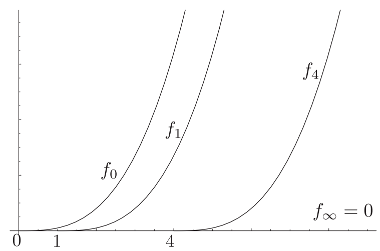

Fig. 10. Here we show four out of infinity independent solutions: f_0, f_1, f_4 and $f_\infty \equiv 0$.

$$\frac{dx}{dt} = 3x^{\frac{2}{3}},\ x(0) = 0, \tag{21}$$

where for every $r \in [0, \infty]$, the function

$$f_r(x) := \begin{cases} 0, & \text{if } x \leq r, \\ x^3, & \text{if } x > 0, \end{cases}$$

is a solution of (21); he shows also that the Lipschitz condition is not necessary for uniqueness.

To prove the existence, he uses a sequence of polygonal approximations (piecewise solutions of differential inequalities) that eventually uniformly converges to a solution. In doing so, Peano realizes that the existence of a selection φ of a multivalued map Φ is not granted by the axioms of the set theory, but in the specific problem, with which he is confronted, Peano goes around the obstacle by picking the least element $\varphi(t)$ of $\Phi(t)$ with respect to the lexicographic order of $\mathbb{R}^n$. He observes that a *principle of infinite arbitrary choices* is not a consequence of the axioms.

A year later, Picard publishes a paper [52, (1891)] proving the existence of solution (but non uniqueness) of a vector equation of type (18) under the Lipschitz condition (19). The latter is superfluous in view of the result of Peano and Peano publishes in [43, (1892)] a comment in this vein.

11. Measure theory

The interest of Peano in measure theory is rooted in his criticism of the definition of *area* (1882), of *integral* (1883) and of *derivative* (1884).

This criticism leads him to an innovative measure theory, which is exposed systematically and fully in a chapter of *Applicazioni geometriche* [35, (1887)], where he refounds the notion of Riemann integral by means of inner and outer measures, as he anticipated in his juvenile work [31, (1883)]. Peano in [35, (1887)], and later Jordan in the paper [24, (1892)] and in the second edition of *Cours d'Analyse* [25, (1893)], develop the well known concepts of classical measure theory, namely, measurability, change of variables, fundamental theorems of calculus, with some methodological differences between them.

The mathematical tools employed by Peano were really advanced at that time (and maybe are even nowadays), both on a geometrical and a topological level. Peano used extensively the geometric vector calculus introduced by Grassmann. The geometric notions include oriented areas and volumes (called geometric forms).

Peano's measure theory is based on solid grounds of logic, set theory and topology. In this context he introduces the notions of closure, interior and boundary of sets (see 5.1).

11.1. Abstract measures and their differentiation.

Most innovative ingrediant of the approach of Peano is the introduction of *abstract measures* and their *differentiation.*

We use the term *abstract measures* to designate *"distributive" set functions* of Peano, which are a functional counterpart of distributive families (6). Most evident distributive set functions are those *finitely additive.*

Peano defined integral with respect to set-function and derivative of measure with respect to another measure, which constituted a first modern measure theory preceding that of *Lebesgue.*

By retracing research on *coexistent magnitudes* (*grandeurs coexistantes*) by Cauchy [5, (1841)], Peano in *Applicazioni geometriche del calcolo infinitesimale* [35, (1887)] defines the "density" (strict derivative) of a "mass" (a distributive set function) with respect to a "volume" (a positive distributive set function), proves its continuity (whenever the strict derivative exists) and shows the validity of the *mass-density paradigm*: "mass" is recovered from "density" by integration with respect to "volume".

It is remarkable that Peano's strict derivative provides a consistent mathematical ground to the concept of "infinitesimal ratio" between two magnitudes, successfully used since Kepler. In this way the classical (pre-Lebesgue) measure theory reaches a complete and definitive form in Peano's *Applicazioni geometriche* ([20]).

[20] A pioneering role of that book is remarked by J. Tannery [58, (1887)]: "Chapter V is titled: Geometric magnitudes. This chapter is probably the most relevant and interesting, the one that marks the difference of the Book of Peano with respect to other classical

In order to grasp the essence of Peano's contribution and to compare it with analogous results by Cauchy, Lebesgue, Radon and Nikodym, we present it in a particular significant case.

Peano's strict derivative of a set function (for instance, the "density" of a "mass" μ with respect to the "volume") at a point $\bar{x}$ is computed, when it exists, as the limit of the quotient of the "mass" with respect to the "volume" of a cube Q, when $Q \to \bar{x}$ (that is, the supremum of the distances of the points of the cube Q from $\bar{x}$ tends to 0). In formula, Peano's strict derivative $g_P(\bar{x})$ of a mass μ at $\bar{x}$ is given by:

$$g_P(\bar{x}) := \lim_{Q \to \bar{x}} \frac{\mu(Q)}{\mathrm{vol}_n(Q)}. \tag{22}$$

On the other hand, Cauchy's derivative [5, (1841)] is obtained as the limit of the ratio between "mass" and "volume" of a cube Q *including* the point $\bar{x}$, when $Q \to \bar{x}$. In formula, *Cauchy's derivative* $g_C(\bar{x})$ of a mass μ at $\bar{x}$ is given by:

$$g_C(\bar{x}) := \lim_{\substack{Q \to \bar{x} \\ \bar{x} \in Q}} \frac{\mu(Q)}{\mathrm{vol}_n(Q)}. \tag{23}$$

Observe that (23) is analoguous to derivative (9), while (22) to strict derivative (10).

Lebesgue's derivative of set functions is computed *à la Cauchy*. Notice that Lebesgue considers finite σ-additive and absolutely continuous measures as "masses", while Peano considers distributive set functions. Lebesgue's derivative exists (i.e., the limit (23) there exists for *almost every* $\bar{x}$), it is measurable and the reconstruction of a "mass" as the integral of the derivative is assured by absolute continuity of the "mass" with respect to volume. On the contrary, Peano's strict derivative does not necessarily exist, but when it exists, it is continuous and the mass-density paradigm holds:

$$\mu(Q) = \int_Q g_P \, d\,\mathrm{vol}_n \, .$$

The constructive approaches to differentiation of set functions corresponding to the two limits (22) and (23) are opposed to the approach given by Radon [54, (1913)] and Nikodym [30, (1930)], who define the derivative in a more abstract and wider context than those of Lebesgue and Peano. As in the case of Lebesgue, a Radon-Nikodym derivative exists; its existence is assured by assuming absolute continuity and σ-additivity of the measures.

Treatises: definitions concerning *sets of points*, exterior, interior and limit points of a given set, distributive functions (coexistent magnitudes in the sense of Cauchy), exterior, interior and proper length (or area or volume) of a set, the extension of the notion of integral to a set, are stated in an abstract, very precise and very clear way."

11.2. **Sweeping-tangent theorem.** The fashionable Mamikon's sweeping-tangent theorem says that *the area of a tangent sweep of a curve is equal to the area of its corresponding tangent cluster* (see the figure below)

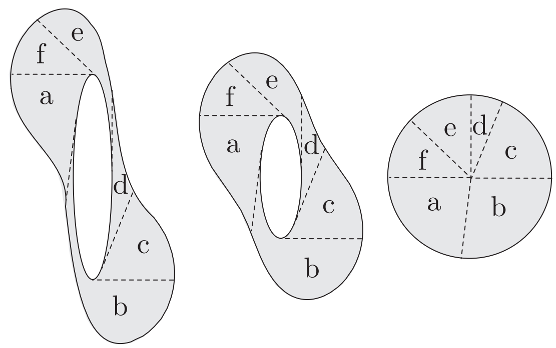

Fig. 11. The three figures have the same area, because they are swept by the same tangent vector to the inner ellipsis (or point). The areas marked by the same letter have the same area as well.

This theorem, published in [28, (1981)], has numerous applications, as it enables one to obtain the areas of complicated figures almost without calculation, by reducing the problem to that of the area of some simple figures. In [15] the authors Greco, Mazzucchi and Pagani discovered with stupefaction that in [35, (1887) p. 242] Peano considerably generalizes the Mamikon's theorem to be, and mentions it as one of the following four special cases (of which Mamikon's theorem corresponds to 3):

(1) A moves along a straight line and the angle of AB with that line is constant;
(2) A is fixed;
(3) AB is tangent at A to the curve described by A;
(4) AB is of constant length and normal to the curve described by its midpoint.

In fact, Peano uses the Grassmann external algebra to give a formula for the area of plane figures that are *described by a segment AB of variable length that never passes twice through the same point.* He gives the following formula for that area:

$$\frac{1}{2}\int_a^b \left|\det\begin{pmatrix} v_1(t) & v_2(t) \\ v_1'(t) & v_2'(t) \end{pmatrix}\right| dt,$$

where $v_1(t), v_2(t)$ are the components of the vector $B(t) - A(t)$ and $t \in [a, b]$. It is clear from this formula, that the area depends only on a vector and not on particular couples $A(t), B(t)$.

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