COMPLETENESS NUMBER OF FAMILIES OF SUBSETS OF CONVERGENCE SPACES

SZYMON DOLECKI

In honor of Professor Eva Lowen Colebunders

Abstract. Compactoid and compact families generalize both convergent filters and compact sets. This concept turned out to be useful in various quests, like Scott topologies, triquotient maps and extensions of the Choquet active boundary theorem.

The completeness number of a family in a convergence space is the least cardinality of collections of covers for which the family becomes complete. 0-completeness amounts to compactness, finite completeness to relative local compactness and countable completeness to Čech completeness. Countably conditional countable completeness amounts to pseudocompleteness of Oxtoby. Conversely, each completeness class of families can be represented as a class of conditionally compactoid families. In this framework, the theorem of Tikhonov for compactoid filters becomes a special case of the theorem on the completeness number of products.

A characterization of completeness in terms of non-adherent filters not only provides a unified language for convergence and completeness, but also clarifies preservation mechanisms of completeness number under various operations.

1. Introduction

Completeness of a convergence is a notion relative to that of *fundamental* (or *Cauchy*) filter. An abstract approach to completeness consists in declaring a class $\mathbb{C}$ of filters to be *Cauchy* filters whenever if $\mathcal{F} \in \mathbb{C}$ and $\mathcal{F} \subset \mathcal{G}$ then $\mathcal{G} \in \mathbb{C}$, if $\mathcal{F}, \mathcal{G} \in \mathbb{C}$ then $\mathcal{F} \cap \mathcal{G} \in \mathbb{C}$ and if all principal ultrafilters belong to $\mathbb{C}$, was adopted in the book [13] of Eva Lowen-Colebunders.

Two types of completeness of topological spaces have been mainly considered in the literature. The first qualifies a topology as *complete* if each fundamental filter is *convergent* (for example, metric completeness), the second, if each fundamental filter is *adherent* (for instance, Čech completeness). In some cases the two notions coincide; this happens, for example, in metric spaces where fundamental filters are defined as those that contain elements of arbitrarily small diameter.

In this paper, I shall study the second type of completeness in a broader framework of convergence spaces. Traditionally, fundamental filters have

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been defined in terms of collections of covers. In [4] I proposed to relate fundamental filters to collections of non-adherent families, a *non-adherent family* being a dual concept of that of *cover*. An advantage of this dual approach is to make evident the reasons of preservation of completeness under several types operations.

A *preconvergence* ξ on a set X is a relation between $\mathbb{F}(X)$ (the set of filters on X) and X, denoted by $x \in \lim_\xi \mathcal{F}$, such that $\mathcal{F} \subset \mathcal{G}$ implies that $\lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G}$. A preconvergence ξ is a *convergence* if $x \in \lim_\xi \{x\}^\uparrow$, where $\{x\}^\uparrow := \{A \subset X : x \in A\}$ is the principal ultrafilter of x. Then we say that $\mathcal{F}$ *converges* to x, equivalently, x is a *limit* of $\mathcal{F}$ ([1,2]). We denote by $|\xi|$ the underlying set of ξ.

We say that A *meshes* B

$$A\#B$$

if $A \cap B \neq \varnothing$. This simple but useful relation is of course symmetric. It extends to families of sets: $\mathcal{A}\#\mathcal{B}$ means that $A\#B$ for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$. If $\mathcal{A} = \{A\}$ then we write $A\#\mathcal{B}$ to abridge $\{A\}\#\mathcal{B}$. The *grill* $\mathcal{A}^\#$ of a family of subsets of a set X was defined by G. Choquet in [1] as

$$\mathcal{A}^\# := \{H \subset X : H\#\mathcal{A}\}.$$

The *adherence* of a family $\mathcal{H}$ of subsets of a convergence space is defined by

$$\operatorname{adh}_\xi \mathcal{H} := \bigcup\nolimits_{\mathcal{F}\#\mathcal{H}} \lim_\xi \mathcal{F}.$$

We say that a filter $\mathcal{H}$ on $|\xi|$ is *adherent* if $\operatorname{adh}_\xi \mathcal{H} \neq \varnothing$, and *non-adherent* if $\operatorname{adh}_\xi \mathcal{H} = \varnothing$.

A family $\mathcal{P}$ of subsets of a convergence space X is said to be a *cover* of a subset A of X

$$\mathcal{P} \succ_\xi A \tag{1.1}$$

provided that $\mathcal{F} \cap \mathcal{P} \neq \varnothing$ ([3]) for each filter $\mathcal{F}$ such that $A\# \lim_\xi \mathcal{F}$ ([4]).

2. Compactoid families

Compact families constitute a common generalization of compact sets and convergent filters. The concept of compact filters was studied in [11],[7] and an akin notion was applied in [18], but already Urysohn considered what can be called sequentially compact sequences [16]. In [8] it became clear that one needs to extend the concept of compactness to arbitrary families of sets in order to characterize open sets for the *Scott convergence* on the hyperspace of open sets (dually, the *upper Kuratowski convergence* on the

[1]Some authors give additional conditions, like $\lim(\mathcal{F} \cap \mathcal{G}) \subset \lim \mathcal{F} \cap \lim \mathcal{G}$. I call such convergences *prototopologies.*

[2]If $\mathcal{B}$ is a filter-base on X, then we write $x \in \lim \mathcal{B}$ whenever $x \in \lim \mathcal{B}^\uparrow$, where $\mathcal{B}^\uparrow := \{F \subset X : \exists (B \in \mathcal{B})\ B \subset F\}$.

[3]That is, there is $P \in \mathcal{P}$ such that $P \in \mathcal{F}$.

[4]In particular, $\mathcal{P}$ is a cover of a topological space X if and only if $\bigcup_{P \in \mathcal{P}} \operatorname{int} P \supset X$.

hyperspace of closed sets). Later it turned out that compact families are essential in very useful characterizations of triquotient maps [15].

Let ξ be a convergence on a set X and let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of X. We say that $\mathcal{A}$ is *ξ-compact at $\mathcal{B}$* if for every filter $\mathcal{H}$ on X,

$$\mathcal{H}\#\mathcal{A} \Longrightarrow \operatorname{adh}_\xi \mathcal{H} \in \mathcal{B}^{\#}. \tag{2.1}$$

In particular, $\mathcal{A}$ is *compactoid* whenever $\mathcal{A}$ is compact at X, $\mathcal{A}$ is *compact* if it is compact at itself. It was observed in [3] that compactness is a pseudotopological notion, that is, a family is ξ-compact (at another family) if and only if it is $S\xi$-compact, where $S\xi$ stands for the pseudotopological modification of ξ. A fundamental fact about compactness is the following generalization of the Tikhonov theorem. Let J be an non-empty set and let ξ_j be a convergence on a non-empty set X_j for each $j \in J$ ([5]).

Theorem 2.1 (Tikhonov)**.** *A filter $\mathcal{F}$ is $\prod_{j\in J} \xi_j$-compactoid if and only if $p_j[\mathcal{F}]$ is ξ_j-compactoid for each $j \in J$.*

In fact, Theorem 2.1 is a consequence of the commutation of the pseudotopological modifier with arbitrary products applied to the characteristic preconvergences ([5]).

If $\mathbb{H}$ is a class of filters (possibly depending on the convergence), then $\mathbb{H}(\xi)$ denotes the set of filters on $|\xi|$ belonging to $\mathbb{H}$. In particular, if $\mathbb{H}$ is *independent* (of the convergence), that is, whenever $\mathbb{H}(\xi) = \mathbb{H}(\zeta)$ provided that $|\xi| = |\zeta|$, then we write $\mathbb{H}X$ rather than $\mathbb{H}(\xi)$, where $X := |\xi|$. Often it is clear from the context which the underlying set of a filter $\mathcal{H}$ and which convergence on that set is considered, so that it is enough to write $\mathcal{H} \in \mathbb{H}$ to determine the set of studied filters.

If $\mathbb{H}$ is a class of filters, then a family $\mathcal{A}$ is said to be *$\mathbb{H}$-compact at $\mathcal{B}$* if (2.1) holds for every filter $\mathcal{H} \in \mathbb{H}X$. In particular, for the class $\mathbb{F}_1$ (of countably based filters), $\mathbb{F}_1$-compactness corresponds to *countable compactness*, $\mathbb{F}_{1\wedge}$ (the class of countably deep filters) $\mathbb{F}_{1\wedge}$-compactness corresponds to *Lindelöf property*, that of $\mathbb{F}_0$, the class of principal filters, $\mathbb{F}_{1\wedge}$-compactness is *finite compactness*, that is trivial in the case of sets, but is interesting and useful for families (see, e.g., [2]). As $\mathbb{F}$ stands for the class of all filters, $\mathbb{F}$-compactness is equivalent to compactness. Incidentally, *sequential ξ-compactness* is equivalent to countable $\operatorname{Seq}\xi$-compactness, where $\operatorname{Seq}\xi$ is the sequential modification of ξ.

Continuous maps preserve $\mathbb{H}$-compactness for the classes $\mathbb{H}$ of filters fulfilling

$$\mathcal{H} \in \mathbb{H}(\tau) \Longrightarrow f^{-}[\mathcal{H}] \in \mathbb{H}(\xi)$$

for $f \in C(\xi, \tau)$. This is the case of (idependent) classes $\mathbb{F}, \mathbb{F}_0, \mathbb{F}_1$ and $\mathbb{F}_{1\wedge}$.

In general, $\mathbb{H}$-compactness is not preserved by products as in Theorem 2.1. This is a well known fact already in very special cases, for instance, the

[5]If $\mathcal{A}$ is a family of subsets of X and $f : X \to Y$, then $f[\mathcal{A}] := \{f(A) : A \in \mathcal{A}\}$. If X_j is a set, then $p_k : \prod_{j\in J} X_j \to X_k$ is the projection on the k-th component.

product of two countably compact topologies need not be countably compact (see, e.g., [9, Example 3.10.19]). In fact, there is an intimate relation between $\mathbb{H}$-compactness and $\mathbb{H}$-*adherence-determined* convergences, that is, the fixed points of the $\mathbb{H}$-*adherence modifier* defined by

$$\lim_{A_{\mathbb{H}}\xi} \mathcal{F} := \bigcap\nolimits_{\mathbb{H}\ni\mathcal{H}\#\mathcal{F}} \operatorname{adh}_\xi \mathcal{H}.$$

The $\mathbb{H}$-*adherence modifier* $A_{\mathbb{H}}$ associates to each (pre)convergence ξ, the finest $\mathbb{H}$-adherence-determined (pre)convergence $A_{\mathbb{H}}\xi$ ([6]).

For example, $\mathbb{F}$-adherence-determined convergences coincide with *pseudotopologies*, $\mathbb{F}_1$-adherence-determined with *paratopologies*, $\mathbb{F}_0$-adherence-determined with *pretopologies*. It turns out that $\mathbb{H}$-compactness is preserved by a product whenever $A_{\mathbb{H}}$ commutes with that product.

A family $\mathcal{A}$ of subsets of X is said to be *isotone* if $B \supset A \in \mathcal{A}$ implies $B \in \mathcal{A}$. Let us denote by $\kappa(\mathbb{H})$ the set of $\mathbb{H}$-compact isotone families.

Theorem 2.2. *The class of all $\mathbb{H}$-compact isotone families on a convergence space X fulfills the axioms of open sets of a topology on the hyperspace 2^X, that is,*

$$\varnothing, 2^X \in \kappa(\mathbb{H}), \tag{2.2}$$

$$\mathcal{A} \subset \kappa(\mathbb{H}) \Longrightarrow \bigcup \mathcal{A} \in \kappa(\mathbb{H}), \tag{2.3}$$

$$\mathcal{A} \subset \kappa(\mathbb{H}), \operatorname{card} \mathcal{A} < \infty \Longrightarrow \bigcap \mathcal{A} \in \kappa(\mathbb{H}). \tag{2.4}$$

Proof. Let $\mathbb{H}$ be an arbitrary non-empty class of filters and let ξ be a convergence. If $\varnothing$ is the empty family of subsets of $|\xi|$, then $\operatorname{adh}_\xi \mathcal{H} \in \varnothing^{\#}$ is true for every filter $\mathcal{H} \in \mathbb{H}$ (since there is no $A \in \varnothing$, the condition $A \cap \operatorname{adh}_\xi \mathcal{H} \neq \varnothing$ is true for each $A \in \varnothing$). On the other hand, since $\varnothing \in 2^{|\xi|}$, no filter $\mathcal{H}$ meshes with $2^{|\xi|}$, thus the condition is emptily fulfilled.

If $\mathcal{H} \in \mathbb{H}$ is a filter such that $\mathcal{H}\# \bigcup_{i\in I} \mathcal{A}_i$, then $\mathcal{H}\#\mathcal{A}_i$ for each $i \in I$ so that $\operatorname{adh}_\xi\mathcal{H} \in \mathcal{A}_i^{\#}$, because $\mathcal{A}_i$ is $\mathbb{H}$-compact for each $i \in I$, that is, $\operatorname{adh}_\xi\mathcal{H} \in \left(\bigcup_{i\in I} \mathcal{A}_i\right)^{\#}$.

Suppose that $\operatorname{adh}_\xi\mathcal{H} \notin \left(\bigcap_{i\in I} \mathcal{A}_i\right)^{\#} = \bigcup_{i\in I} \mathcal{A}_i^{\#}$ ([7]) for some $\mathcal{H} \in \mathbb{H}$,that is, $\operatorname{adh}_\xi\mathcal{H} \notin \mathcal{A}_i^{\#}$ for each $i \in I$. Then for each $i \in I$ there is $H_i \in \mathcal{H}$ with $H_i \notin \mathcal{A}_i^{\#}$. As I is finite, $\bigcap_{i\in I} H_i \in \mathcal{H}$ and $\bigcap_{i\in I} H_i \notin \bigcup_{i\in I} \mathcal{A}_i^{\#} = \left(\bigcap_{i\in I} \mathcal{A}_i\right)^{\#}$, we infer that $\mathcal{H}$ does not mesh $\bigcap_{i\in I} \mathcal{A}_i$. ∎

Often one considers the restriction

$$\kappa_{\mathcal{O}}(\mathbb{H}) := \{\mathcal{A} \cap \mathcal{O} : \mathcal{A} \in \kappa(\mathbb{H})\},$$

where $\mathcal{O}$ is a family of open sets. A subfamily $\mathcal{A}$ of $\mathcal{O}$ is called *openly isotone* if $A \in \mathcal{A}$ and $A \subset O \in \mathcal{O}$, then $O \in \mathcal{A}$.

[6] $A_{\mathbb{H}}$ is a concrete reflector under some natural conditions on a class $\mathbb{H}$.

[7] This equality holds for isotone families (see, e.g., [6, (2.1)]).

Corollary 2.3. *The class of all $\mathbb{H}$-compact openly isotone families of open sets fulfills the axioms of open sets of a topology on the hyperspace of all open sets.*

In particular, if $K \subset X$ and $\mathcal{O}_X(K)$ stands for the family of open sets including K, then $\mathcal{O}_X(K)$ is a compact family if and only if K is a compact set. Therefore for each family $\mathcal{L}$ of compact sets, $\bigcup_{K\in\mathcal{L}} \mathcal{O}_X(K)$ is a compact family. A topological space X is called *consonant* if each openly isotone compact family is of this form [7] ([8]).

3. Completeness

If ξ is a convergence and $\mathbb{P}$ is a collection of families of subsets of $|\xi|$, then a filter $\mathcal{F}$ (on $|\xi|$) is called $\mathbb{P}$-*fundamental* if $\mathcal{F} \cap \mathcal{P} \neq \varnothing$ for each $\mathcal{P} \in \mathbb{P}$. We denote by $\mathbb{F}_{\mathbb{P}}$ the set of $\mathbb{P}$-fundamental filters.

A convergence is called $\mathbb{P}$-*complete* if each $\mathbb{P}$-fundamental filter is adherent (equivalently, if each $\mathbb{P}$-fundamental ultrafilter is convergent, or else whenever each $\mathbb{P}$-fundamental filter is compactoid) ([9]). Notice that if a convergence is $\mathbb{P}$-complete and $\mathbb{S}$ is a collection such that for each $\mathcal{P} \in \mathbb{P}$ there exists a refinement ([10]) S of $\mathcal{P}$ such that $\mathcal{S} \in \mathbb{S}$, then it is also $\mathbb{S}$-complete.

For a family $\mathcal{P}$ of subsets of X, let

$$\mathcal{P}^{\cup} := \left\{\bigcup\nolimits_{Q\in\mathcal{Q}} Q : \mathcal{Q} \subset \mathcal{P}, \operatorname{card}\mathcal{Q} < \infty\right\},\ \mathcal{P}^{\downarrow} := \left\{Q : \underset{P\in\mathcal{P}}{\exists} Q \subset P\right\}.$$

Consequently, $\mathcal{P}^{\cup\downarrow} = \mathcal{P}^{\downarrow\cup}$ is the least (possibly degenerate) *ideal* including $\mathcal{P}$. Let $\mathbb{P}_{\cup} := \{\mathcal{P}^{\cup} : \mathcal{P} \in \mathbb{P}\}$ and $\mathbb{P}_{\cup\downarrow} := \left\{\mathcal{P}^{\cup\downarrow} : \mathcal{P} \in \mathbb{P}\right\}$. It is straightforward that

Proposition 3.1. *A convergence is $\mathbb{P}$-complete if and only if it is $\mathbb{P}_{\cup}$-complete if and only if it is $\mathbb{P}_{\cup\downarrow}$-complete.*

A convergence ξ is said to be κ-*complete* if there exists a collection $\mathbb{P}$ of *covers* with $\operatorname{card}\mathbb{P} \leq \kappa$ such that ξ is $\mathbb{P}$-complete. The least cardinal κ for which ξ is κ-complete is called the *completeness number* $\operatorname{compl}(\xi)$ of ξ. As will become clear from our dual approach,

$$\operatorname{compl}(\xi) \leq 2^{2^{\operatorname{card}|\xi|}} \tag{3.1}$$

for each convergence ξ.

Every filter is $\varnothing$-fundamental (where $\varnothing$ stands for the empty collection of families of sets). Therefore ([11])

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[8] Much research has been done since [7] on consonant topologies.

[9] Let ξ, θ be convergences on X. A collection $\mathbb{P}$ is called θ-*openly* ξ-*complete* if every θ-open $\mathbb{P}$-fundamental filter is ξ-adherent. Every ξ-complete collection is θ-openly ξ-complete for every θ. If $\xi = \theta$ is fixed, then we say *openly complete*. In [10] Frolík uses the term *complete* for what I call here *openly complete*.

[10] $\mathcal{R}$ is called a *refinement* of $\mathcal{P}$ if for each $R \in \mathcal{R}$ there is $P \in \mathcal{P}$ such that $R \subset P$.

[11] Mind that no separation axiom is used in our definitions of compactness and local compactness.

Proposition 3.2. *A convergence is compact if and only if its completeness number is* 0.

A convergence is called *locally compactoid* if each convergent filter contains a compactoid set. If $\mathcal{P}$ is a family of subsets of $|\xi|$, then ξ is $\{\mathcal{P}\}$-complete whenever $\operatorname{adh}_\xi \mathcal{H} \neq \varnothing$ for every $P \in \mathcal{P}$ and each filter $\mathcal{H}$ such that $P \in \mathcal{H}$. Hence

Proposition 3.3. *A convergence is locally compactoid if and only if its completeness number is finite (equivalently* 1*).*

Equivalently, a convergence is locally compactoid if and only if it admits a cover consisting of compactoid sets.

I call a convergence *countably complete* if its completeness number is countable (or 0). In these terms, a topology is Čech-complete whenever it is completely regular and countably complete.

Of course, the space of *rational numbers* $\mathbb{Q}$ is not complete (in the traditional terminology), that is, the completeness number of $\mathbb{Q}$ is not countable. A set is called *dominating* if it is cofinal ([12]) in $(\omega^\omega, \leq^*)$ ([13]) and the *dominating number* $\mathfrak{d}$ is the least cardinality of a dominating set ([14]). We shall see that

Theorem 3.4. *The completeness of the space of rational numbers is* $\mathfrak{d}$.

4. Conditional completeness

If $\mathbb{P}$ is a collection and $\mathbb{H}$ is a class of filters, then

$$\mathbb{H}_\mathbb{P} := \mathbb{F}_\mathbb{P} \cap \mathbb{H}$$

denotes the collection of $\mathbb{P}$-fundamental filters that belong to $\mathbb{H}$. A convergence is called $\mathbb{H}$-*conditionally* $\mathbb{P}$-*complete* if each $\mathbb{P}$-fundamental filter from $\mathbb{H}$ is adherent. It is said to be $\mathbb{H}$-*conditionally* κ-*complete* if there exists a collection $\mathbb{P}$ of *covers* with $\operatorname{card} \mathbb{P} \leq \kappa$ such that it is $\mathbb{H}$-conditionally $\mathbb{P}$-complete. Finally, the least cardinal κ for which a convergence ξ is $\mathbb{H}$-conditionally κ-complete is called the $\mathbb{H}$-*conditional completeness number* of ξ and is denoted by

$$\operatorname{compl}_\mathbb{H}(\xi).$$

Of particular interest is $\mathbb{F}_1$-conditional completeness number, where $\mathbb{F}_1$ stands for the class of countably based filters. In other words, $\mathbb{F}_1$-conditional completeness is expressed in terms of fundamental filters that are countably based.

[12] A subset G of an ordered set X is called *cofinal* if for every $x \in X$ there exists $g \in G$ such that $x \leq g$.

[13] Where $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n.

[14] It is known that the least cardinal of a cofinal subset of $(\omega^\omega, \leq)$ is equal to $\mathfrak{d}$ (see [17]).

Example 4.1 ($\mathbb{F}_1$-conditional countable completeness)**.** Oxtoby [14] defines a *pseudobase* as $\mathcal{B} \subset \mathcal{O}_X \setminus \{\varnothing\}$ such that for each $O \in \mathcal{O}_X$ there is $B \in \mathcal{B}$ with $B \subset O$. He calls a topology *pseudocomplete* if

1. for each $\varnothing \neq O \in \mathcal{O}_X$ there is $\varnothing \neq P \in \mathcal{O}_X$ such that $\mathrm{cl}_X P \subset O$;
2. there exists a sequence of pseudobases $\{\mathcal{B}(n) : n \in \mathbb{N}\}$ such that if $U_n \in \mathcal{B}(n)$ and $\mathrm{cl}_X U_{n+1} \subset U_n$ for each n, then $\bigcap_n U_n \neq \varnothing$ ([15]).

It is clear that each pseudocomplete topology is an $\mathbb{F}_1$-conditionally countably complete convergence.

5. Cocompletness

If $\mathcal{A}$ is a family of subsets of X, then we set $\mathcal{A}_c := \{X \setminus A : A \in \mathcal{A}\}$ and if $\mathbb{A}$ is a collection of families, then we write $\mathbb{A}_\neg := \{\mathcal{A}_c : \mathcal{A} \in \mathbb{A}\}$. Let us recall a fundamental fact relating covers and non-adherent families ([3],[4]). If ξ is a convergence, $A \subset |\xi|$ and $\mathcal{P} \subset 2^{|\xi|}$ then

$$\mathcal{P} \succ_\xi A \Longleftrightarrow A \cap \mathrm{adh}_\xi \mathcal{P}_c = \varnothing,$$

where $\succ_\xi$ was defined in (1.1).

Let $\mathbb{G}$ be a collection (of families of subsets) of $|\xi|$. A filter $\mathcal{F}$ is called $\mathbb{G}$-*cofundamental* if $\mathcal{F}\#\mathcal{G}$ implies that $\mathcal{G} \notin \mathbb{G}$. A filter is $\mathbb{G}$-cofundamental if and only if it is $\mathbb{G}_\neg$-fundamental. Indeed a filter $\mathcal{F}$ does not mesh a family $\mathcal{G}$ whenever there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = \varnothing$, equivalently $F \subset G^c$, that is, $G^c \in \mathcal{F}$ so that $\mathcal{F} \cap \mathcal{G}_c \neq \varnothing$.

A convergence is called $\mathbb{G}$-*cocomplete* if each $\mathbb{G}$-cofundamental filter is compactoid. Of course, a convergence is $\mathbb{G}$-cocomplete if and only if it is $\mathbb{G}_\neg$-complete. Accordingly, $\mathbb{F}_{\mathbb{G}_\neg}$ is the class of cofundamental filters and thus

Corollary 5.1. *A convergence is $\mathbb{G}$-cocomplete if and only if it is $\mathbb{G}_\neg$-complete if and only if it is $\mathbb{F}_{\mathbb{G}_\neg}$-compact.*

Let us illustrate a clarifying role of the cocompleteness point of view.

Example 5.2 (countably complete non-locally-compact space)**.** The space of irrational numbers is countably complete, but not locally compact. Let $X := \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers with the subspace topology. Of course, X is dense but not open in $\mathbb{R}$. As $\mathbb{R}$ is topological, X is not locally relatively compact.

For every $q \in \mathbb{Q}$, the filter $\mathcal{N}(q) \vee X$ is non-adherent in X. As well, the filters $\mathcal{N}(+\infty) \vee X$ and $\mathcal{N}(-\infty) \vee X$ are non-adherent ([16]). Then X is cocomplete with respect to the countable collection

$$\{\mathcal{N}(q) \vee X : q \in \mathbb{Q} \cup \{-\infty, +\infty\}\},$$

[15] Notice that if $\mathcal{P}$ is a cover, then $\mathcal{P}^\downarrow$ is a pseudobase. The converse is not true. For example, all the intervals (a,b) such that $a < b$ and $0 \notin (a,b)$ is such a pseudobase.

[16] $\mathcal{N}(+\infty)$ is the filter on $\mathbb{R}$ generated by $\{(n,\infty) : n \in \mathbb{N}\}$ and $\mathcal{N}(-\infty)$ is the filter on $\mathbb{R}$ generated by $\{(-\infty, n) : n \in \mathbb{N}\}$.

because if $\mathcal{U}$ is a free ultrafilter on X, then either there is a bounded set in $\mathcal{U}$, and thus $\lim_{\mathbb{R}} \mathcal{U} \in \mathbb{R}$, or $\mathcal{U} \geq \mathcal{N}(+\infty)$ or $\mathcal{U} \geq \mathcal{N}(-\infty)$. So if $\mathcal{H}$ meshes with no filter in $\{\mathcal{N}(q) \vee X : q \in \mathbb{Q} \cup \{-\infty, +\infty\}\}$, then $\mathcal{H}$ contains a bounded set, hence $\varnothing \neq \operatorname{adh}_{\mathbb{R}} \mathcal{H} \subset X$, and thus $\operatorname{adh}_X \mathcal{H} = \operatorname{adh}_{\mathbb{R}} \mathcal{H} \cap X$.

An essential and illuminating fact about cocompleteness is the following. Recall that for a given filter $\mathcal{G}$, the symbol $\beta(\mathcal{G})$ denotes the set of all ultrafilters finer than $\mathcal{G}$.

Lemma 5.3. *A convergence is κ-complete if and only if there exists a collection $\mathbb{G}$ of filters such that* $\operatorname{card} \mathbb{G} \leq \kappa$ *and the set of non-convergent ultrafilters is equal to $\bigcup_{\mathcal{G} \in \mathbb{G}} \beta(\mathcal{G})$.*

The completeness number of a convergence on X is at most that of the cardinality of the set of non-convergent ultrafilters, hence not greater than the cardinality of ultrafilters on X, that is, $2^{2^{\operatorname{card} X}}$, hence (3.1).

The Arhangel'skii-Frolík theorem is an easy consequence of Lemma 5.3, as shown in [4].

6. Subspaces

A subset of a convergence space is called a *G_κ-subset* if it is the intersection of κ many open sets. Notice that, traditionally $G_{\aleph_0}$ *is* called G_δ.

A convergence ξ is called *weakly diagonal* if $\operatorname{adh}_\xi \mathcal{H}$ is closed for each filter $\mathcal{H}$ ([12]). A convergence ξ is called *regular* if

$$x \in \lim_\xi \mathcal{F} \Longrightarrow x \in \lim_\xi \{\operatorname{adh}_\xi F : F \in \mathcal{F}\}. \tag{6.1}$$

The following two theorems were proved in [4], but I reproduce here the demonstration of one of them in order to show advantages of the cocompleteness viewpoint.

Theorem 6.1. *A dense κ-complete subset of a Hausdorff weakly diagonal convergence is a G_κ-subset.*

Proof. Let X be a Hausdorff weakly diagonal convergence space, Y be a dense subset of X, and $\mathbb{G}$ be a a collection of non-adherent filters on Y such that $\operatorname{card} \mathbb{G} = \kappa$ and Y is $\mathbb{G}$-cocomplete. If $\mathcal{G} \in \mathbb{G}$ then $\operatorname{adh}_X \mathcal{G}$ is disjoint from Y and closed in X, because X is weakly diagonal. As Y is $\mathbb{G}$-cocomplete and Y is dense in X,we infer that $X \setminus Y = \bigcup_{\mathcal{G} \in \mathbb{G}} \operatorname{adh}_X \mathcal{G}$, hence Y is a G_κ. ∎

Theorem 6.2. *Every G_κ-subset of a regular λ-complete convergence space is $\kappa\lambda$-complete.*

The *compact covering number* $\operatorname{kc}(X)$ of a topological space X is defined as the least cardinal of a family $\mathcal{L}$ of compact subsets of X such that $\bigcup_{L \in \mathcal{L}} L = X$.

Lemma 6.3. *The compact covering number of the set of irrational numbers is $\mathfrak{d}$.*

Proof. Let $h : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{N}^{\mathbb{N}}$ be a homeomorphism. For every compact subset K of $\mathbb{R} \setminus \mathbb{Q}$, let $f_K(n) := \sup \{h(x)(n) : x \in K\}$. As $h(K)$ is compact in the pointwise topology, $f_K \in \mathbb{N}^{\mathbb{N}}$. Observe that a family $\mathcal{L}$ is a cover of $\mathbb{R} \setminus \mathbb{Q}$ if and only if $\{f_L : L \in \mathcal{L}\}$ is cofinal in $(\mathbb{N}^{\mathbb{N}}, \leq)$. Hence $\mathrm{kc}(\mathbb{P}) = \mathfrak{d}$. ∎

As the set of irrational numbers is homeomorphic to $\mathbb{R} \cap (0,1) \setminus \mathbb{Q}$, there exists a family of $\mathbb{R}$-closed subsets of $\mathbb{R} \setminus \mathbb{Q}$ the union of which is $\mathbb{R} \setminus \mathbb{Q}$. Hence, by Theorem 6.1, $\mathbb{Q}$ is $G_{\mathfrak{d}}$, so that Theorem 3.4 holds.

If $\mathbb{G}$ is a collection and $\mathbb{H}$ is a class of filters, then $\mathbb{H}_{\mathbb{G}_{\neg}}$ denotes the collection of $\mathbb{G}$-cofundamental filters that belong to $\mathbb{H}$. A convergence is called $\mathbb{H}$-*conditionally* $\mathbb{G}$-*cocomplete* if each $\mathbb{G}$-cofundamental filter from $\mathbb{H}$ is adherent. It follows that ξ is $\mathbb{H}$-conditionally κ-complete if and only if there exists a collection $\mathbb{G}$ of non-adherent filters with $\operatorname{card}\mathbb{G} \leq \kappa$ such that ξ is $\mathbb{H}$-conditionally $\mathbb{G}$-cocomplete.

7. Complete and cocomplete families

More generally, let ξ be a convergence on a set X, $\mathbb{H}$ a class of filters and $\mathbb{P}$ a collection of families of subsets of X. A family $\mathcal{A}$ is said to be $\mathbb{H}$-*conditionally* $\mathbb{P}$-*complete at* a family $\mathcal{B}$ if $\mathrm{adh}_{\xi}\,\mathcal{H} \in \mathcal{B}^{\#}$ for each filter $\mathcal{H} \in \mathbb{H}_{\mathbb{P}}$ such that $\mathcal{H}\#\mathcal{A}$. It is clear that

Proposition 7.1. *$\mathcal{A}$ is $\mathbb{H}$-conditionally $\mathbb{P}$-complete at $\mathcal{B}$ if and only if $\mathcal{A}$ is $\mathbb{H}_{\mathbb{P}}$-compact at $\mathcal{B}$.*

Proposition 7.1 is a simple observation, but its scope is broad. In particular, by virtue of Theorem 2.2, it implies that for each $\mathbb{H}$ and $\mathbb{P}$,

Theorem 7.2. *The class of all $\mathbb{H}$-conditionally $\mathbb{P}$-complete isotone families fulfills the axioms of open sets of a hyperspace topology.*

Corollary 7.3. *The class of all $\mathbb{H}$-conditionally $\mathbb{P}$-complete openly isotone families of open sets fulfills the axioms of open sets of a hyperspace topology.*

The $\mathbb{H}$-*conditional completeness number* of $\mathcal{A}$ with respect to $\mathcal{B}$

$$\mathrm{compl}_{\mathbb{H}}^{\xi}(\mathcal{A}, \mathcal{B})$$

is the least cardinal κ such that there exists a collection $\mathbb{P}$ of covers of ξ such that $\operatorname{card}\mathbb{P} = \kappa$ and $\mathcal{A}$ is $\mathbb{H}$-conditionally $\mathbb{P}$-complete at $\mathcal{B}$. In particular, if $\mathcal{B} = \{|\xi|\}$, then we abridge $\mathrm{compl}_{\mathbb{H}}^{\xi}(\mathcal{A})$ or even

$$\mathrm{compl}_{\mathbb{H}}(\mathcal{A}),$$

when a convergence is implicit.

Of course, if $\mathbb{H} = \mathbb{F}$ is the class of all filters, $\mathbb{H}$-conditional completeness becomes completeness. Then we set

$$\mathrm{compl}(\mathcal{A}) := \mathrm{compl}_{\mathbb{F}}(\mathcal{A}).$$

Dually, if $\mathbb{H}$ is a class of filters and $\mathbb{P}$ is a collection of families of subsets of X, then $\mathcal{A}$ is said to be $\mathbb{H}$-*conditionally* $\mathbb{G}$-*cocomplete at* a family $\mathcal{B}$ if

$\operatorname{adh}_\xi \mathcal{H} \in \mathcal{B}^{\#}$ for each filter $\mathcal{H} \in \mathbb{H}$ such that $\mathcal{H}\#\mathcal{A}$. Accordingly, the $\mathbb{H}$-conditional completeness number of $\mathcal{A}$ with respect to $\mathcal{B}$ is the least cardinal κ such that there exists a collection $\mathbb{G}$ of non-adherent filters such that $\operatorname{card}\mathbb{G} = \kappa$ and $\mathcal{A}$ is $\mathbb{H}$-conditionally $\mathbb{G}$-cocomplete at $\mathcal{B}$.

8. Product theorems

In [4, Theorem 7.1] it is proved (on using cocompleteness) that the completeness number of product of convergences is equal to the sum of the completeness numbers of the component convergences. The following theorem extends that result to filters in products.

Theorem 8.1. *Let ξ_α be a convergence for $\alpha < \kappa$. The completeness number of a filter $\mathcal{F}$ at the product $\prod_{\alpha<\kappa} |\xi_\alpha|$ is*

$$\operatorname{compl}(\mathcal{F}) = \sum\nolimits_{\alpha<\kappa} \operatorname{compl}(p_\alpha[\mathcal{F}]).$$

Proof. For every $\alpha < \kappa$ such that $\operatorname{compl}(p_\alpha[\mathcal{F}]) > 0$, let $\mathbb{P}_\alpha$ be a collection of covers of ξ_α such that $\operatorname{card}\mathbb{P}_\alpha = \operatorname{compl}(p_\alpha[\mathcal{F}])$. For each $\mathcal{P} \in \mathbb{P}_\alpha$, let

$$\mathcal{S}_{\mathcal{P}} := \{P \times \prod\nolimits_{\alpha\neq\gamma<\kappa} X_\gamma : P \in \mathcal{P}\}.$$

Then $\mathcal{S}_{\mathcal{P}}$ is a cover of $\prod_{\gamma<\kappa} \xi_\gamma$, because if $\lim \mathcal{H} \neq \varnothing$ then $\lim_{\xi_\gamma} p_\gamma(\mathcal{H}) \neq \varnothing$ for each $\gamma < \kappa$, hence $p_\alpha(\mathcal{H}) \cap \mathcal{P} \neq \varnothing$, that is, $\mathcal{H} \cap \mathcal{S}_{\mathcal{P}} \neq \varnothing$.

If $\mathbb{S} := \{\mathcal{S}_{\mathcal{P}} : \mathcal{P} \in \mathbb{P}_\alpha, \operatorname{compl}(p_\alpha[\mathcal{F}]) > 0, \alpha < \kappa\}$, then $\mathcal{F}$ is $\mathbb{S}$-complete. Indeed, if $\mathcal{U} \geq \mathcal{F}$ is an $\mathbb{S}$-fundamental ultrafilter, then $p_\alpha[\mathcal{U}]\#p_\alpha[\mathcal{F}]$ and $\mathcal{U} \cap \mathcal{S}_{\mathcal{P}} \neq \varnothing$ for each $\alpha < \kappa$ such that $\operatorname{compl}(p_\alpha[\mathcal{F}]) > 0$ and for every $\mathcal{P} \in \mathbb{P}_\alpha$, that is, $p_\alpha(\mathcal{U}) \cap \mathcal{P} \neq \varnothing$. Therefore $\lim_{\xi_\alpha} p_\alpha(\mathcal{U}) \neq \varnothing$, because $p_\alpha[\mathcal{F}]$ is $\mathbb{P}_\alpha$-complete; if $\operatorname{compl}(p_\alpha[\mathcal{F}]) = 0$, then $\lim_{\xi_\alpha} p_\alpha(\mathcal{U}) \neq \varnothing$, because $p_\alpha[\mathcal{F}]$ is compactoid. It follows that $\lim_\xi \mathcal{U} \neq \varnothing$. On the other hand,

$$\begin{aligned} \operatorname{card}\mathbb{S} &= \sum\nolimits_{\alpha<\gamma} \operatorname{card}\mathbb{P}_\alpha = \sum\nolimits_{\alpha<\gamma} \operatorname{compl}(p_\alpha[\mathcal{F}]) \\ &= \sup_{\alpha<\kappa} \operatorname{compl}(p_\alpha[\mathcal{F}]) \cdot \operatorname{card}\{\alpha < \kappa : \operatorname{compl}(p_\alpha[\mathcal{F}]) > 0\} \end{aligned}$$

∎

In particular, by taking the principal filter of the whole product, we get [4, Proposition 7.1]:

$$\begin{aligned} \operatorname{compl}\left(\prod\nolimits_{\alpha<\kappa} \xi_\alpha\right) &= \sum\nolimits_{\alpha<\kappa} \operatorname{compl}(\xi_\alpha) \\ &= \sup_{\alpha<\kappa} \operatorname{compl}(\xi_\alpha) \cdot \operatorname{card}\{\alpha < \kappa : \operatorname{compl}(\xi_\alpha) > 0\}. \end{aligned}$$

If $\operatorname{compl}(p_\alpha[\mathcal{F}]) = 0$ for each $\alpha < \kappa$, then we recover Theorem 2.1.

Theorem 8.2. *A filter $\mathcal{F}$ is $\prod_{j\in J} \xi_j$-compactoid if and only if $p_j[\mathcal{F}]$ is ξ_j-compactoid for each $j \in J$.*

Corollary 8.3 (Tikhonov)**.** *The product $\prod_{\alpha<\kappa} \xi_\alpha$ is compact if and only if ξ_α is compact for every $\alpha < \kappa$.*

On the other hand,

Corollary 8.4. *The product $\prod_{\alpha<\kappa}\xi_\alpha$ is locally compactoid if and only if ξ_α is locally compactoid for finitely many $\alpha<\kappa$ and compact for other $\alpha<\kappa$.*

Theorem 8.1 does not extend to conditional completeness. For example, $\mathbb{F}_1$-conditional countable completeness is productive.

Theorem 8.5. *If $\mathcal{F}$ is a filter and $p_j[\mathcal{F}]$ is $\mathbb{F}_1$-conditionally countably complete for each $j\in J$, then $\mathcal{F}$ is $\mathbb{F}_1$-conditionally countably complete.*

Proof. Let $\{\xi_j : j\in J\}$ be convergences, let $\xi := \prod_{j\in J}\xi_j$ and let $\mathcal{F}$ be a filter on $\prod_{j\in J}|\xi_j|$ such that $p_j[\mathcal{F}]$ is $\mathbb{F}_1$-conditionally countably complete. For each $j\in J$, consider a sequence $\{\mathcal{P}_j(n) : n\in\mathbb{N}\}$ of ideal covers of ξ_j such that $\mathcal{P}_j(n+1)$ is a refinement of $\mathcal{P}_j(n)$ and such that $p_j[\mathcal{F}]$ is $\mathbb{F}_1$-conditionally complete with respect to $\{\mathcal{P}_j(n) : n\in\mathbb{N}\}$. Define

$$\mathcal{P}(n) := \left\{\prod_{j\in J}P_j^n : P_j^n\in\mathcal{P}_j(n), \operatorname{card}\left\{P_j^n\neq|\xi_j|\right\}<\infty\right\}.$$

Then $\mathcal{F}$ is $\mathbb{F}_1$-conditionally $\{\mathcal{P}(n) : n\in\mathbb{N}\}$-complete. Indeed, if $\mathcal{H}\in\mathbb{F}_1$ is such that $\mathcal{H}\#\mathcal{F}$ and $\mathcal{H}\cap\mathcal{P}(n)\neq\varnothing$ for each n, that is, there is a decreasing sequence $(H_n)_n$ of elements of $\mathcal{H}$ such that

$$H_n = \prod_{j\in J}P_j^n,$$

so that $P_j^n\supset P_j^{n+1}$ for each $j\in J$ and $n\in\mathbb{N}$ and $\{P_j^n\}_n\#p_j[\mathcal{F}]$. It follows that, for each $j\in J$, there is $x_j\in\operatorname{adh}_{\xi_j}\{P_j^n\}_n$, that is, there is an ultrafilter $\mathcal{U}_j$ on $|\xi_j|$ such that $x_j\in\lim_{\xi_j}\mathcal{U}_j$ and $\mathcal{U}_j\#\{P_j^n\}_n\vee p_j[\mathcal{H}]$. Equivalently, $p_j^-[\mathcal{U}_j]\#\mathcal{H}$ for each $j\in J$, hence $\mathcal{H}\#\bigvee_{j\in J}p_j^-[\mathcal{U}_j]$ and there is an ultrafilter $\mathcal{W}$ such that

$$\mathcal{H}\vee\bigvee_{j\in J}p_j^-[\mathcal{U}_j]\leq\mathcal{W}.$$

Thus, if $f(j) := x_j$ for each $j\in J$, then $f\in\lim_\xi\mathcal{W}$ and thus $f\in\operatorname{adh}_\xi\mathcal{H}$. ∎

Corollary 8.6. *Each product of $\mathbb{F}_1$-conditionally countably complete convergences is countably $\mathbb{F}_1$-conditionally complete.*

Oxtoby proved in [14] that an arbitrary product of pseudocomplete topologies is pseudocomplete, which is a consequence of Corollary 8.6. It is immediate that a regular $\mathbb{F}_1$-conditionally countably complete topology has the Baire property, hence every product of regular $\mathbb{F}_1$-conditionally countably complete topologies has the Baire property, although the product of two topologies with the Baire property need not have the Baire property.

9. Preservation of completeness number by maps

Compactness is preserved by continuous maps, but local (relative) compactness is not. These two special cases of completeness (0-completeness and 1-completeness) behave differently under continuous maps. The cocompleteness approach provides an explanation of this difference. Moreover, it enables us to understand the mechanism of preservation of completeness numbers.

If $f : X \to Y$ and $\mathbb{G}$ is a collection of families of subsets of X, then let $f\{\mathbb{G}\} := \{f[\mathcal{G}] : \mathcal{G} \in \mathbb{G}\}$. Similarly, if $\mathbb{H}$ is a collection of families of subsets of Y, then let $f^{-}\{\mathbb{H}\} := \{f^{-}[\mathcal{H}] : \mathcal{H} \in \mathbb{H}\}$.

If ξ and τ are convergences, then $C(\xi, \tau)$ stands for the set of all maps that are continuous from ξ to τ. Let $\mathcal{A}$ be a family on $|\xi|$ and $\mathcal{B}$ a family on $|\tau|$.

Proposition 9.1. *If $\mathcal{A}$ is $\mathbb{G}$-cocomplete at $f^{-}[\mathcal{B}]$ and $f \in C(\xi, \tau)$ and is surjective, then $f[\mathcal{A}]$ is $f\{\mathbb{G}\}$-cocomplete at $\mathcal{B}$.*

Proof. Let $\mathcal{H}\#\mathcal{A}$ and $\mathcal{H}\neg\# f[\mathcal{G}]$ for each $\mathcal{G} \in \mathbb{G}$. Equivalently, $f^{-}[\mathcal{H}]\#\mathcal{A}$ and $f^{-}[\mathcal{H}]\neg\#\mathcal{G}$ for each $\mathcal{G} \in \mathbb{G}$, hence $\mathrm{adh}_\xi f^{-}[\mathcal{H}] \in f^{-}[\mathcal{B}]^{\#}$, equivalently $f(\mathrm{adh}_\xi f^{-}[\mathcal{H}]) \in \mathcal{B}^{\#}$. We infer that $f(\mathrm{adh}_\xi f^{-}[\mathcal{H}]) \subset \mathrm{adh}_\tau \mathcal{H}$, because $f \in C(\xi, \tau)$ and surjective. ∎

Corollary 9.2. *If ξ is $\mathbb{G}$-cocomplete and $f \in C(\xi, \tau)$ and surjective, then τ is $f\{\mathbb{G}\}$-cocomplete.*

A map does not increase the completeness number if it preserves cocomplete collections and maps non-adherent filters onto non-adherent filters. By Proposition 9.1, continuous maps preserve cocomplete collections. On the other hand, a map $f : |\xi| \to |\tau|$ maps non-adherent filters onto non-adherent filters if and only of

$$\mathrm{adh}_\tau f[\mathcal{G}] \neq \varnothing \Longrightarrow \mathrm{adh}_\xi \mathcal{G} \neq \varnothing. \tag{9.1}$$

for each filter $\mathcal{G}$ on $|\xi|$. It was shown in [3] that a surjective map $f : |\xi| \to |\tau|$ is perfect ([17]) if and only if for every $y \in |\tau|$ and each filter $\mathcal{G}$ on $|\xi|$,

$$y \in \mathrm{adh}_\tau f[\mathcal{G}] \Longrightarrow f^{-}(y) \cap \mathrm{adh}_\xi \mathcal{G} \neq \varnothing.$$

Therefore,

Theorem 9.3. *If $f \in C(\xi, \tau)$ is a surjective perfect map, then the completeness number of $\mathcal{A}$ at $f^{-}[\mathcal{B}]$ is equal to the completeness number of $f[\mathcal{A}]$ at $\mathcal{B}$.*

Proof. By Proposition 9.1, if $\mathcal{A}$ is $\mathbb{G}$-cocomplete at $f^{-}[\mathcal{B}]$ and $f \in C(\xi, \tau)$, then $f[\mathcal{A}]$ is $f\{\mathbb{G}\}$-cocomplete at $\mathcal{B}$, hence $\mathrm{card}(\mathbb{G}) \geq \mathrm{card}\, f\{\mathbb{G}\}$.

If $\mathbb{G}$ is a collection of ξ-non-adherent filters and f is perfect then by (9.1) $f\{\mathbb{G}\}$ is a collection of τ-non-adherent filters. It follows that the completeness of $\mathcal{A}$ at $f^{-}[\mathcal{B}]$ is greater than or equal to the completeness of $f[\mathcal{A}]$ at $\mathcal{B}$.

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[17] This definition does not require continuity. In other words, a map is said to be *perfect* if the preimage of each point is relatively compact.

Conversely, if $f[\mathcal{A}]$ is $\mathbb{H}$-cocomplete at $\mathcal{B}$ and $\mathbb{H}$ is a collection of τ-non-adherent filters, then, by the continuity of f, the collection $f^{-}\{\mathbb{H}\}$ consists of ξ-non-adherent filters. Indeed, if on the contrary, $x \in \mathrm{adh}_{\xi} f^{-}[\mathcal{H}]$ then there exists $\mathcal{U} \in \beta(f^{-}[\mathcal{H}])$ such that $x \in \lim_{\xi} \mathcal{U}$. By continuity, $f(x) \in \lim_{\tau} f[\mathcal{U}] \subset \mathrm{adh}_{\tau} \mathcal{H}$, which is a contradiction, because $f[\mathcal{U}] \# \mathcal{H}$.

To see that $\mathcal{A}$ is $f^{-}\{\mathbb{H}\}$-cocomplete at $f^{-}[\mathcal{B}]$, let $\mathcal{F} \# \mathcal{A}$ and $\mathcal{F} \neg \# f^{-}[\mathcal{H}]$ (equivalently $f[\mathcal{F}] \neg \# \mathcal{H}$) for every $\mathcal{H} \in \mathbb{H}$. As $f[\mathcal{A}]$ is $\mathbb{H}$-cocomplete at $\mathcal{B}$, $\mathrm{adh}_{\tau} f[\mathcal{F}] \in \mathcal{B}^{\#}$, that is, $\mathrm{adh}_{\tau} f[\mathcal{F}] \# B$ for each $B \in \mathcal{B}$, and since f is perfect, by (9.1), $\mathrm{adh}_{\xi} \mathcal{F} \# f^{-}(B)$. This means that the completeness of $f[\mathcal{A}]$ at $\mathcal{B}$ is greater than or equal to the completeness $\mathcal{A}$ at $f^{-}[\mathcal{B}]$. ∎

As a corollary, we get

Theorem 9.4. *If $f \in C(\xi, \tau)$ is surjective and perfect, then* $\mathrm{compl}(\xi) = \mathrm{compl}(\tau)$.

If $\mathrm{compl}(\xi) = 0$, then (9.1) is fulfilled (for each map f and every τ), because each filter on $|\xi|$ is ξ-adherent.

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Département de Mathématiques, Université de Bourgogne, B. P. 47870, 21078 Dijon, France

E-mail address: dolecki@u-bourgogne.fr