NON NORMALITY NUMBERS

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Abstract. Non normality number and strong non normality number of a topological space are introduced to the effect that a topology is normal if and only if its non normality number is 1 if and only if its strong non normality number is 1. It is proved that for every cardinal κ, there exists a completely regular topology of non normality and strong non normality κ; for every uncountable regular cardinal κ, there exists a (completely regular) Moore space of non normality and cardinality κ. On the other hand, for every cardinals $\kappa < \lambda$ there exists a completely regular topology of strong non normality κ and non normality greater than λ. As an answer to a question of U. Marconi, it is proved that the non normality number of every separable regular topology with a closed discrete subset of cardinality continuum, is at least continuum.

1. Introduction

Roughly speaking, the non normality number indicates how much non normal is a topology. The concept has been introduced in [1] on the occasion of study of kernels of upper semicontinuous relations.

If we denote by $\mathcal{N}(A)$ the *neighborhood filter* of A, [1] then a space [2] is not normal if and only if there exist two (non empty) disjoint closed sets A_0, A_1 such that the filter supremum $\mathcal{N}(A_0) \vee \mathcal{N}(A_1)$ is non degenerate. The *non normality (number)* $\nu(X)$ of a space X is the supremum of cardinals κ such that there exists a disjoint family $\mathcal{A}$ of non empty closed subsets of X with $|\mathcal{A}| = \kappa$, and the supremum of the neighborhood filters of the elements of

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[1] The neighborhood filter of a non empty set A is generated by all the open sets that include A; in particular, $\mathcal{N}(x)$ stands for the neighborhood filter of x.

[2] In this paper a *space* means a topological Hausdorff space.

$\mathcal{A}$,

$$\bigvee_{A \in \mathcal{A}} \mathcal{N}(A) \tag{1.1}$$

is non degenerate (that is, $O_0 \cap O_1 \cap \ldots \cap O_n \neq \varnothing$ for every finite subset $\{A_0, A_1, \ldots, A_n\}$ of $\mathcal{A}$ and each choice of open sets $O_0 \supset A_0, O_1 \supset A_1, \ldots, O_n \supset A_n$). The *strong non normality (number)* $s\nu(X)$ is the supremum of cardinals κ such that there exists a disjoint family $\mathcal{A}$ of non empty closed subsets of X with $|\mathcal{A}| = \kappa$, and $\bigcap_{A \in \mathcal{A}} O_A \neq \varnothing$ for every choice $O_A \in \mathcal{N}(A)$ with $A \in \mathcal{A}$. If the supremum in the definitions above is attained, then we say that the (strong) non normality is *attained.* In these terms, a (non empty) space X is normal if and only if $\nu(X) = 1$ if and only if $s\nu(X) = 1$. In general $s\nu(X) \leq \nu(X) \leq |X|$ and if $\nu(X)$ is finite or non attained $\aleph_0$, then both the non normalities coincide. We shall also consider intermediate non normality numbers: if ζ is a cardinal, then the *ζ-non-normality (number)* $\nu_\zeta(X)$ is the supremum of cardinals κ such that there exists a disjoint family $\mathcal{A}$ of non empty closed subsets of X with $|\mathcal{A}| = \kappa$ such that for every $\mathcal{A}_0 \subset \mathcal{A}$ with $|\mathcal{A}_0| < \zeta$, one has $\bigcap_{A \in \mathcal{A}_0} O_A \neq \varnothing$ for every choice $O_A \in \mathcal{N}(A)$. Of course, $\nu_\zeta(X) \leq \nu_{\aleph_0}(X) = \nu(X)$ for $\aleph_0 \leq \zeta$, and $s\nu(X) = \kappa$ whenever $\nu_{\kappa^+}(X) = \kappa$, where κ^+ is the least among the cardinals greater than κ.

It is immediate that for every ζ, if X is a closed subspace of Y, then $\nu_\zeta(X) \leq \nu_\zeta(Y)$ (and the inequality can be strict), and if f is a closed continuous map, then $\nu_\zeta(X) \geq \nu_\zeta(f(X))$, but this need not hold for a (continuous) open map. It follows again from the known facts about normality that in general neither $\nu(f^-(Y)) \leq \nu(Y)$ for open prefect maps, nor $\nu(X \times Y) \leq \nu(X) \times \nu(Y)$. Actually, there exists a normal space X such that $\nu(X^2) = |X| = 2^{\aleph_0}$.

In this paper we prove that for every cardinals $\kappa \leq \lambda$ there exist a completely regular space of non normality and strong non normality κ, and a completely regular space of strong non normality κ and non normality greater than λ. Also, if $\zeta < \kappa$ are infinite regular cardinals, then there exists a completely regular space of ζ-non-normality and cardinality κ; in particular, for every regular uncountable cardinal κ there exists a Moore space of non normality and cardinality κ. On the other hand, if a space of density δ admits a closed discrete subset of cardinality 2^δ then its non normality is at least 2^δ.

2. Each cardinal is a non normality number

Example 2.1. *Let $(\xi_\alpha)_{\alpha<\kappa}$ be uncountable regular cardinals (equipped with the order topology) such that $\kappa < \xi_0$ and for every $0 < \beta < \kappa$,*

$$\prod_{\alpha<\beta} \xi_\alpha < \xi_\beta. \tag{2.1}$$

Consider $X_\kappa = \prod_{\alpha<\kappa}[0,\xi_\alpha]$ *and let*

$$Z_\alpha = \prod_{\beta<\alpha}\{\xi_\beta\} \times [0,\xi_\alpha[\times \prod_{\alpha<\beta<\kappa}\{\xi_\beta\}.$$

For every α *and each* $x \in Z_\alpha$ *the neighborhood filter of* x *is that of the box topology of* X_κ*, and all the other elements of* X_κ *are isolated. As every element of* X_κ *admits a neighborhood base consisting of clopen sets, the topology is completely regular.*

Example 2.2. *We shall modify the topology of* X_κ *of Example 2.1 so that for each* α *and every element* $(\zeta_\beta)_{\beta<\kappa}$ *of* Z_α *(hence* $\zeta_\alpha < \xi_\alpha$ *and* $\zeta_\beta = \xi_\beta$ *for each* $\beta \neq \alpha$*) basic neighborhoods are of the form* $\prod_{\beta<\alpha}[\gamma_\beta,\xi_\beta] \times \{\zeta_\alpha\} \times \prod_{\alpha<\beta<\kappa}[\gamma_\beta,\xi_\beta]$*. This topology is completely regular, because all the neighborhood filters admit bases of clopen sets.*

Lemma 2.3. *In the space of Example 2.2, if* F_α *is an unbounded subset of* Z_α *and* O_α *is an open set that includes* F_α *for each* $\alpha < \kappa$*, then* $\bigcap_{\alpha<\kappa} O_\alpha \neq \varnothing$*.*

Proof. Indeed, for every $\alpha, \beta < \kappa$ such that $\alpha \neq \beta$, and for each $x = (x_\gamma)_{\gamma<\kappa} \in F_\alpha$, there is $h^\alpha_\beta(x) < \xi_\beta$ for which

$$\prod_{\alpha<\beta}[h^\alpha_\beta(x),\xi_\beta] \times \{x_\alpha\} \times \prod_{\alpha<\beta<\kappa}[h^\alpha_\beta(x),\xi_\beta] \subset O_\alpha.$$

If $\beta > \alpha$, then

$$h^\alpha_\beta = \sup_{x\in F_\alpha} h^\alpha_\beta(x) < \xi_\beta.$$

If $\beta < \alpha$, then there is $\varphi(\beta) < \xi_\beta$ such that the set

$$A_\varphi = \bigcap_{\beta<\alpha}\{x \in F_\alpha : h^\alpha_\beta(x) \le \varphi(\beta)\}$$

is unbounded, for otherwise $\sigma_\varphi = \sup A_\varphi < \xi_\alpha$ for every $\varphi \in \prod_{\beta<\alpha}\xi_\beta$, and thus $\sup\{\sigma_\varphi : \varphi \in \prod_{\beta<\alpha}\xi_\beta\} < \xi_\alpha$ by the regularity of ξ_α and by (2.1); on the other hand, F_α is the union of A_φ with φ running over $\prod_{\beta<\alpha}\xi_\beta$, which yields a contradiction. Therefore there exists an unbounded subset C_α of F_α such that for every $\beta < \alpha$,

$$h^\alpha_\beta = \sup_{x\in C_\alpha} h^\alpha_\beta(x) < \xi_\beta.$$

Because for every $\beta < \kappa$, the cardinal ξ_β is regular and greater than κ,

$$h_\beta = \sup_{\kappa>\alpha\neq\beta} h^\alpha_\beta < \xi_\beta.$$

As a result, $\prod_{\beta<\alpha}[h_\beta,\xi_\beta] \times C_\alpha \times \prod_{\alpha<\beta<\kappa}[h_\beta,\xi_\beta] \subset O_\alpha$ for every $\alpha < \kappa$. It follows that if $\zeta_\alpha \in C_\alpha$ is such that $h_\alpha < \zeta_\alpha < \xi_\alpha$, then $(\zeta_\alpha)_{\alpha<\kappa} \in \bigcap_{\alpha<\kappa} O_\alpha$. ∎

Theorem 2.4. *For every cardinal κ, there exists a completely regular space X_κ such that $s\nu(X_\kappa) = \nu(X_\kappa) = \kappa$ and both non normalities are attained.*

Proof. Consider the space X_κ of Example 2.1. Because the sets Z_α are closed and the topology is coarser than that of Example 2.2, by Lemma 2.3, the strong non normality of X_κ is at least κ. We will prove that if $\mathcal{A}$ is a disjoint family of closed subsets of X_κ such that (1.1) is non degenerate, then its cardinality is not greater than κ. Because no regular cardinal greater than $\aleph_0$ includes two disjoint unbounded closed subsets, there are at most κ elements A of $\mathcal{A}$ such that there exists α for which $A \cap Z_\alpha$ is unbounded in Z_α. Hence if $|\mathcal{A}| > \kappa$, then there exists $A_0 \in \mathcal{A}$ such that for every $\alpha < \kappa$ there is a non limit ordinal $\zeta_\alpha < \xi_\alpha$ with

$$A_0 \cap (\prod_{\beta<\alpha} \{\xi_\beta\} \times [\zeta_\alpha, \xi_\alpha[\times \prod_{\alpha<\beta<\kappa} \{\xi_\beta\}) = \varnothing.$$

Let A_1 be another element of $\mathcal{A}$. Because each Z_α is normal, there exist disjoint open subsets P_0^α and P_1^α of Z_α such that $A_0 \cap Z_\alpha \subset P_0^\alpha$ and $A_1 \cap Z_\alpha \subset P_1^\alpha$, and moreover P_0^α is disjoint from $\prod_{\beta<\alpha}\{\xi_\beta\} \times [\zeta_\alpha, \xi_\alpha[\times \prod_{\alpha<\beta<\kappa}\{\xi_\beta\}$. Then let O_0 be the union of $A_0 \setminus \bigcup_{\alpha<\kappa} Z_\alpha$ and of open boxes

$$O_0^\alpha = \bigcup_{y \in P_0^\alpha} \prod_{\beta<\kappa} [\gamma_\beta^\alpha(y), \delta_\beta^\alpha(y)]$$

disjoint from A_1 such that $\delta_\alpha^\alpha(y) = y_\alpha$ (the α-component of y), $[\gamma_\alpha^\alpha(y), \delta_\alpha^\alpha(y)] \subset P_0^\alpha$ and $\zeta_\beta < \gamma_\beta^\alpha(y) < \delta_\beta^\alpha(y) = \xi_\beta$ for $\beta \neq \alpha$. Similarly, let O_1 be the union of $A_1 \setminus \bigcup_{\alpha<\kappa} Z_\alpha$ and of open boxes

$$O_1^\alpha = \bigcup_{z \in P_1^\alpha} \prod_{\beta<\kappa} [\varepsilon_\beta^\alpha(z), \eta_\beta^\alpha(z)]$$

disjoint from A_0 and such that $\eta_\alpha^\alpha(z) = z_\alpha$ (the α-component of z), $[\varepsilon_\alpha^\alpha(z), \eta_\alpha^\alpha(z)] \subset P_1^\alpha$ and $\zeta_\beta < \varepsilon_\beta^\alpha(z) < \eta_\beta^\alpha(z) = \xi_\beta$ for $\beta \neq \alpha$. The sets O_0, O_1 are open, because $X_\kappa \setminus \bigcup_{\alpha<\kappa} Z_\alpha$ is discrete; they are disjoint, because if $x \in O_0^{\alpha_0} \cap O_1^{\alpha_1}$, then either $\alpha_0 = \alpha_1 = \alpha$ and thus there exist $y \in A_0 \cap Z_\alpha$ and $z \in A_1 \cap Z_\alpha$ such that $[\gamma_\alpha^\alpha(y), \delta_\alpha^\alpha(y)] \cap [\varepsilon_\alpha^\alpha(z), \eta_\alpha^\alpha(z)] \neq \varnothing$, hence $P_0^\alpha \cap P_1^\alpha \neq \varnothing$ contrary to the hypothesis, or $\alpha_0 \neq \alpha_1$, then $[\gamma_{\alpha_0}^{\alpha_0}(y), \delta_{\alpha_0}^{\alpha_0}(y)] \cap [\varepsilon_{\alpha_0}^{\alpha_1}(z), \eta_{\alpha_0}^{\alpha_1}(z)] = \varnothing$ because $\delta_{\alpha_0}^{\alpha_0}(y) < \zeta_{\alpha_0}$ and $\zeta_{\alpha_0} < \varepsilon_{\alpha_0}^{\alpha_1}(z)$ for each $y \in A_0 \cap Z_\alpha$ and $z \in A_1 \cap Z_\alpha$. ∎

Theorem 2.5. *For every infinite cardinal κ there exists a completely regular space X_κ of density κ such that $s\nu(X_\kappa) \leq \kappa$ and $\nu(X_\kappa) = 2^\kappa$.*

Proof. By Hewitt-Marczewski-Pondiczery theorem there is a dense subset S of cardinality κ of $\{0,1\}^{2^\kappa}$ (endowed with the product topology); on the other hand, there is a discrete subspace D of cardinality 2^κ of $\{0,1\}^{2^\kappa}$ disjoint from S. We consider $X = S \cup D$ with the topology in which all the elements of S are isolated, while those of D have the neighborhoods inherited from $\{0,1\}^{2^\kappa}$. Because $|X| = 2^\kappa$ (hence the cardinality of each disjoint family is at most 2^κ) and by Theorem 3.5, $\nu(X) = 2^\kappa$. On the other hand, if $\mathcal{A}$ is a disjoint collection of closed sets such that $\bigcap_{A\in\mathcal{A}} O_A \neq \varnothing$ for every choice of open sets $O_A \supset A$, then $|\mathcal{A}| < \kappa$. Indeed, it is not restrictive to assume that $A \subset D$ for every $A \in \mathcal{A}$, because S is open and discrete. Furthermore, since D is closed and discrete, it is enough to consider the case where $O_A \setminus S = A$, and thus $\bigcap_{A\in\mathcal{A}} O_A \subset S$. If now $|\mathcal{A}| \geq \kappa$ and $f : S \to \mathcal{A}$ is an injective map, then $\{O_{f(x)} \setminus \{x\} : x \in S\}$ is a family of open sets such that $f(x) \subset O_{f(x)}$ for every $x \in S$, and $\bigcap_{x\in S}(O_{f(x)}\setminus\{x\}) = \varnothing$. ∎

Theorem 2.6. *For infinite cardinals $\kappa < \lambda$ there exists a completely regular space X_κ such that $s\nu(X_\kappa) = \kappa$ and is attained, and $\nu(X_\kappa) \geq \lambda$.*

Proof. Consider the space of Example 2.2 and add the assumption that $\lambda \leq \xi_0$. By Lemma 2.3, $s\nu(X_\kappa) \geq \kappa$. Conversely, if $\mathcal{A}$ is a disjoint family of closed sets and $\bigcap_{A\in\mathcal{A}} O_A \neq \varnothing$ for every choice of open sets $O_A \supset A$ with $A \in \mathcal{A}$, then $|\mathcal{A}| \leq \kappa$. Indeed, if for each $A \in \mathcal{A}$ and every $\alpha < \kappa$, we consider $O_A^\alpha = \pi_\alpha^{-1}(A \cap Z_\alpha)$ (where π_α is the projection on the α-th component) and $O_A^\kappa = A\setminus\bigcup_{\alpha<\kappa} Z_\alpha$, then $O_A = \bigcup_{\alpha\leq\kappa} O_A^\alpha$ is an open set that includes A; if now $x \in \bigcap_{A\in\mathcal{A}} O_A$ then for every A there is $\psi(A) \leq \kappa$ with $x \in O_A^{\psi(A)}$. Since for each $\alpha \leq \kappa$ the sets $O_{A_0}^\alpha \cap O_{A_1}^\alpha = \varnothing$ whenever A_0 and A_1 are distinct elements of $\mathcal{A}$, we infer that $\psi : \mathcal{A} \to \kappa + 1$ is injective, so that $|\mathcal{A}| \leq \kappa$.

We will find a disjoint family $\mathcal{A}$ of closed sets with $|\mathcal{A}| = \lambda$ and such that (1.1) is non degenerate. Since $\lambda \leq \xi_\alpha$ for every α we can find a disjoint family $\{E_\beta^\alpha : \beta < \lambda\}$ of subsets of ξ_α such that $|E_\beta^\alpha| = \xi_\alpha$. It follows that every E_β^α is unbounded in ξ_α. Let $A_\beta^\alpha = \prod_{\gamma<\alpha}\{\xi_\gamma\} \times E_\beta^\alpha \times \prod_{\alpha<\gamma<\kappa}\{\xi_\gamma\}$ and define

$$A_\beta = \bigcup_{\alpha<\kappa} A_\beta^\alpha.$$

Each A_β is closed as the union of a locally finite family of closed sets, and the family $\{A_\beta : \beta < \lambda\}$ is disjoint. If $\beta_1 < \beta_2 < \ldots < \beta_n$ and $O_i \supset A_{\beta_i}$ are open sets, then a fortiori $O_i \supset A_{\beta_i}^i$, hence $\bigcap_{i=1}^n O_i \neq \varnothing$ by Lemma 2.3. ∎

3. When non normality is equal to cardinality

Theorem 2.4 establishes the existence, for each cardinal κ, of a completely regular space of non normality and strong non normality equal κ. However the construction used in the proof yields a space of very big cardinality. If we reconsider the problem for regular (uncountable) cardinals, then it is possible to construct a space of prescribed non normality equal to its cardinality.

We shall generalize a construction of G. M. Reed [6] and apply it to subsets of predecessors of fixed cofinality of a given regular cardinal. Let us remind that if κ is a regular uncountable cardinal, then the family $\mathcal{D}_\kappa$ of closed unbounded subsets of κ is a filter base, and that a subset S is *stationary* if it meshes with every element of $\mathcal{D}_\kappa$. It is known [4, Lemma 7.4] that if $\mathcal{L} \subset \mathcal{D}_\kappa$ is of cardinality less than κ, then $\bigcap\mathcal{L} \in \mathcal{D}_\kappa$. Dually,

Lemma 3.1. [5, p. 78] *If $0 < \zeta < \kappa$ and $\bigcup_{\beta<\zeta} E_\beta$ is stationary in κ, then there is $\beta < \zeta$ such that E_β is stationary.*

We shall also use the fact [5] that if ζ is an infinite regular cardinal smaller than κ, then the set $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \zeta\}$ is stationary in κ.

Theorem 3.2. *If κ is an uncountable regular cardinal, then there exists a completely regular space X of cardinality κ such that $\nu_\zeta(X) = \kappa$ for every regular cardinal $\zeta < \kappa$.*

Proof. Consider $X = \kappa \times (\kappa + 1)$ and for every non-zero limit ordinal $\sigma < \kappa$ let $\{\beta^\sigma_\gamma < \sigma : \gamma < \mathrm{cf}(\sigma)\}$ be a set of ordinals such that $\sigma = \sup_{\gamma<\mathrm{cf}(\sigma)} \beta^\sigma_\gamma$. For $\gamma < \mathrm{cf}(\sigma)$,

$$G_\gamma(\sigma) = \{(\sigma, \kappa)\} \cup \bigcup_{\gamma\leq\eta<\mathrm{cf}(\sigma)} ([\beta^\sigma_\eta, \sigma] \times \{\eta\}), \tag{3.1}$$

is declared to be a neighborhood base of (σ, κ). All other elements are isolated. This is a completely regular space of cardinality κ and thus $\nu_\zeta(X) \leq \kappa$ for each infinite regular cardinal ζ less than κ. We claim that $\nu_\zeta(X) = \kappa$. Then the subset $S(\zeta)$ of κ, of elements of cofinality ζ, is stationary. By the Solovay theorem [4, Theorem 85] $S(\zeta) = \bigcup_{\alpha<\kappa} S_\alpha$, where $\{S_\alpha : \alpha < \kappa\}$ is a disjoint collection of stationary sets. On the other hand, $\{S_\alpha \times \{\kappa\} : \alpha < \kappa\}$ is a disjoint collection of closed subsets of X. If $\alpha < \kappa$ and O_α is an open set that includes $S_\alpha \times \{\kappa\}$, then there is a map $f_\alpha : S_\alpha \to \zeta$ such that the neighborhood $G_{f_\alpha(\sigma)}(\sigma)$ of (σ, κ) is a subset of O_α and $\beta^\sigma_{f_\alpha(\sigma)} < \sigma$ for every $\sigma \in S_\alpha$. By Lemma 3.1, there exists $\gamma(\alpha) < \zeta$ such that $W_\alpha = \{\sigma \in S_\alpha : f_\alpha(\sigma) = \gamma(\alpha)\}$ is stationary. For every subset A of κ with $|A| < \zeta$, let $\gamma_A = \sup\{\gamma(\alpha) : \alpha \in A\}$. Then $\gamma_A < \zeta$ and $\bigcup_{\sigma\in W_\alpha}([\beta^\sigma_{\gamma_A}, \sigma] \times \{\gamma_A\}) \subset O_\alpha$ for each $\alpha \in A$. Because W_α is stationary and $\beta^\sigma_{\gamma_A} < \sigma$ for every $\sigma \in W_\alpha$, hence in virtue of the Fodor theorem [4, Theorem 22], there exist $\delta_\alpha < \kappa$ and a stationary (hence unbounded) subset Y_α of W_α such that $\beta^\sigma_{\gamma_A} = \delta_\alpha$ for every $\sigma \in Y_\alpha$. Because Y_α is unbounded, $\bigcup_{\sigma\in Y_\alpha}[\delta_\alpha, \sigma] = [\delta_\alpha, \kappa[$ and thus

$$[\delta_\alpha, \kappa[\times\{\gamma_A\} \subset \bigcup_{\sigma\in S_\alpha} G_{f_\alpha(\sigma)}(\sigma) \subset O_\alpha$$

for each $\alpha \in A$ and $\sup_{\alpha\in A} \delta_\alpha < \kappa$. Therefore $\varnothing \neq \{\gamma : \sup_{\alpha\in A} \delta_\alpha < \gamma < \kappa\} \times \{\gamma_A\} \subset \bigcap_{\alpha\in A} O_\alpha$. ∎

If we simplify the construction in the proof above by taking $X = \kappa\times\omega_0$, by declaring isolated all the elements except for those of the form (σ, ω_0) with σ of countable cofinality, and for which the neighborhood is given by (3.1), then we get a (completely) regular topology that admits a development, that is, a *Moore space.*

Corollary 3.3. *For each uncountable regular cardinal κ, there exists a completely regular Moore space which attained non normality and cardinality are both κ.*

Let κ be weakly inaccessible, that is, regular uncountable limit cardinal. Then $\kappa = \sup_{\alpha<\kappa} \zeta_\alpha$, where $\mathrm{cf}(\zeta_\alpha) = \zeta_\alpha$ for every $\alpha < \kappa$. It follows from Theorem 3.2 that there exists a completely regular space X such that $\sup_{\mathrm{cf}(\zeta)=\zeta<\kappa} \nu_\zeta(X) = \kappa = |X|$. This implies neither that $\nu_\kappa(X) = \kappa$ nor the existence of a completely regular space X for which $s\nu(X) = |X| = \kappa$. The existence of weakly inaccessible cardinals is not provable in **ZFC**. Does there exist in **ZFC** (for each regular κ) a completely regular space X such that $s\nu(X) = |X| = \kappa$?

One of the classical examples of a non normal completely regular space is the Niemytzki plane [2, Example 1.5.10].

Example 3.4. *The* Niemytzki plane *is the upper half plane X in which the elements with non zero ordinate have Euclidean neighborhoods, while for every $r \in \mathbb{R}$, a neighborhood base of $(r, 0)$ consists of closed discs $V(r, \varepsilon)$ of radius $\varepsilon > 0$ that are tangent to $L = \{(s, 0) : s \in \mathbb{R}\}$ at $(r, 0)$. It was proved in* [1] *that its non normality is continuum. Let us show that the strong non normality is (non attained) $\aleph_0$. As the non normality is infinite, the strong normality is at least $\aleph_0$. Notice that because $\{(s, t) : s \in \mathbb{R}, t > 0\}$ is normal, if $\mathcal{A}$ is a disjoint family of closed subsets of X and there is a family $\{Q_A : A \in \mathcal{A}\}$ of open sets such that $A \cap L \subset Q_A$ for each $A \in \mathcal{A}$ and $\bigcap_{Q_A \in \mathcal{A}} Q_A = \varnothing$, then there is a family $\{O_A : A \in \mathcal{A}\}$ of open sets such that $A \subset O_A$ for each $A \in \mathcal{A}$ and $\bigcap_{Q_A \in \mathcal{A}} O_A = \varnothing$. Therefore in order to get an upper bound of the strong non normality of X, it suffices to consider disjoint families of subsets of L (necessarily closed, because L is closed and discrete). If (A_n) is a disjoint sequence of subsets of L, then $B_n = \bigcup_{(r,0)\in A_n} V(r, \frac{1}{n})$ is a neighborhood of A_n for each $n < \omega$, and $\bigcap_{n<\omega} B_n = \varnothing$.*

The Niemytzki plane is separable and includes a closed discrete subset of cardinality continuum. Umberto Marconi (University of Padua) conjectured that the non normality of each separable space that includes a closed discrete subset of cardinality continuum, is at least continuum. This conjecture is confirmed below for regular spaces.

By $\beta(\mathcal{F})$ we denote Stone transform of a filter $\mathcal{F}$ on a discrete space X that is, the set of all ultrafilters that are finer than $\mathcal{F}$. In particular, if $A \subset X$ then $\beta(A)$ stands for the set of all ultrafilters that contain A.

Theorem 3.5. *The non normality of a regular infinite space of density κ that admits a closed discrete subset of cardinality 2^κ, is at least the attained 2^κ.*

Proof. Let X be a regular space, S a dense subset of cardinality κ, and D a closed, discrete subset of cardinality 2^κ. The generality is not lost if we assume that $S \cap D = \varnothing$. It is enough to show that there exists a disjoint family $\mathcal{A}$ of subsets of D (as D is closed and discrete, these subsets are necessarily closed) such that the cardinality of $\mathcal{A}$ is 2^κ, and (1.1) is non degenerate in $S \cup D$ with the induced topology. For every $x \in D$, let $\mathcal{U}(x)$ be an ultrafilter on S such that $\mathcal{U}(x) \supset \mathcal{N}(x)$. Define on $S \cup D$ the following space: the elements of S are isolated and, for every $x \in D$, the only free filter that converges to x is $\mathcal{U}(x)$. The new topology is finer than the topology originally induced from X, hence D is closed, discrete in the new topology. It follows that $S \cup D$ is regular, and thus the natural embedding into βS is homeomorphic (hence $S \cup D$ is completely regular).

There exists $p \in \beta S \setminus S$ such that $|U \cap D| = 2^\kappa$ for every $U \in \mathcal{N}(p)$. In fact, if for every $p \in \mathrm{cl}_{\beta S} D \setminus S$, there existed $U_p \in \mathcal{N}(p)$ with $|U_p \cap D| < 2^\kappa$, then by the compactness of $\mathrm{cl}_{\beta S} D$, the set D would be covered by a finite union of sets of cardinality less than 2^κ, what contradicts $|D| = 2^\kappa$.

Let $\{V_\zeta : \zeta < \lambda\}$ be a neighborhood base of p ($\lambda \leq 2^\kappa$ because the weight of βS is 2^κ) and let $\varphi : 2^\kappa \times \lambda \to 2^\kappa$ be a one-to-one map. Let $W_{\varphi(\alpha,\zeta)} = V_\zeta$ for every $\alpha < 2^\kappa$. Then there exists a set $\{p_\xi : \xi < 2^\kappa\}$ of distinct elements such that $p_\xi \in D \cap W_\xi$ for every $\xi < 2^\kappa$. Indeed, let $p_0 \in W_0 \cap D$ be arbitrary, and suppose that we have already constructed $\{p_\xi : \xi < \delta\}$. As the set $W_\delta \cap D$ is of cardinality 2^κ, and the set $\{p_\xi : \xi < \delta\}$ is of cardinality less than 2^κ, there exists $p_\delta \in W_\delta \cap D \setminus \{p_\xi : \xi < \delta\}$. Therefore, if $D_\alpha = \{p_\xi : \xi = \varphi(\alpha, \zeta), \zeta < \lambda\}$, then $p \in \bigcap_{\alpha < 2^\kappa} \mathrm{cl}\, D_\alpha$.

Consequently, if O_α is an open subset of $S \cup D$ that includes D_α, then $\beta(O_\alpha \cap S)$ is a clopen set that includes D_α, that is, $p \in \beta(O_\alpha \cap S)$. For each finite choice $\alpha_1, \alpha_2, \ldots \alpha_m$, the intersection $\bigcap_{1 \leq k \leq m} \beta(O_{\alpha_k} \cap S)$ is a neighborhood of p, hence $\bigcap_{1 \leq k \leq m} O_{\alpha_k} \supset \bigcap_{1 \leq k \leq m} \beta(O_{\alpha_k} \cap S) \cap S \neq \varnothing$. It follows that $\mathcal{A} = \{D_\alpha : \alpha < 2^\kappa\}$ is a family of closed subsets $S \cup D$ of such that (1.1) is non degenerate, thus a fortiori it is non degenerate with respect to the original topology. ∎

By Theorem 2.5 for every cardinal κ there exists a completely regular topology fulfilling the assumptions of Theorem 3.5.

Corollary 3.6. *The non normality of every regular separable space with a closed discrete subset of cardinality continuum, is at least (the attained) continuum.*

It follows that the Sorgenfrey line is a (perfectly) normal space X such that $\nu(X^2) = 2^{\aleph_0}$, because its square is a separable space which diagonal is a closed discrete subset of cardinality $2^{\aleph_0}$.

Is the non normality of a space of density κ and of extent 2^κ equal to 2^κ? It is known [7] that if B is a subset of real numbers, then $M(B)$, the *Moore space derived from B*,[3] is normal if and only if B is a Q-set, that is, such that its every subset is relative F_σ, and that there is the least cardinal $\mathfrak{s}\mathfrak{s}$ such that $\kappa \geq \mathfrak{s}\mathfrak{s}$ if and only if there exists a Q-set of cardinality κ [3]. On the other hand, if $2^{\aleph_0} = 2^{\aleph_1}$, then there is a separable normal T_1 space with an uncountable closed discrete subspace [7, Example E] .

[3]The subspace of the Niemytzki plane with $\mathbb{R} \times \{0\}$ replaced by $B \times \{0\}$.

The non normality of a separable space with a closed discrete subset of cardinality $\mathfrak{ss} \le \kappa < 2^{\aleph_0}$ need not be κ, because $\mathfrak{ss} = \omega_1$ is compatible with $2^{\aleph_0} = 2^{\aleph_1}$ [4].

4. Topless products of ordinals

A classical example of a non normal, completely regular space is $[0, \omega_0] \times [0, \omega_1] \setminus \{(\omega_0, \omega_1)\}$ endowed with its natural topology. It follows from Proposition 4.2 that the non normality (strong non normality) is 2.

Let $(\xi_\alpha)_{\alpha<\kappa}$ be regular cardinals fulfilling the condition of Example 2.1. Let $Y = \prod_{\alpha<\kappa}[0, \xi_\alpha]$ and $X = Y \setminus \{\infty\}$ where $\infty = (\xi_\alpha)_{\alpha<\kappa}$ endowed with the box topology.

Lemma 4.1. *If A is a closed subset of X (in the box topology) and $\infty \in \operatorname{cl}_Y A$, then there is $\alpha_0 < \kappa$ such that $\infty \in \operatorname{cl}_Y(A \cap (\prod_{\alpha_0 \alpha<\alpha_0}\{\xi_\alpha\} \times [0, \xi_{\alpha_0}] \times \prod_{\alpha_0<\alpha<\kappa}\{\xi_\alpha\}))$.*

Proof. Indeed, let λ be the least cardinal for which there is a rearrangement of κ such that

$$\infty \in \operatorname{cl}_Y(A \cap (\prod_{\alpha<\lambda}[0, \xi_\alpha] \times \prod_{\lambda\le\alpha<\kappa}\{\xi_\alpha\}).$$

Therefore if $\mu < \lambda$, then because of the closedness of A, for each $\alpha < \mu$ there exist $\zeta_\alpha < \xi_\alpha$ and a neighborhood W of $\prod_{\alpha<\mu}[\zeta_\alpha, \xi_\alpha] \times \prod_{\mu\le\alpha<\kappa}\{\xi_\alpha\}$ such that $A \cap W = \varnothing$. Hence for every $\mu < \beta < \kappa$ and $\vartheta \in \prod_{\alpha<\mu}[\zeta_\alpha, \xi_\alpha]$ there is $f_\beta(\vartheta) < \xi_\beta$ such that $\{\vartheta\} \times \prod_{\mu\le\beta<\kappa}[f_\beta(\vartheta), \xi_\beta] \subset W$. Because

$$\zeta_\beta = \sup\{f_\beta(\vartheta) : \vartheta \in \prod_{\alpha<\mu}[\zeta_\alpha, \xi_\alpha]\} < \xi_\beta,$$

we conclude that $A \cap \prod_{0\le\alpha<\kappa}[\zeta_\alpha, \xi_\alpha] = \varnothing$, what means that $\infty \notin \operatorname{cl}_Y A$. ∎

Proposition 4.2. *If $\omega_0 < \xi_0$ and (2.1) holds, then for $m \le \omega_0$ the non normality and the strong non normality of $\prod_{0\le n<m}[0, \xi_n] \setminus \{\infty\}$ (in the product topology) is m.*

Proof. If A is a closed subset of $\prod_{n<m}[0, \xi_n] \setminus \{\infty\}$ in the product topology, then it is closed for the box topology. Hence by Lemma 4.1 either there exists n_0 such that $\infty \in \operatorname{cl}_Y(A \cap (\prod_{n<n_0}\{\xi_n\} \times [0, \xi_{n_0}] \times \prod_{n_0<n<m}\{\xi_n\}))$ or for each $n < m$ there exist non limit ordinals $\zeta_n < \xi_n$ such that $A \cap \prod_{n<m}[\zeta_n, \xi_n] = \varnothing$. With respect to the product topology, the sets $F_n = \{(x_k)_{k<m} : x_n < \zeta_n\}$ are closed (hence compact) subsets of $\prod_{n<m}[0, \xi_n]$, and thus

$$\bigcup_{n<m} F_k = \prod_{n<m}[0, \xi_n] \setminus \prod_{n<m}[\zeta_n, \xi_n]$$

is Lindelöf and (completely) regular, hence normal. Therefore if $\mathcal{A}$ is a disjoint family of closed subsets of $\prod_{n<m}[0, \xi_n] \setminus \{\infty\}$, then there is at most

[4] We are grateful to professor Peter Nyikos (University of South Carolina, Columbia) for this observation that answers a question formulated in a preliminary version.

one $A \in \mathcal{A}$ which is unbounded in $\prod_{k<n}\{\xi_k\} \times [0,\xi_n] \times \prod_{n<k<m}\{\xi_k\}$ for $0 \leq n < m$, and if (1.1) is non degenerate, then because of the normality of $\prod_{n<m}[0,\xi_n] \setminus \prod_{n<m}[\zeta_n,\xi_n]$ every A in $\mathcal{A}$ is unbounded within $\prod_{k<n}\{\xi_k\} \times [0,\xi_n] \times \prod_{n<k<m}\{\xi_k\}$ for some n. It follows that the non normality of X is not greater than m. On the other hand, by Lemma 2.3 the strong non normality of X is m. ∎

Even if the cardinals ξ_n in the construction above are not distinct, the non normality of a topless cube can be equal to the cube dimension. For example, for each $n < \omega$ the non normality of $\prod_{1 \leq k \leq n}[0,\omega_1] \setminus \{\infty\}$ is n.

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