WEAK REGULARITY AND CONSECUTIVE TOPOLOGIZATIONS AND REGULARIZATIONS OF PRETOPOLOGIES

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In Memory of Jan Pelant

Abstract. L. Foged proved that a weakly regular topology on a countable set is regular. In terms of convergence theory, this means that the topological reflection $T\xi$ of a regular pretopology ξ on a countable set is regular. It is proved that this still holds if ξ is a regular σ-compact pretopology. On the other hand, it is proved that for each $n < \omega$ there is a (regular) pretopology ρ (on a set of cardinality c) such that $(RT)^k\rho > (RT)^n\rho$ for each $k < n$ and $(RT)^n\rho$ is a Hausdorff compact topology, where R is the reflector to regular pretopologies. It is also shown that there exists a regular pretopology of Hausdorff RT-order $\geq \omega_0$. Moreover, all these pretopologies have the property that all the points except one are topological and regular.

1. Introduction

The notion of weak base of a topology τ is equivalent to that of a base of a pretopology, the topologization of which is equal to τ. This fact enables reciprocal transfer and crossbreeding of results between general topology and convergence theory. As we shall see, the framework of convergence theory will enable much richer investigations than it might have been possible in topological terms.

If τ is a topology on X and $\mathcal{B} = \{\mathcal{B}(x) : x \in X\}$ is a collection of filter bases such that the filter generated by $\mathcal{B}(x)$ is finer than the neighborhood filter of x for every $x \in X$, and a subset O of X is open if and only if for every $x \in O$ there is $B \in \mathcal{B}(x)$ such that $x \in B \subset O$, then we say that $\mathcal{B}$ is a *weak base of* τ [2]. A weak base is *Hausdorff* if $x_0 \neq x_1$ implies the existence of $B_0 \in \mathcal{B}(x_0)$ and $B_1 \in \mathcal{B}(x_1)$ such that $B_0 \cap B_1 = \varnothing$. A topology is called *weakly regular* if it admits a Hausdorff weak base $\mathcal{B}$ of closed sets.

Under what additional conditions is a weakly regular topology regular?

Nyikos and Vaughan attribute to Foged [17, Theorem 2.4] the following

Theorem 1.1. *Each weakly regular topology on a countable set is normal, hence regular.*

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Later we provide several examples to the effect that the assumption of countability of the underlying set cannot be dropped. Of course, every (Hausdorff) compact topology is normal, and thus, trivially, each weakly regular (Hausdorff) compact topology is regular. This suggests a possible extension of the Foged theorem.

Let $\mathcal{B}$ be a weak base of a topology τ on X. We say that a family $\mathcal{Q}$ of subsets of X is a $\mathcal{B}$-*cover* if for every $x \in X$ there is $Q \in \mathcal{Q}$ and $B \in \mathcal{B}(x)$ such that $B \subset Q$. A topological space is *compact with respect to* a weak base $\mathcal{B}$ if for every $\mathcal{B}$-cover there exists a finite subfamily, which is a $\mathcal{B}$-cover. If a topological space is compact with respect to a weak base then it is compact, but not conversely (see the topology $T\pi$ of Example 4.6). If a topological space can be represented as a countable union of $\mathcal{B}$-compact sets, then it is called σ-*compact with respect to* $\mathcal{B}$.

Theorem 1.2. *If a topology is weakly regular and σ-compact with respect to the same weak base, then it is normal.*

A *pretopology* π on a set X is a collection of filters $\{\mathcal{V}_\pi(x) : x \in X\}$ such that $x \in V$ for each $V \in \mathcal{V}_\pi(x)$ and every $x \in X$. In particular, each topology τ defines a pretopology via its neighborhood system $\{\mathcal{N}_\tau(x) : x \in X\}$. A pretopology π is *finer* than a pretopology ρ, or ρ is *coarser* than π (in symbols, $\pi \geq \rho$) if $\mathcal{V}_\pi(x) \supset \mathcal{V}_\rho(x)$ for every $x \in X$. The finest topology among those that are coarser than a pretopology π is denoted by $T\pi$. We shall see that $\mathcal{B}$ is a weak base for a topology τ on X if and only if there exists a pretopology π such that $T\pi = \tau$ and $\mathcal{B}(x)$ is a filter base of $\mathcal{V}_\pi(x)$ for each $x \in X$. In these terms, a topology τ is weakly regular whenever there exists a regular pretopology π with $T\pi = \tau$. Therefore it is convenient to investigate questions related to weak bases in the framework of pretopologies.

The category of pretopologies is a *topological category* over the category of sets (see [1]): there exists a forgetful functor $|\cdot|$ that associates, to every pretopology ξ, the underlying set $|\xi|$, and to every morphism (that is, continuous map) $\varphi : \zeta \to \tau$ the underlying map $|\varphi| : |\zeta| \to |\tau|$. It is known that every *concrete endofunctor* F in a topological category[1] is determined by its action on objects of the category [1]. A map F on objects of such a category is the restriction of a concrete endofunctor if and only if (i) $|F\xi| = |\xi|$ (ii) $\xi \leq \zeta$ implies $F\xi \leq F\zeta$ and (iii) $f^-(F\tau) \leq F(f^-\tau)$ for all pretopologies ξ, ζ and τ, and for each map f [9], where $f^-\tau$ stands for the initial pretopology of the pretopology τ with respect to f. Therefore, in our studies, it is enough to consider concrete endofunctors as maps on objects. In particular, the categories of topologies and regular convergences are concretely reflective subcategories of the category of pretopologies. For brevity's sake we shall call the topological reflector the *topologizer* T (the topological reflection $T\xi$ of ξ will be called the *topologization* of ξ), and the reflector to regular pretopologies the *regularizer* R (the regular reflection $R\xi$ of ξ will be called the *regularization* of ξ) (see also [14], [15]).

If π is a pretopology, then neither $RT\pi$ need be topological, nor $TR\pi$ need be regular.

How long can one iterate alternatively the topologization and the regularization before getting to a stand?

We will show that for every ordinal $\gamma \leq \omega_0 + 1$ there is a regular pretopology π such that the γ-th iteration of RT is the first to yield a Hausdorff regular topology,

[1]An endofuctor F is *concrete* if $|Ff| = |f|$ for every morphism f of the category.

which is moreover compact if $\gamma < \omega_0$. If this is still true for an arbitrary ordinal γ, remains an open question.

It is remarkable that the pretopologies, that we use to prove the iteration results mentioned above, are topological and regular everywhere with the exception of a single point. Our construction is based on a concatenation of spaces of the type $\{\infty\}\cup\omega\cup\mathcal{A}$, where $\mathcal{A}$ is a maximal almost disjoint family on ω admitting a Simon's partition, on which a pretopology is constructed with the aid of that partition.

2. Pretopologies, regularity, topologicity

Families $\mathcal{F},\mathcal{H}$ (of subsets of a given set) *mesh* ($\mathcal{F}\#\mathcal{H}$) if $F\cap H\neq\varnothing$ for every $F\in\mathcal{F}$ and for each $H\in\mathcal{H}$. The operation $\#$ is related to the notion of the *grill* $\mathcal{H}^{\#}$ of a family $\mathcal{H}$, which was defined by Choquet [3] as $\mathcal{H}^{\#}=\bigcap_{H\in\mathcal{H}}\{G : G\cap H\neq\varnothing\}$.

A pretopology ξ is defined by assigning a *vicinity filter* $\mathcal{V}_\xi(x)$ to every $x\in|\xi|$ so that $x\in V$ for each $V\in\mathcal{V}_\xi(x)$. The associated convergence of filters is defined by

$$x\in\lim_\xi\mathcal{F}\Leftrightarrow\mathcal{V}_\xi(x)\subset\mathcal{F}.$$

A pretopology ζ is *finer* than a pretopology ξ (in symbols, $\zeta\geq\xi$) if they are defined on the same set X and if $\mathcal{V}_\xi(x)\subset\mathcal{V}_\zeta(x)$ for each $x\in X$. A pretopology is *Hausdorff* if for every pair of distinct elements, the corresponding vicinity filters do not mesh. A subset A of $|\xi|$ is *compact* (respectively, *cover-compact*) if for every family $\mathcal{P}$ of subsets of X such that for each $x\in A$ there is $P\in\mathcal{P}\cap\mathcal{V}_\xi(x)$, there exists a finite subfamily $\mathcal{P}_0$ which is a set-theoretic cover of $|\xi|$ (respectively, such that for each $x\in A$ there is $P\in\mathcal{P}_0\cap\mathcal{V}_\xi(x)$). A cover-compact set is compact, but not conversely (see e.g., [4, Example 8.4]); however if ξ is a topology then the converse also holds.

We notice that if $\mathcal{B}$ is a weak base of a topology, then a set is compact with respect to $\mathcal{B}$ if and only if it is cover-compact for the pretopology determined by $\mathcal{B}$.

An element x belongs to the *adherence* $\mathrm{adh}_\xi H$ of a set H whenever $H\in\mathcal{V}_\xi(x)^{\#}$. If $\mathcal{F}$ is a filter on $|\xi|$, then the symbol $\mathrm{adh}_\xi^\natural\mathcal{F}$ denotes the filter generated by $\{\mathrm{adh}_\xi F : F\in\mathcal{F}\}$.

A pretopology ξ is *regular* if ([2])

$$\mathcal{V}_\xi(x)\subset\mathrm{adh}_\xi^\natural\mathcal{V}_\xi(x) \tag{2.1}$$

for each $x\in|\xi|$. An element x of $|\xi|$ is said to be *regular* [5] if (2.1) holds. Of course, a pretopology is regular if and only if all its elements are regular. The category of regular pretopologies is a concretely reflective subcategory of the category of pretopologies. In particular, the corresponding reflector R, called the *regularizer*, associates with every pretopology ξ the finest regular pretopology $R\xi$ that is coarser than ξ.

A subset O of $|\xi|$ is *open* if $O\in\mathcal{V}_\xi(x)$ for every $x\in O$. A set N is a *neighborhood* of x if there exists an open set O such that $x\in O\subset N$. The family $\mathcal{N}_\xi(x)$ of neighborhoods of x is a filter. A set is *closed* if its complement is open. The least closed set that includes a set A is called the *closure* of A and is denoted by $\mathrm{cl}_\xi A$. It is straightforward that $x\in\mathrm{cl}_\xi A$ if and only if $A\in\mathcal{N}_\xi(x)^{\#}$.

The family of all open sets of a pretopology ξ fulfills all the axioms of open sets of a topology. The corresponding topology is denoted by $T\xi$, where the *topologizer*

[2] This definition is that of Fischer [10]. It is equivalent to that of Grimeisen [12] for pseudotopologies, *a fortiori* for pretopologies.

T is the reflector to the concretely reflective subcategory of topologies. An element x of a pretopological space (X, ξ) is *topological* if $\mathcal{N}_\xi(x) = \mathcal{V}_\xi(x)$.

If $\mathcal{W}(y)$ is a family of subsets of X for every $y \in Y$, and if $\mathcal{F}$ is a family of subsets of Y, then the *contour* of $\mathcal{W}$ along $\mathcal{F}$ is defined by[3]

$$\mathcal{W}(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} \bigcap_{y \in F} \mathcal{W}(y). \tag{2.2}$$

An element of $x \in |\xi|$ is called *topological* if $\mathcal{V}_\xi(x) \subset \mathcal{V}_\xi(\mathcal{V}_\xi(x))$ [6]. Clearly a pretopology ξ is a topology if and only if every $x \in |\xi|$ is topological.

Regular topologies form a concretely reflective subcategory of the category of pretopologies. It turned out [5, Proposition 4.4] that each regular pretopology ξ is *topologically regular*, that is, such that

$$\mathcal{V}_\xi(x) \subset \mathrm{cl}_\xi^\natural \mathcal{V}_\xi(x)$$

for each $x \in |\xi|$, where $\mathrm{cl}_\xi^\natural \mathcal{F}$ denotes the filter generated by $\{\mathrm{cl}_\xi F : F \in \mathcal{F}\}$. However, neither RT nor TR is the reflector to the subcategory of regular topologies. The compositions of two concrete reflectors R, T are contractive functors ([4]), but neither of them is idempotent. If F is a concrete contractive functor of a topological category, then its iterations on an object π are defined by induction $F^0\pi = \pi$ and for $\gamma > 0$,

$$F^\gamma \pi = F(\bigwedge\nolimits_{\alpha < \gamma} F^\alpha \pi).$$

Because each set is well-ordered (in ZFC), for every π there is the least γ (called the *F-order* of π) such that $F^\gamma \pi = F^{\gamma+1}\pi$. If γ is the F-order of π and moreover $F\pi$ is Hausdorff (compact), then we shall say that π is of *Hausdorff F-order* γ (respectively, *compact F-order* γ).

In particular, we can iterate RT (and TR) and for each pretopology there is the least γ such that $(RT)^\gamma \pi$ (respectively $(RT)^\gamma \pi$) is a regular topology. Therefore the *RT-order* and *TR-order* of a pretopology are well-defined.

The case of F being a composition of two contractive functors leads to an additional subtlety in the definition of F-order. If for example, the RT-order of π is $n < \omega$, then $(RT)^{k-1}\pi > T(RT)^{k-1}\pi > (RT)^k \pi$ for each $0 < k < n$, for otherwise either $(RT)^{k-1}\pi$ would be a (regular) topology (and thus $(RT)^{k-1}\pi = (RT)^\alpha \pi$ for each $\alpha \geq k$) or $T(RT)^{k-1}$ would be regular (and a topology) and thus $T(RT)^{k-1}\pi = (RT)^\alpha \pi$ for each $\alpha \geq k$. However it may happen either that $T(RT)^{n-1}\pi > (RT)^n \pi$ or that $T(RT)^{n-1}\pi = (RT)^n \pi$. In the latter case, we shall say that the RT-order of π is *degenerate*.

3. Interplay between regularity and topologicity

If $\mathcal{B}$ is a Hausdorff weak base for a topology τ on X, and for each $x \in X$, we denote by $\mathcal{V}_\pi(x)$ the filter generated by $\mathcal{B}(x)$, then we have defined a pretopology π such that $T\pi = \tau$. If $\mathcal{B}$ consists of closed sets, then π is Hausdorff topologically regular (equivalently, regular). A subset of X is π-compact if and only if it is $\mathcal{B}$-compact.

In these terms, having in mind [5, Proposition 4.4], Theorem 1.1 can be reformulated as follows:

[3] It seems that this notion was first introduced by Kowalsky for filters in [16] under the name of *diagonal operation*.

[4] A concrete functor F is contractive if $F\xi \leq \xi$.

Theorem 3.1. *The topologization of a Hausdorff regular pretopology on a countable set is normal (hence regular).*

As we have said, the assumption in Theorem 3.1 that the underlying set is countable cannot be dropped. In Example 4.6 we construct a regular pretopology ξ, the topologization of which is Hausdorff but not regular. This pretopology is defined on $\{\infty\} \cup \omega \cup \mathcal{A}$, where $\mathcal{A}$ is an arbitrary maximal almost disjoint family on ω. Recall that a is the least cardinal number, for which there is a maximal almost disjoint family of that cardinality.

Therefore,

Theorem 3.2. *There exists a pretopology ξ on a set of cardinality a such that $RT\xi$ is Hausdorff, and*

$$R\xi = \xi > T\xi > RT\xi.$$

In other terms,

Corollary 3.3. *There exists a Hausdorff, non-regular, weakly regular topology on a set of cardinality a.*

The class of pretopologies, for which the regularity implies the regularity of their topologization is larger than those with countable underlying set. For example,

Theorem 3.4. *Each Hausdorff cover-compact regular pretopology π is topological, hence $T\pi = \pi$ is normal (thus regular).*

Actually, this is a special case of a more general fact (due to M. P. Kac [13]; see also [11, 3.17.9]) that every Hausdorff compact regular pseudotopology is a topology. ([5]) Of course, each Hausdorff compact topology is normal, *a fortiori* regular. But the assumption of Hausdorffness in Theorem 3.4 regards a pretopology, and not its topologization. In terms of weak bases, Theorem 3.4 becomes

Corollary 3.5. *If $\mathcal{B}$ is a Hausdorff weak base of closed subsets of a topology that is compact with respect to $\mathcal{B}$, then $\mathcal{B}$ is a base of the topology.*

Here is a common generalization of Theorems 3.1 and 3.4.

Theorem 3.6. *If π is a Hausdorff regular σ-cover-compact pretopology, then $T\pi$ is normal.*

Proof. Let $X = \bigcup_{0 \leq n < \omega} K_n$, where each K_n is a cover-compact set repeated infinitely many times. Let A_0, B_0 be two closed disjoint sets. Suppose that we have constructed ascending sequences of closed sets $A_0, A_1, \ldots, A_n, \ldots$ and $B_0, B_1, \ldots, B_n, \ldots$ such that $A_n \cap B_n = \varnothing$.

If $K_n \cap A_n \neq \varnothing$ then let $\operatorname{cl} Q_n = Q_n \in \mathcal{V}(K_n \cap A_n)$ be disjoint from B_n; set $A_{n+1} = A_n \cup Q_n$. Otherwise $A_{n+1} = A_n$. If $K_n \cap B_n \neq \varnothing$ then let $\operatorname{cl} R_n = R_n \in \mathcal{V}(K_n \cap B_n)$ be disjoint from A_{n+1}; set $B_{n+1} = B_n \cup R_n$. Otherwise $B_{n+1} = B_n$. Let $A = \bigcup_{n<\omega} A_n$ and $B = \bigcup_{n<\omega} B_n$. Then A, B are disjoint. To show that A is open, let $x \in A$. Then there exists $n < \omega$ such that $x \in A_n$. Let $k \geq n$ be the first integer such that $x \in A_n \cap K_k$. Thus $Q_k \in \mathcal{V}(K_k \cap A_k) \subset \mathcal{V}(x)$ and so $A \supset A_{k+1} \in \mathcal{V}(x)$. It follows that A is open. Likewise B is open. □

[5]A convergence is a *pseudotopology* provided that $x \in \lim \mathcal{F}$ if and only if $x \in \lim \mathcal{U}$ for every ultrafilter $\mathcal{U} \supset \mathcal{F}$.

In the language of weak bases, Theorem 3.6 becomes Theorem 1.2.

The topologization of a σ-cover-compact pretopology is σ-compact. We do not know if one can weaken the assumption of Theorem 3.6 to the σ-compactness of $T\pi$. In other words, is a Hausdorff σ-compact weakly regular topology normal (regular)?

In contrast,

Proposition 3.7. *There exists a topology τ on a countable set such that $\tau > R\tau > TR\tau$ and $TR\tau$ is Hausdorff and regular.*

This fact follows from Example 3.9 below. Indeed, the phenomenon is somewhat more general.

We denote by $\mathcal{F} \vee \mathcal{G}$ the *supremum* and by $\mathcal{F} \wedge \mathcal{G}$ the *infimum* of two filters $\mathcal{F}$ and $\mathcal{G}$. If $\mathcal{G}$ is the principal filter generated by G, then we abridge $\mathcal{F} \vee G$ and $\mathcal{F} \wedge G$ respectively.

Let $\{X_v : v \in V\}$ be an infinite family of disjoint infinite sets such that $v \in X_v$ for each $v \in V$ and let $\mathcal{F}$ be a free filter on V. Consider a pretopology ξ_v on X_v for each $v \in V$. Then the *topologizing module* is a pretopology $\tau = \tau(\mathcal{F}; \xi_v : v \in V)$ on $X := \{\infty\} \cup \bigcup_{v \in V} X_v$ such that its restriction to $\bigcup_{v \in V} X_v$ is the coproduct $\bigoplus_{v \in V} \xi_v$ and $\mathcal{V}_\tau(\infty)$ is generated by $\mathcal{F} \wedge \{\infty\}$.

Lemma 3.8. *If $\tau = \tau(\mathcal{F}; \zeta_v : v \in V)$ is a topologizing module such that $\zeta_v > R\zeta_v$ and $R\zeta_v$ is a Hausdorff regular topology for each $v \in V$, then $\tau > R\tau > TR\tau$ and $TR\tau$ is a Hausdorff regular topology.*

Proof. As τ restricted to V is discrete, $\mathcal{V}_\tau(\infty)$ has a base of τ-closed sets, that is, $\mathcal{V}_\tau(\infty) = \mathcal{V}_{R\tau}(\infty)$. Therefore $R\tau(\mathcal{F}; \zeta_v : v \in V) = \tau(\mathcal{F}; R\zeta_v : v \in V)$ and is strictly coarser than τ. All the elements of X with the exception of ∞ are topological in $R\tau$. Therefore $\mathcal{V}_{TR\tau}(\infty) = \mathcal{V}_{R\tau}(\mathcal{V}_\tau(\infty))$, the contour of $\mathcal{V}_{R\tau} = \mathcal{V}_{TR\tau}$ along $\mathcal{V}_\tau(\infty) = \mathcal{V}_{R\tau}(\infty)$. □

Example 3.9. *Let W and X_w be countably infinite sets for each $w \in W$. Let $\mathcal{W}$ be the cofinite filter of W and $\mathcal{X}(w)$ be the cofinite filter of X_w for each $w \in W$. We define a pretopology ζ on the disjoint union $X := \{\infty\} \cup \bigcup_{w \in W} X_w$ so that $\mathcal{V}_\zeta(w)$ is generated by $\mathcal{X}(w) \wedge \{w\}$ and $\mathcal{V}_\zeta(\infty)$ is generated by $\mathcal{X}(\mathcal{W}) \wedge \{\infty\}$. All the other elements are isolated. This is a topology. All the points except ∞ are regular, and $\mathcal{V}_{R\zeta}(\infty) = \mathcal{V}_\zeta(\infty) \wedge \mathcal{W}$. By applying $\tau = \tau(\mathcal{F}; \zeta_v : v \in V)$ with the cofinite filter $\mathcal{F}$ of a countably infinite set V, with X_v being a copy of X and ζ_v a copy of ζ for each $v \in V$, we are in the assumptions of Lemma 3.8. Of course, the underlying set of τ is countably infinite.*

As we shall see below, similar constructions with the inverted role of T and R give rise to regular pretopologies, the topologizations of which are necessarily regular.

Theorem 3.10. *Let ξ_v be a regular pretopology on X_v and $\infty_v \in X_v$ so that all the elements of $X_v \setminus \{\infty_v\}$ are topological and $T\zeta_v$ is a Hausdorff regular topology for each $v \in V$. Let ρ be a regular Hausdorff pretopology on a disjoint union $X := \{\infty\} \cup \bigcup_{v \in V} X_v$ such that $\rho|_{\bigcup_{v \in V} X_v} = \bigoplus_{v \in V} \xi_v$. Then $T\rho$ is regular.*

Proof. Suppose that there is an ultrafilter $\mathcal{U} \# \operatorname{cl}_\rho^\natural \mathcal{V}_{T\rho}(\infty)$ and such that $\mathcal{U}$ does not converge to ∞ in $T\rho$. It follows that $\{\infty_v : v \in V\} \notin \mathcal{U}$. As $\mathcal{U} \# \operatorname{cl}_\rho^\natural \mathcal{V}_{T\rho}(\infty)$

is equivalent to $\mathcal{V}_{T\rho}(\mathcal{U})\#\mathcal{V}_{T\rho}(\infty)$ and all the elements of $\bigcup\{X_v \setminus \{\infty_v\} : v \in V\}$ are topological, we infer that $\mathcal{V}_\rho(\mathcal{U})\#\mathcal{V}_{T\rho}(\infty)$, that is, $\mathcal{V}_\rho(\mathcal{U})\#\mathcal{V}_{T\rho}(\mathcal{V}_\rho(\infty))$, because $\mathcal{V}_{T\rho}(\mathcal{V}_\rho(\infty)) = \mathcal{V}_{T\rho}(\infty)$. It follows that $\mathrm{cl}^\natural_\rho \mathcal{V}_\rho(\mathcal{U})\#\mathcal{V}_\rho(\infty)$, hence $\mathcal{V}_\rho(\mathcal{U})\#\mathcal{V}_\rho(\infty)$, because the elements of $\bigcup\{X_v \setminus \{\infty_v\} : v \in V\}$ are topological. Therefore $\mathcal{U}\#\mathcal{V}_\rho(\mathcal{V}_\rho(\infty))$ and thus $\mathcal{U} \geq \mathcal{V}_{T\rho}(\infty)$, contrary to the assumption. □

4. Modules

To construct pretopologies of prescribed (finite) RT-order, we will use some modifications of the Mrówka-Isbell topology. If A is an infinite subset of ω, then $\mathcal{E}(A)$ denotes the cofinite filter of $A \subset \omega$, that is, $\mathcal{E}(A)$ is the filter generated by the free sequence of the elements of A. Recall that a family $\mathcal{A}$ of infinite subsets of ω is *almost disjoint* (AD) if any two of its elements have finite intersection. If $\mathcal{A}$ is an AD family, then the *Mrówka-Isbell* topology $\tau = \tau_\mathcal{A}$ is defined on a disjoint union $\omega \cup \mathcal{A}$ so that $\mathcal{N}_\tau(A) := \{\{A\} \cup E : E \in \mathcal{E}(A)\}$ is the neighborhood filter of A (seen as an element of $\mathcal{A}$) for every $A \in \mathcal{A}$, and that all the elements of ω are isolated. The topology $\tau_\mathcal{A}$ is locally compact and Hausdorff (because $\mathcal{A}$ is almost disjoint). The Alexandrov compactification of $\tau_\mathcal{A}$ (on a disjoint union $X := \omega \cup \mathcal{A} \cup \{\infty\}$) is called the *Franklin compact* (of $\mathcal{A}$) [18].

We shall consider a disjoint union $X := \omega \cup \mathcal{A} \cup \{\infty\}$, where $\mathcal{A}$ is a *maximal almost disjoint* (MAD) on ω, and a free filter $\mathcal{F}$ on X such that $\mathcal{A} \in \mathcal{F}$. We call a *module of* $\mathcal{F}$ (*over* $\mathcal{A}$) the finest pretopology $\mu = \mu(\mathcal{A}, \mathcal{F})$ such that $\mathcal{E}(A)$ converges to A for every $A \in \mathcal{A}$, and $\mathcal{F}$ converges to ∞. Consequently, $\mathcal{V}_\mu(\infty) = \{\infty\} \wedge \mathcal{F}$ and $\mathcal{V}_\mu(A) = \{A\} \wedge \mathcal{E}(A)$. Of course, the restriction to $\omega \cup \mathcal{A}$ of a module is equal to τ.

Each module is a regular pretopology. More precisely, each $x \in X \setminus \{\infty\}$ is regular and topological for $(RT)^\alpha \mu, T(RT)^\alpha \mu, (TR)^\alpha \mu$ and $R(TR)^\alpha \mu$ for each ordinal α, because the vicinity filters of such an x remain invariant under regularization and topologization. All these pretopologies are Hausdorff.

Notice that $\mathcal{V}_{T\mu}(A) = \mathcal{V}_\mu(A)$ for every $A \in \mathcal{A}$, and

$$\mathcal{V}_{T\mu}(\infty) = \mathcal{V}_\mu(\infty) \wedge \mathcal{E}(\mathcal{F}) = \{\infty\} \wedge \mathcal{F} \wedge \mathcal{E}(\mathcal{F}),$$

where the *contour* $\mathcal{E}(\mathcal{F})$ is defined by (2.2), so that $\mu > T\mu$. The regularization $RT\mu$ of $T\mu$ is described by

$$\mathcal{V}_{RT\mu}(\infty) = \mathrm{cl}^\natural_{T\mu}(\mathcal{V}_{T\mu}(\infty)) = \mathrm{cl}^\natural_\mu(\mathcal{V}_\mu(\infty) \wedge \mathcal{E}(\mathcal{F})) = \mathcal{V}_\mu(\infty) \wedge \mathrm{cl}^\natural_\mu \mathcal{E}(\mathcal{F}). \tag{4.1}$$

Whether $RT\mu$ is strictly coarser than $T\mu$ or not, depends on the module. Consequently the topologization (of a module) can be described with the aid of contours.

It is often insightful to perceive this operation in terms of the Čech-Stone compactification of ω. The (free) Stone transform β^* of the contour $\mathcal{E}(\mathcal{F})$ fulfills

$$\beta^*(\mathcal{E}(\mathcal{F})) = \bigcap_{F \in \mathcal{F}} \mathrm{cl}_\beta \left(\bigcup_{A \in F} \beta^* A\right), \tag{4.2}$$

hence is the *upper Kuratowski limit* of $\mathcal{F}$. The *residual filter* (on ω) of an AD family $\mathcal{A}$ is the contour $\mathcal{E}(\mathcal{F}_\mathcal{A})$ of the cofinite filter $\mathcal{F}_\mathcal{A}$ of $\mathcal{A}$. In the case of $\mathcal{F}_\mathcal{A}$, (4.2) becomes

$$\beta^*(\mathcal{E}(\mathcal{F}_\mathcal{A})) = \mathrm{cl}_\beta \bigcup \{\beta^* A : A \in \mathcal{A}\} \setminus \bigcup \{\beta^* A : A \in \mathcal{A}\}.$$

If moreover $\mathcal{A}$ is maximal, then $\beta^*(\mathcal{E}(\mathcal{F}_\mathcal{A})) = \beta^*\omega \setminus \bigcup \{\beta^* A : A \in \mathcal{A}\}$.

The regularization of this special type of pretopologies can be described in terms of a set-theoretic operation $\mathrm{adh}^{\natural}_{\mathcal{A}}$. If $\mathcal{H}$ is a filter on ω, then $\mathrm{adh}^{\natural}_{\mathcal{A}}\mathcal{H}$ is the filter (on $\mathcal{A}$) generated by

$$\{\mathrm{adh}_{\mathcal{A}} H : H \in \mathcal{H}\},$$

where $\mathrm{adh}_{\mathcal{A}} H = \{A \in \mathcal{A} : \mathrm{card}\,(A \cap H) = \infty\}$. Notice that if $\mu = \mu(\mathcal{A}, \mathcal{F})$, then

$$\mathcal{V}_{RT\mu}(\infty) = \mathcal{V}_{\mu}(\infty) \wedge \mathcal{E}(\mathcal{F}) \wedge \mathrm{adh}^{\natural}_{\mathcal{A}}\mathcal{E}(\mathcal{F}).$$

Of course, if $\mathcal{F}_0, \mathcal{F}_1$ are filters on $\mathcal{A}$ then $\mathcal{F}_0 \le \mathcal{F}_1$ implies that $\mathcal{E}(\mathcal{F}_0) \le \mathcal{E}(\mathcal{F}_1)$, and $\mathcal{H}_0 \le \mathcal{H}_1$ implies that $\mathrm{adh}^{\natural}_{\mathcal{A}}\mathcal{H}_0 \le \mathrm{adh}^{\natural}_{\mathcal{A}}\mathcal{H}_1$. Therefore the operation

$$\mathrm{Adh}_{\mathcal{A}}\mathcal{F} := \mathrm{adh}^{\natural}_{\mathcal{A}}\mathcal{E}(\mathcal{F}) \tag{4.3}$$

is isotone. Finally,

$$\mathcal{F} \ge \mathrm{adh}^{\natural}_{\mathcal{A}}\mathcal{E}(\mathcal{F}). \tag{4.4}$$

Indeed if $B \in \mathrm{adh}^{\natural}_{\mathcal{A}}(\mathcal{E}(\mathcal{F}))$, that is, there is $F \in \mathcal{F}$ and for each $A \in F$, there is $E_A \in \mathcal{E}(A)$ such that $\left\{D \in \mathcal{A} : \mathrm{card}\left(D \cap \bigcup_{A\in F} E_A\right) = \infty\right\} \subset B$. Hence $F \subset B$, and thus, (4.4) holds.

By (4.4) $\mathrm{Adh}_{\mathcal{A}}$ is contractive, thus can be iterated till it becomes stationary. How long does this iteration last for a given free filter $\mathcal{F}$ on $\mathcal{A}$? This is another way of asking about the RT-order of a certain pretopology constructed with the aid of $\mathcal{A}$ and of $\mathcal{F}$.

Lemma 4.1. *If $\mathcal{F}_{\mathcal{A}}$ is the cofinite filter of a MAD family $\mathcal{A}$, then $\mathcal{F}_{\mathcal{A}} = \mathrm{adh}^{\natural}_{\mathcal{A}}\mathcal{E}(\mathcal{F}_{\mathcal{A}})$.*

Proof. If $\mathcal{A}_0$ is a finite subset of $\mathcal{A}$, then $\bigcup_{D\in\mathcal{A}_0} D \cap A$ is finite for each $A \in \mathcal{A} \setminus \mathcal{A}_0$, hence there is $W \in \mathcal{E}(\mathcal{F}_{\mathcal{A}})$ disjoint from $\bigcup_{D\in\mathcal{A}_0} D$ and thus $\mathcal{A}_0 \cap \mathrm{adh}^{\natural}_{\mathcal{A}} W = \varnothing$ showing that $\mathcal{F}_{\mathcal{A}} \le \mathrm{adh}^{\natural}_{\mathcal{A}}\mathcal{E}(\mathcal{F}_{\mathcal{A}})$. □

This means that the cofinite filter of $\mathcal{A}$ is a fixed point of $\mathrm{Adh}_{\mathcal{A}}$.

Corollary 4.2. *If $\mathcal{F}$ is a free filter on a MAD family $\mathcal{A}$, then $\mathrm{Adh}^{\alpha}_{\mathcal{A}}\mathcal{F}$ is free for each ordinal α.*

Proof. A filter $\mathcal{F}$ on $\mathcal{A}$ is free whenever it is finer than the cofinite filter of $\mathcal{A}$, that is, $\mathcal{F} \ge \mathcal{F}_{\mathcal{A}}$. Hence $\mathrm{Adh}_{\mathcal{A}}\mathcal{F} \ge \mathrm{Adh}_{\mathcal{A}}\mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A}}$ by Lemma 4.1, that is, $\mathrm{Adh}_{\mathcal{A}}\mathcal{F}$ is free. □

Corollary 4.3. *For each module $\mu = \mu(\mathcal{A}, \mathcal{F})$ and each ordinal α, the pretopology $(RT)^{\alpha}\mu$ is Hausdorff.*

Example 4.4. *Let $\mathcal{F}_{\mathcal{A}}$ be the cofinite filter of a MAD family $\mathcal{A}$ on ω.*[6] *Its topologization $T\mu$ is homeomorphic to the Alexandrov compactification of the Mrówka-Isbell topology. Therefore the RT-order of the corresponding module is 1, and is degenerate in the sense that $RT\mu = T\mu$. In fact, each free ultrafilter on $\omega \cup \mathcal{A} \cup \{\infty\}$ is either finer than $\mathcal{F}_{\mathcal{A}}$, finer than the cofinite filter $\mathcal{E}(\mathcal{A})$ of A for some $A \in \mathcal{A}$ or finer than the residual filter $\mathcal{E}(\mathcal{F}_{\mathcal{A}})$. Therefore $T\mu$ is a (Hausdorff) compact topology, and in particular, is regular.*

[6]Then the module μ of $\mathcal{F}$ is a Fréchet α_1 pretopology, because all the vicinity filters of non-isolated elements are cofinite filters. It follows (see e.g., [7]) that $T\mu$ is a sequential topology.

Remark 4.5. *Of course,* $\mathrm{card}\,(A\cap H)=\infty$ *if and only if* $\beta^*A\cap\beta^*H\neq\varnothing$*. Consequently, the contour* $\mathcal{E}(\mathrm{adh}^\natural_{\mathcal{A}}\mathcal{H})$ *is a filter on* ω*, and its Stone transform is the upper Kuratowski limit*

$$\bigcap\nolimits_{H\in\mathcal{H}}\mathrm{cl}_\beta\bigcup\{\beta^*A:\beta^*A\cap\beta^*H\neq\varnothing\}\,.$$

Therefore $\beta^*\mathcal{H}\subset\beta^*\left(\mathcal{E}(\mathrm{adh}^\natural_{\mathcal{A}}\mathcal{H})\right)$*, because* $\beta^*\mathcal{H}=\bigcap_{H\in\mathcal{H}}\beta^*H$*, that is,* $\mathcal{E}(\mathrm{adh}^\natural_{\mathcal{A}}\mathcal{H})\leq\mathcal{H}$*.*

Example 4.6. *If* $\mathcal{A}$ *is a MAD family and* $\mathcal{A}_0$ *is a countably infinite subfamily of* $\mathcal{A}$*, and let* $\mathcal{A}_1:=\mathcal{A}\setminus\mathcal{A}_0$*. Denote by* $\mathcal{F}$ *the cofinite filter of* $\mathcal{A}_0$*. We notice that for each* $F\in\mathcal{F}$ *and every choice* $E_A\in\mathcal{E}(A)$ *with* $A\in F$*, there exists* $H\subset\bigcup_{A\in F}E_A$ *such that* $H\cap E_A$ *is a singleton for each* $A\in\mathcal{A}_0$*. Because* $\mathcal{A}$ *is maximal, there exists* $A_H\in\mathcal{A}$ *such that* $H\cap A_H$ *is infinite, hence* $A_H\in\mathcal{A}_1$*, and thus* $A_H\in\mathrm{cl}_\mu\left(\bigcup_{A\in F}E_A\right)$*, which means that* $\mathrm{cl}^\natural_\mu\,\mathcal{E}(\mathcal{F})$ *meshes with* $\mathcal{A}_1$*. Consequently,* $T\mu>RT\mu$*. Actually, the restriction to* $\mathcal{A}_1$ *of every element of* $\mathrm{cl}^\natural_\mu\,\mathcal{E}(\mathcal{F})$ *is uncountable, because the restriction of* $\mathcal{A}$ *to each* $B\in\mathcal{E}(\mathcal{F})$

$$\mathcal{A}\vee_\infty B:=\{A\cap B:\mathrm{card}\,(A\cap B)=\infty,A\in\mathcal{A}\}$$

is infinite and maximal almost disjoint. The pretopology $RT\mu$ *is Hausdorff, because the restriction* $\mathrm{cl}^\natural_\mu\,\mathcal{E}(\mathcal{F})\vee_\infty\mathcal{A}_1$ *of* $\mathrm{cl}^\natural_\mu\,\mathcal{E}(\mathcal{F})$ *to* $\mathcal{A}_1$ *is free. Actually it is easy to see that* $\mathrm{cl}^\natural_\mu\,\mathcal{E}(\mathcal{F})\vee_\infty\mathcal{A}_1$ *is finer than the cocountable filter of* $\mathcal{A}_1$*. Indeed, if* $\mathcal{B}$ *is a countable subfamily of* $\mathcal{A}_1$ *then for each* $A\in\mathcal{A}_0\cup\mathcal{B}$ *there is* $E_A\in\mathcal{E}(A)$ *so that* $\{E_A:A\in\mathcal{A}_0\cup\mathcal{B}\}$ *consists of disjoint sets. Therefore* $\mathrm{cl}_\mu\left(\bigcup_{A\in F}E_A\right)$ *is disjoint from* $\mathcal{B}$*.*

It is essential for the precision of estimates of the *RT-order* of pretopologies constructed later, to find a module π of non-degenerate (Hausdorff) *RT*-order 1, that is, such that $\pi>T\pi>RT\pi=TRT\pi$ and the latter topology is Hausdorff.

In [18] P. Simon showed that there exists a maximal almost disjoint family $\mathcal{A}$ on ω that can be split into two subfamilies $\mathcal{A}_0,\mathcal{A}_1$ so that, if S is an infinite subset of ω such that $\mathcal{A}_j\vee_\infty S$ is maximal, then $\mathcal{A}_j\vee_\infty S$ is finite (for $j\in\{0,1\}$). We call such $\mathcal{A}_0,\mathcal{A}_1$ a *Simon's partition* of $\mathcal{A}$.

Theorem 4.7. *Let* $\mathcal{A}=\mathcal{A}_0\cup\mathcal{A}_1$ *be a Simon's partition. If* $\mathcal{F}_0$ *is the cofinite filter of* $\mathcal{A}_0$*, then the module* $\mu(\mathcal{A},\mathcal{F}_0)$ *fulfills* $\mu>T\mu>RT\mu=TRT\mu$*.*

Proof. The main point is that the contours of the residual filters of $\mathcal{A}_0,\mathcal{A}_1$ and $\mathcal{A}$ are all equal. Indeed, if H is an infinite subset of ω, then $\mathcal{A}_0\vee_\infty H$ is infinite if and only if $\mathcal{A}_1\vee_\infty H$ is infinite, because if $\mathcal{A}_j\vee_\infty H$ is infinite (for $j=0,1$), then it is not maximal, but $\mathcal{A}\vee_\infty H$ is maximal. This means that the boundaries of $\bigcup_{A\in\mathcal{A}_0}\beta^*A$ and $\bigcup_{A\in\mathcal{A}_1}\beta^*A$ are equal, that is, the residual filters of the respective cofinite filters $\mathcal{F}_0$ and $\mathcal{F}_1$ are equal. Since $\mathcal{F}_0\wedge\mathcal{F}_1$ is the cofinite contour $\mathcal{F}$ of $\mathcal{A}$, we have $\mathcal{E}(\mathcal{F}_0)=\mathcal{E}(\mathcal{F})$. As a result, $\mathrm{adh}^\natural_{\mathcal{A}}\,\mathcal{E}(\mathcal{F}_0)$ is the cofinite filter of $\mathcal{A}$ so that $RT\mu$ is homeomorphic to the Alexandrov compactification of the Mrówka-Isbell topology. □

What is the RT-order of the module $\mu(\mathcal{A},\mathcal{F})$ *for a given filter* F *on a MAD family* $\mathcal{A}$*? What is the supremum of the RT-orders of all the modules of a given MAD family* $\mathcal{A}$?

As a by-product of our main quest, we shall provide some elements of reply. A systematic study of the questions above deserves a separate paper.

5. Concatenation of modules

Are there Hausdorff regular pretopologies of every (Hausdorff) RT-order? What are the least cardinalities of the underlying sets of such pretopologies? In a preliminary version of this paper [8] we believed to have answered positively to the (first) question. The proof however contained a gap. We know now that methods based on well-capped trees are not adequate in a construction of a regular pretopology, the topologization of which is not regular. Nevertheless, by using other methods we prove in this section that there exist Hausdorff regular pretopologies (with the underlying sets of cardinality not greater than c) of every RT-order less than or equal to $\omega_0 + 1$. Moreover, for each $\gamma \leq \omega_0$ there is a module of RT-order γ.

Theorem 5.1. *For every cardinal $n < \omega_0$ there is a regular pretopology π (on a set of cardinality not greater than c) of non-degenerate RT-order n and such that $(RT)^n\pi$ is a Hausdorff compact topology.*

Proof. Let $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ be a Simon's partition of a maximal almost disjoint family on ω. Let $\{W_k : k < \omega\}$ be a family of disjoint countably infinite sets, and let $f_k : \omega \to W_k$ be a bijection for every $k < \omega$. Then $f_k(\mathcal{A}) := \{f_k(A) : A \in \mathcal{A}\}$ is a MAD family on W_k. If $\gamma \leq \omega$ let X_γ be a disjoint union

$$X_\gamma := \{\infty\} \cup \bigcup\nolimits_{k<\gamma} W_k \cup \bigcup\nolimits_{k<\gamma} f_k(\mathcal{A}). \tag{5.1}$$

For each $k < \gamma$, the subset $W_k \cup f_k(\mathcal{A})$ of X_γ is endowed with the Mrówka-Isbell topology, that is, for each $A \in \mathcal{A}$ the cofinite filter $\mathcal{E}(f_k(A))$ converges to $f_k(A)$, and all the elements of W_k are isolated. We define a pretopology π_γ by identifying $f_{k-1}(\mathcal{A}_1)$ with $f_k(\mathcal{A}_1)$ for each odd $0 < k < \gamma$ and $f_{k-1}(\mathcal{A}_0)$ with $f_k(\mathcal{A}_0)$ for each even $0 < k < \gamma$, and by setting $\mathcal{V}_{\pi_\gamma}(\infty) = \{\infty\} \wedge \mathcal{F}_0$, where $\mathcal{F}_0$ is the cofinite filter on $f_0(\mathcal{A}_0)$.

For $n < \omega_0$ the pretopology π_n is regular of TR-order n. In fact, if $n = 1$ then π_1 is the module of Theorem 4.7 and $RT\pi_1$ is a Hausdorff compact topology that differs from $T\pi_1$ only at ∞, namely $\mathcal{V}_{RT\pi_1}(\infty) = \mathcal{V}_{T\pi_1}(\infty) \wedge \mathcal{F}_1$ where $\mathcal{F}_1$ is the cofinite filter of $f_0(\mathcal{A}_1)$. Proceeding by induction, we assume that $0 < n < \omega$ and π_n satisfies the requirements of the theorem for $n = n$ so that

$$\mathcal{V}_{(RT)^n\pi_n}(\infty) = \mathcal{V}_{T(RT)^{n-1}\pi_n}(\infty) \wedge \mathcal{F}_n,$$

where $\mathcal{F}_n$ is the cofinite filter of $f_{n-1}(\mathcal{A}_j)$ for $j = 1$ if n is odd and $j = 0$ if n is even. As π_n is the restriction of π_{n+1} to X_n, by Theorem 4.7 with $\mathcal{A}_0$ replaced by $f_n(\mathcal{A}_0)$ if n is even and by $f_n(\mathcal{A}_1)$ if n is odd, we see that $\mathcal{V}_{T(RT)^n\pi_{n+1}}(\infty) = \mathcal{V}_{(RT)^n\pi_{n+1}}(\infty) \wedge \mathcal{E}(\mathcal{F}_n)$ and $\mathcal{V}_{(RT)^{n+1}\pi_{n+1}}(\infty) = \mathcal{V}_{T(RT)^n\pi_{n+1}}(\infty) \wedge \mathcal{F}_{n+1}$ where $\mathcal{F}_{n+1}$ is the cofinite filter of $f_n(\mathcal{A}_j)$, where $j = 1$ if n is even and $j = 0$ if n is odd. Clearly $(RT)^{n+1}\pi_{n+1}$ is a Hausdorff compact topology. □

Remark 5.2. *Consider the pretopology $\pi = \pi_{\omega_0}$ from the proof of Theorem 5.1 on X_{ω_0}. The vicinity filter of ∞ in the pretopological infimum $\bigwedge_{n<\omega_0}(RT)^n\pi$ is the intersection of all $\mathcal{F}_n$ and of all $\mathcal{E}(\mathcal{F}_n)$ for $n < \omega_0$. Hence π is a regular topology.*

Corollary 5.3. *There exists a regular pretopology (on a set of cardinality not greater than c) of Hausdorff RT-order ω_0.*

Corollary 5.4. *For every cardinal $n < \omega_0$ there is a topology (on a set of cardinality not greater than c) of Hausdorff compact TR-order n.*

To see this, it is enough to put $\rho = T\pi$, where π fulfills the conditions of Theorem 5.1.

Actually, the proof of Theorem 5.1 enables us to replace an unspecified pretopology fulfilling the conditions of Theorem 5.1 by a module.

Theorem 5.5. *For every $n < \omega_0$ there is a module π such that $(RT)^n\pi$ is a Hausdorff compact topology and $(RT)^k\pi > T(RT)^k\pi > (RT)^n\pi$ for each $k < n$.*

Proof. Let X_n be given by (5.1), and let $\mathcal{B}$ be the family of subsets of $W := \bigcup_{k<n} W_k$ consisting of all the elements of $f_0(\mathcal{A}_0), f_{n-1}(\mathcal{A}_j)$ (where $j = 1$ if n is odd and $j = 0$ if n is even) and of the unions $f_k(A) \cup f_{k+1}(A)$ where $0 \leq k < n$ and $A \in \mathcal{A}_1$ if k is even and $A \in \mathcal{A}_0$ if k is odd.

The so defined $\mathcal{B}$ is a MAD family on W. Therefore the pretopology π_n on X_n defined in the proof of Theorem 5.1 is in fact the module of $\mathcal{F}_0$ (the cofinite filter of $f_0(\mathcal{A}_0)$ over $\mathcal{B}$). □

Theorem 5.6. *There is a module of Hausdorff RT-order greater than or equal to ω_0.*

Proof. Let X_ω be given by (5.1), and let $\mathcal{B}$ be the family of subsets of $W := \bigcup_{k<\omega} W_k$ consisting of all the elements of $f_0(\mathcal{A}_0)$ and of the unions $f_k(A) \cup f_{k+1}(A)$ where $0 \leq k < n$ and $A \in \mathcal{A}_1$ if k is even and $A \in \mathcal{A}_0$ if k is odd. Of course, $\mathcal{B}$ is almost disjoint but not maximal. Let $\mathcal{A}_\infty$ be a family on W such that $\mathcal{B} \cup \mathcal{A}_\infty$ is MAD. Let μ be the module of the cofinite filter of $f_0(\mathcal{A}_0)$ over $\mathcal{B}$ on W. Then $(RT)^n \mu > T\,(RT)^n \mu > (RT)^{n+1} \mu$ for each $n < \omega$. The infimum $\mu_\infty := \bigwedge_{n<\omega_0} (RT)^n\mu$ (in the lattice of pretopologies) turns out to be topological. This follows from the equality

$$\mathcal{E}(\bigwedge\nolimits_{n<\omega_0} \mathcal{F}_n) = \bigwedge\nolimits_{n<\omega_0} \mathcal{E}(\mathcal{F}_n).$$

In fact, if $B \in \bigwedge_{n<\omega_0} \mathcal{E}(\mathcal{F}_n)$ then for each $n < \omega$ there is $F_n \in \mathcal{F}_n$ such that $B \in \mathcal{E}(A)$ for each $A \in F_n$. In other words, there is $F = \bigcup_{n<\omega} F_n \in \bigwedge_{n<\omega_0} \mathcal{F}_n$ such that $B \in \mathcal{E}(A)$ for each $A \in F$, that is, $B \in \mathcal{E}(\bigwedge_{n<\omega_0} \mathcal{F}_n)$. The converse is always true. We shall see that $\mu_\infty > R\mu_\infty$. Indeed, if $w_k \in W_k$ for each $k < \omega$, then $\{w_k : k < \omega\}$ has infinite intersection with an element A of $\mathcal{A}_\infty$. It follows that the trace $\mathrm{adh}^\natural_{\mathcal{A}_\infty} \mathcal{N}_{\mu_\infty}(\infty)$ of $\mathrm{adh}^\natural_{\mu_\infty} \mathcal{N}_{\mu_\infty}(\infty)$ on $\mathcal{A}_\infty$ is non-degenerate. We have proved that $(RT)^n\mu > (RT)^\omega\mu$ for each $n < \omega$. □

Actually, it can be shown that $\mathrm{adh}^\natural_{\mathcal{A}_\infty} \mathcal{N}_{\mu_\infty}(\infty)$ is finer than the cocountable filter of $\mathcal{B} \cup \mathcal{A}_\infty$. The construction in the proof above enables us to make one more step.

Proposition 5.7. *There is a module of Hausdorff RT-order equal to or greater than $\omega_0 + 1$.*

Proof. Take the "mirror image" with respect to $\mathcal{A}_\infty$ of the module from the proof of Theorem 5.6, that is, let h be a one-to-one map defined on W, and consider a disjoint union

$$X := \{\infty\} \cup W \cup \mathcal{B} \cup \mathcal{A}_\infty \cup h(\mathcal{B}) \cup h(W).$$

We define the following pretopology ζ on X. For each $B \in \mathcal{B}$, let $\mathcal{V}_\zeta(B) := \{B\} \wedge \mathcal{E}(B)$ and $\mathcal{V}_\zeta(h\,(B)) := \{h\,(B)\} \wedge \mathcal{E}(h\,(B))$. For every $A \in \mathcal{A}_\infty$ we set $\mathcal{V}_\zeta(A) := \{A\} \wedge \mathcal{E}(A \cup h(A))$ and $\mathcal{V}_\zeta(\infty) := \{\infty\} \wedge f_0(\mathcal{A}_0)$. As the so defined family $\mathcal{B} \cup$

$\mathcal{A}_\infty \cup h(\mathcal{B})$ on $W \cup h(W)$ is MAD, ζ is in fact a module. From the proof of Theorem 5.6, it follows that $\mathcal{H} := \mathrm{adh}^{\natural}_{\mathcal{A}_\infty} \mathcal{N}_{\mu_\infty}(\infty)$ converges to ∞ for $(RT)^{\omega_0} \zeta$ but not in $T\left(\bigwedge_{n<\omega_0}(RT)^n\zeta\right) = \bigwedge_{n<\omega_0}(RT)^n\zeta$. Therefore, $\mathcal{E}(\mathcal{H})$ has non-degenerate trace on $h(W)$ and converges to ∞ for $T\,(RT)^{\omega_0}\,\zeta$ but not in $(RT)^{\omega_0}\,\zeta$, so that $(RT)^{\omega_0}\,\zeta > (RT)^{\omega_0+1}\,\zeta$. □

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