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A UNIFIED THEORY OF FUNCTION SPACES AND HYPERSPACES: LOCAL PROPERTIES

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Abstract. Every convergence (in particular, every topology) τ on the hyperspace $C(X,\$)$ *preimage-wise* determines a convergence $\tau^{\Uparrow}$ on $C(X,Z)$, where X,Z are topological spaces and \$ is the *Sierpiński topology*, so that $f\in\lim_{\tau^{\Uparrow}}\mathcal{F}$ if and only if $f^{-1}(U)\in\lim_{\tau}\mathcal{F}^{-1}(U)$ for every open subset U of Z. Classical instances of function space structures that are determined this way by their hyperspace counterparts are the *pointwise, compact-open and Isbell* topologies, and the *natural* (that is, *continuous*) convergence.

It is shown that several fundamental local properties hold for a hyperspace convergence τ on $C(X,\$)$ at X if and only if they hold for $\tau^{\Uparrow}$ on $C(X,\mathbb{R})$ at the origin, provided that the underlying topology of X have some $\mathbb{R}$-separation properties. This concerns character, tightness, fan tightness, strong fan tightness, and various Fréchet properties (from the simple through the strong to that for finite sets) and corresponds to various covering properties (like Lindelöf, Rothberger, Menger) of the underlying space X.

This way, many classical results are unified, extended and improved. Among new surprising results: the tightness and the character of the natural convergence coincide and are equal to the Lindelöf number of the underlying space; the Fréchet property coincides with the Fréchet property for finite sets for the hyperspace topologies generated by compact networks.

1. Introduction

Study of the interplay between properties of a topological space X and those of the associated space $C(X,Z)$ of continuous functions from X to another topological space Z, endowed with convergence structures, is one of the central themes of topology, and an active area interfacing *topology* and *functional analysis*.

A *convergence* on a set Y is a relation $y \in \lim \mathcal{F}$ between the elements y of Y and the filters $\mathcal{F}$ on Y such that $\mathcal{F} \subset \mathcal{G}$ implies $\lim \mathcal{F} \subset \lim \mathcal{G}$ and $y \in \lim \{y\}^{\uparrow}$ for every $y \in Y$, where $\{y\}^{\uparrow}$ stands for the principal ultrafilter determined by y.

A subset O of Y is said to be *open* for this convergence, whenever $O \cap \lim \mathcal{F} \neq \varnothing$ entails $O \in \mathcal{F}$, and a convergence is *topological* if $y \in \lim \mathcal{F}$ provided that $O \in \mathcal{F}$ for each open set O that contains y.

A convergence is called a *pseudotopology* if $x \in \lim \mathcal{F}$ provided that $x \in \lim \mathcal{U}$ for every ultrafilter $\mathcal{U}$ that is finer than $\mathcal{F}$.

A convergence is called a *pretopology* if the intersection of a family of filters converging to x converges to x. For further information on convergence theory, see for example [13].

The convergences on $C(X,Z)$ that are most commonly studied, are the *finite-open* (or *pointwise*) *topology* $p(X,Z)$, the *compact-open topology* $k(X,Z)$, the *Isbell topology* $\kappa(X,Z)$, the *natural convergence* $[X,Z]$ (often called the *continuous convergence*) and its topological modification, the *natural topology* $T\,[X,Z]$. The *pointwise topology* $p(X,Z)$ is the coarsest convergence on $C(X,Z)$, for which the natural coupling

$$\langle \cdot,\cdot \rangle : X \times C(X,Z) \to Z \tag{1.1}$$

is pointwise continuous for each $x \in X$. The *natural convergence* $[X,Z]$ is the coarsest structure on $C(X,Z)$, for which (1.1) is jointly continuous ([1]). The natural convergence $[X,Z]$ is pseudotopological whenever Z is.

There are two basic instances of the image space Z: the real line $\mathbb{R}$ with the *usual topology*, and a two-point set with the *Sierpiński topology* $\$:= \{\varnothing, \{1\}, \{0,1\}\}$. Of course, $C(X,\$)$ can be identified with the *hyperspace* of either open or closed subsets of X. The *characteristic* function $\chi_A : X \to \{0,1\}$, defined by $\chi_A(x) = 1$ if and only if $x \in A$, fulfills $\chi_A \in C(X,\$)$ if and only if A is open. The *indicator* function $\psi_A := \chi_{X\setminus A} \in C(X,\$)$ if and only if A is closed. We shall adopt the convention that $C(X,\$)$ is the hyperspace of open subsets of X, while $cC(X,\$)$ is its image by the complementation $c\,(A) := X \setminus A$, that is, the hyperspace of closed subsets of X.

[1]Therefore it satisfies the exponential law

$$[X \times Y, Z] = [Y, [X,Z]],$$

and, as such, has been called *natural convergence* (e.g., [18]), the terminology that we adopt here. The exceptional role of the natural convergence among all function space structures on $C(X,Z)$ was recognized as early as [1] by Arens and Dugundji, and a compelling case for its systematic use in functional analysis was made by Binz in [7] and more recently and thoroughly by Beattie and Butzmann in [6].

Hence, $p(X,\$)$, $k(X,\$)$, $\kappa(X,\$)$ and $[X,\$]$ are convergences on the hyperspace of open sets; of course, they have their homeomorphic counterparts on $cC(X,\$)$.

It was observed in [12] that $p(X,Z)$, $k(X,Z)$, $\kappa(X,Z)$ and $[X,Z]$ can be recovered preimage-wise from the corresponding hyperspace convergences.

More generally, for any convergence τ on $C(X,\$)$ the *preimage-wise* convergence $\tau^{\Uparrow}$ of τ is defined so that $f_0 \in \lim_{\tau^{\Uparrow}} \mathcal{F}$ provided that $h \circ f_0 \in \lim_\tau h \circ \mathcal{F}$ for every $h \in C(Z,\$)$. The name of this convergence is due to the fact that each two-valued map is of the form $h = \chi_U$ and $\chi_U \circ f = f^-(U) := \{z \in Z : f(z) \in U\}$. In particular, if α is a topology on the hyperspace $C(X,\$)$, then $\alpha(X,Z)$ is the preimage-wise topology on $C(X,Z)$ with respect to α. Moreover, $\alpha(X,\$)$ coincides with α.

The relationship between convergences and topologies on *functional spaces* $C(X,\mathbb{R})$ and the corresponding convergences and topologies on *hyperspaces* $C(X,\$)$ is a principal theme of this paper. Because of the preimage-wise character of the studied functional spaces, their properties depend on the corresponding hyperspace properties. On the other hand, certain aspects of functional spaces and hyperspaces are quite different. For instance, all the mentioned convergences on $C(X,\mathbb{R})$ are completely regular, provided that X is completely regular, while their corresponding hyperspace convergences are T_0 but never T_1.

We show that local properties of functional spaces induce the corresponding hyperspace properties (although are not identical to them) provided that real-valued functions separate closed sets from singletons of the underlying topological space.

Convergences on $C(X,\$)$ have usually a simpler structure than their counterparts on $C(X,\mathbb{R})$. On the other hand, local properties of convergences on $C(X,\$)$ are often easily seen to be equivalent to some (hereditary) covering properties of X. Therefore in the study of the interdependence between X and $C(X,\mathbb{R})$, it is very useful to comprehend the relationship between $C(X,\mathbb{R})$ and $C(X,\$)$.

Given a class of filters $\mathbb{D}$, we say that a convergence ξ is $\mathbb{D}$-*based* if whenever $x \in \lim_\xi \mathcal{F}$, there is a filter $\mathcal{D} \in \mathbb{D}$ with $\mathcal{D} \leq \mathcal{F}$ and $x \in \lim_\xi \mathcal{D}$. We say that a convergence property $\mathbf{P}$ is *local* if there exists a class of filters $\mathbb{D}$ such that a convergence τ has $\mathbf{P}$ whenever τ is $\mathbb{D}$-based.

The following classes of convergences are examples of local properties: of *countable character, Fréchet, strongly Fréchet, productively Fréchet, countably tight, countably fan-tight and strongly fan-tight.*

The so-called γ-*connection* of Gruenhage, e.g. [23], is a very particular instance of our preimage-wise approach (it describes the neighborhood filter of the whole space X for the pointwise topology on the hyperspace $C(X,\$)$ of open sets). Jordan exploited the γ-connection in [26] establishing a relation between the neighborhood filter of the zero function in $C_p(X,\mathbb{R})$ and the neighborhood filter of the whole space X in $C_p(X,\$)$.

Jordan's paper is a prefiguration of our theory. Actually the first author realized that Jordan's approach can be easily extended to general topologies $\alpha(X,Z)$, encompassing, among others, the topology of pointwise convergence, the compact-open topology and the Isbell topology [12]. On the other hand, the fact that the natural (or continuous) convergence fits (2.3), that is, that $[X,Z]$ is pre-imagewise with respect to $[X,\$]$, was observed before, e.g. [40].

Even though the relationship between hyperspace structures and function space structures has been identified on a case by case basis, and even as an abstract scheme in [11] for topologies, it seems that no systematic use of this situation is to be found in the literature before [12]. In the present paper, we extend the results of [12] to general convergences, simplify some of the arguments, clarify the role of topologicity, and obtain as by-products a wealth of classical results for function space topologies, as well as new results for the natural convergence. In particular, we obtain the surprising result that the character and tightness of the natural convergence on real valued continuous functions coincide, and are equal to the Lindelöf degree of the underlying space.

2. Preimagewise convergences

Let X,Z be topological spaces and $\$$ be the Sierpiński topological space. We will identify $C(X,\$)$ with the set of open subsets of X. If τ is a convergence on $C(X,\$)$, then $\tau^{\Uparrow}$ is the following convergence on $C(X,Z)$:

$$f_0 \in \lim_{\tau^{\Uparrow}} \mathcal{F} \iff \forall_{U \in C(Z,\$)} f_0^-(U) \in \lim_\tau \mathcal{F}^-(U), \tag{2.1}$$

where $f^-(U) := \{x \in X : f(x) \in U\}, F^-(U) := \{f^-(U) : f \in F\}$ and $\mathcal{F}^-(U) := \{F^-(U) : F \in \mathcal{F}\}$. Then $\tau^{\Uparrow}$ is called the *preimage-wise* convergence of τ.

If for a convergence θ on $C(X,Z)$ there exists a convergence on $C(X,\$)$ with respect to which θ is preimage-wise, then there is a finest convergence $\theta^{\Downarrow}$ on $C(X,\$)$ among those τ for which $\theta = \tau^{\Uparrow}$. Hence, $\tau^{\Uparrow\Downarrow\Uparrow} = \tau^{\Uparrow}$ for each τ.

As we shall show in Proposition 4.5 below, it is not necessary to test that $f^-(U) \in \lim_\tau \mathcal{F}^-(U)$ for every open subset U of Z in (2.1), but only for the elements of a base of the topology on Z that is stable for finite unions. In terms of closed sets, (2.1) becomes

$$f \in \lim_{\tau^{\Uparrow}} \mathcal{F} \iff \forall_{C \in cC(Z,\$)} f^-(C) \in \lim_{c\tau} \mathcal{F}^-(C).$$

In the formula above, analogously to (2.1), it is enough to test $f^-(C) \in \lim_{c\tau} \mathcal{F}^-(C)$ for the elements of a basis of closed sets that is stable for finite intersections.

This is a special case of the following scheme. Each $h \in C(Z,W)$ defines the *lower conjugate* map $h_* : C(X,Z) \to C(X,W)$ given by $h_*(f) := h \circ f$. Each convergence τ on $C(X,W)$ determines on $C(X,Z)$ the coarsest convergence for which h_* is continuous for every $h \in C(Z,W)$. In the particular case when W is the Sierpiński topology \$, then for each $U \in C(Z,\$)$, the image $U_*(f) \in C(X,\$)$ and

$$U_*(f) = f^-(U). \tag{2.2}$$

In other words, if an element U of $C(Z,\$)$ is identified with an open set (via the characteristic function), then in the same way $U_*(f)$ is identified with the preimage of U by f. Therefore θ is *preimage-wise with respect to* τ, if the source

$$(U_* : C_\theta(X,Z) \to C_\tau(X,\$))_{U \in C(Z,\$)} \tag{2.3}$$

is initial, that is, if θ is the coarsest convergence on $C(X,Z)$ making each map $U_* : C(X,Z) \to C_\tau(X,\$)$ continuous ([2]).

The following is an immediate consequence of the definition.

Proposition 2.1. *If* $\mathbf{J}$ *is a reflective category of convergences,* $C_\tau(X,\$)$ *is an object of* $\mathbf{J}$*, then* $C_{\tau^\Uparrow}(X,Z)$ *is also an object of* $\mathbf{J}$.

For instance, if τ is a topology or a pseudotopology, so is $\tau^\Uparrow$.

In the particular important case where $Z = \mathbb{R}$ ([3]), the preimage of a closed set by a continuous function is a *zero set*, because all closed subsets of $\mathbb{R}$ are zero sets. Therefore, a τ-preimage-wise convergence on $C(X,\mathbb{R})$ is determined by the restriction of τ to the cozero sets of X (or the restriction of $c\tau$ to zero sets). More generally, we say that an open subset G of X is Z*-functionally open* if there exist $f \in C(X,Z)$ and $U \in C(Z,\$)$ such that $G = f^-(U)$. Of course, all the elements of $C(X,\$)$ that are not Z*-functionally open* are isolated for $\tau^{\Uparrow\Downarrow}$.

3. Fundamental examples of preimagewise convergences

If $\mathcal{A}$ is a family of subsets of X and U is a subset of Z, then let

$$[\mathcal{A},U] := \bigcup_{A \in \mathcal{A}} \{f \in C(X,Z) : A \subset f^-(U)\}, \tag{3.1}$$

[2] As the category of topological spaces and continuous maps is reflective in that of convergence spaces (and continuous maps), the coarsest convergence making the maps U_* continuous is also the coarsest topology with this property, whenever τ is topological.

[3] More generally, if Z is perfectly normal.

where $f^{-}(U) := \{x \in X : f(x) \in U\}$ is our usual shorthand for $f^{-1}(U)$. In particular, if $\mathcal{A} = \{A\}$, then we abridge

$$[A, U] := [\{A\}, U].$$

In [12] and [15], we studied topologies on $C(X, Z)$ generated by collections α of families of subsets of X (the *Z-dual topology* of α). If α is *non-degenerate*, that is, $\alpha \setminus \varnothing \neq \{\varnothing\}$, then

$$\{[\mathcal{A}, U] : \mathcal{A} \in \alpha, U \in C(Z, \$)\} \tag{3.2}$$

is a subbase for a topology on $C(X, Z)$ denoted by $\alpha(X, Z)$. Such topologies were called *family-open* in [11].

If A is a subset of X then $\mathcal{O}_X(A)$ denotes the collection of open subsets of X that contains A. Observe that

$$[\mathcal{O}_X(D), U] = [D, U], \tag{3.3}$$

whenever U is open. If $\mathcal{A}$ is a collection of subsets of X then $\mathcal{O}_X(\mathcal{A}) := \bigcup_{A \in \mathcal{A}} \mathcal{O}_X(A)$.

A family $\mathcal{A}$ is said to be *openly isotone* if $\mathcal{O}_X(\mathcal{A}) = \mathcal{A}$.

A family $\mathcal{A} \subset C(X, \$)$ is called *compact* if for every $\mathcal{B} \subset C(X, \$)$ such that $\bigcup_{B \in \mathcal{B}} B \in \mathcal{A}$, there exists a finite subcollection $\mathcal{S}$ of $\mathcal{B}$ such that $\bigcup_{B \in \mathcal{S}} B \in \mathcal{A}$. The collection $\kappa(X)$ of all compact openly isotone families form a topology on $C(X, \$)$, known as the *Scott topology* (for the lattice of open subsets of X ordered by inclusion). We denote the corresponding topological space by $C_\kappa(X, \$)$ ([4]).

The *Isbell topology* on $C(X, Z)$ has a subbase composed of $[\mathcal{A}, U]$ where U ranges over $C(Z, \$)$ and $\mathcal{A}$ ranges over $\kappa(X)$.

In view of (3.3) we can, and we will throughout the paper, assume that each $\mathcal{A} \in \alpha$ is *openly isotone*, that is, $\mathcal{A} = \mathcal{O}_X(\mathcal{A})$. The topology of pointwise convergence is obtained when α is equal to $p(X) := \{\bigcup_{F \in \mathcal{F}} \mathcal{O}(F) : \mathcal{F} \subset [X]^{<\infty}\}$, the family of *finitely generated families*. The compact open topology is obtained when α is equal to $k(X) := \{\bigcup_{K \in \mathcal{F}} \mathcal{O}(K) : \mathcal{F} \subset \mathcal{K}(X)\}$, the family of *compactly generated families*, where $\mathcal{K}(X)$ stands for the set of all compact subsets of X. Of course, the Isbell topology is obtained when α is the topology $\kappa(X)$ of compact families.

Even if $\alpha \subset C(X, \$)$ is not a basis for a topology, $\alpha(X, Z) = \alpha^{\cap}(X, Z)$, where $\alpha^{\cap}$ is the collection of finite intersections of elements of α, because $\bigcap_{i=1}^{n} [\mathcal{A}_i, U] = [\bigcap_{i=1}^{n} \mathcal{A}_i, U]$. Therefore, *we can assume that α is a basis for* $\alpha(X, \$)$.

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[4]The homeomorphic image of $C_\kappa(X, \$)$ is the hyperspace $cC_\kappa(X, \$)$ of closed subsets of X endowed with the *upper Kuratowski topology.*

Proposition 3.1. [12] *If* $\alpha \subset C(X,\$)$ *is non-degenerate then* $\alpha(X,Z) = \alpha(X,\$)^{\Uparrow}$.

Proof. If $\mathcal{A} \in \alpha$ and $U \in C(Z,\$)$ then

$$U_*^{-1}(\mathcal{A}) = \{f \in C(X,Z) : f^- U \in \mathcal{A}\} = [\mathcal{A}, U]$$

because $\mathcal{A} = \mathcal{O}_X(\mathcal{A})$. Therefore $\alpha(X,Z)$ is indeed the initial topology for the family of maps $(U_* : C(X,Z) \to C_\alpha(X,\$))_{U \in C(Z,\$)}$. □

By definition, the *natural convergence* $[X,Z]$ on $C(X,Z)$ (also called continuous convergence, e.g., [7], [6]) is the coarsest convergence making the canonical coupling (or evaluation)

$$\langle \cdot,\cdot \rangle : X \times C(X,Z) \to Z \tag{3.4}$$

defined by $\langle x,f \rangle := f(x)$ continuous. In other words, $f \in \lim_{[X,Z]} \mathcal{F}$ if and only if for every $x \in X$, the filter $\langle \mathcal{N}(x), \mathcal{F} \rangle$ converges to $f(x)$ in Z, that is, if $U \in \mathcal{O}_Z(f(x))$ there is $V \in \mathcal{O}_X(x)$ and $F \in \mathcal{F}$ such that $\langle V,F \rangle \subset U$, equivalently, $F \subset [V,U]$. Therefore

Proposition 3.2. $f_0 \in \lim_{[X,Z]} \mathcal{F}$ *if and only if for every open subset* U *of* Z *and* $x \in X$,

$$f_0 \in [x,U] \Longrightarrow \exists_{V \in \mathcal{O}_X(x)}\ [V,U] \in \mathcal{F}, \tag{3.5}$$

if and only if for every open subset U *of* Z *and* $x \in X$,

$$x \in f_0^-(U) \Longrightarrow \exists_{F \in \mathcal{F}} \bigcap_{f \in F} f^-(U) \in \mathcal{O}_X(x). \tag{3.6}$$

In the case where $Z = \$$, the only non-trivial open subset of Z is $\{1\}$ and elements of $C(X,\$)$ are of the form χ_Y for Y open in X. Therefore (3.6) translates into: $Y \in \lim_{[X,\$]} \gamma$ if and only if

$$x \in Y \Longrightarrow \exists_{\mathcal{G} \in \gamma} \bigcap_{G \in \mathcal{G}} G \in \mathcal{O}_X(x),$$

In other words, $Y \in \lim_{[X,\$]} \gamma$ if and only if

$$Y \subset \bigcup_{\mathcal{G} \in \gamma} \operatorname{int}_X \bigcap_{G \in \mathcal{G}} G. \tag{3.7}$$

This convergence is often (e.g., [22]) known as the *Scott convergence* (in the lattice of open subsets of X ordered by inclusion). Its homeomorphic image $c[X,\$]$ on the set of closed subsets of X is known as *upper Kuratowski convergence* ([5]).

Proposition 3.3. *(e.g.,* [40]*)*

$$[X,Z]=[X,\$]^{\Uparrow}.$$

PROOF. In view of (3.6), $f_0 \in \lim_{[X,Z]}\mathcal{F}$ if and only if $f_0^-(U) \subset \bigcup_{F\in\mathcal{F}}\operatorname{int}_X\bigcap_{f\in F}f^-(U)$, equivalently,

$$U_*(f_0)\subset\bigcup_{\mathcal{G}\in U_*(\mathcal{F})}\operatorname{int}_X\bigcap_{G\in\mathcal{G}}G,$$

for every open subset U of Z. In view of (3.7), we conclude that $f_0\in\lim_{[X,Z]}\mathcal{F}$ if and only if $U_*(f_0)\in\lim_{[X,\$]}U_*(\mathcal{F})$ for every $U\in C(Z,\$)$, which concludes the proof. □

It follows that if $\tau\le[X,\$]$ then $\tau^{\Uparrow}\le[X,Z]$. A convergence τ on $C(X,Z)$ is called *splitting* if for any space Y, the continuity of a map $g:X\times Y\to Z$ implies that of the map ${}^tg:Y\to C_\tau(X,Z)$ defined by ${}^tg(y)(x)=g(x,y)$. It is easy to see that τ is splitting if and only if $\tau\le[X,Z]$. Thus, the preimage-wise convergence of a splitting convergence is splitting.

That the natural convergence is not in general topological is a classical fact and one of the main motivations to consider convergence spaces. It is well known (see, e.g., [40], [14]) that the topological reflection $T[X,\$]$ of $[X,\$]$ is equal to the *Scott topology* $\kappa(X,\$)$ and we have seen that $\kappa(X,\$)=\kappa(X)$, the collection of all compact openly isotone families on X.

We do not know if for every topological space X there exists a hyperconvergence τ on $C(X,\$)$ such that $T[X,\mathbb{R}]=\tau^{\Uparrow}$.

4. Hyperconvergences

We focus on convergences τ on $C(X,\$)$ that share basic properties with $[X,\$]$ and topologies of the type $\alpha(X,\$)$ ([6]). In particular, we say that τ is *lower* if

$$A\subset B\in\lim_\tau\gamma\Longrightarrow A\in\lim_\tau\gamma,$$

and *upper regular* if

$$O\in\lim_\tau\gamma\Longrightarrow O\in\lim_\tau\mathcal{O}_X^{\natural}(\gamma),$$

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[5]Explicitely, if C is a closed subset of X and γ is a filter on $cC(X,\$)$ then $C\in\lim_{c[X,\$]}\gamma$ if and only if $\bigcap_{\mathcal{G}\in\gamma}\operatorname{cl}_X\left(\bigcup_{F\in\mathcal{G}}F\right)\subset C$, that is, $\operatorname{adh}_X|\gamma|\subset C$ where $|\gamma|:=\left\{\bigcup_{F\in\mathcal{G}}F:\mathcal{G}\in\gamma\right\}$.

[6]We do not treat here *hit-and-miss* convergences, like the *Vietoris topology* or *Fell topology.*

where $\mathcal{O}_X^\natural(\gamma)$ is generated by $\{\mathcal{O}_X(\mathcal{G}) : \mathcal{G} \in \gamma\}$. Observe that if O_0, O_1 are open subsets of Z and $\mathcal{F}$ is a filter on $C(X,Z)$ then $\mathcal{O}_X^\natural(\mathcal{F}^-(O_0)) \leq \mathcal{O}_X^\natural(\mathcal{F}^-(O_1))$ whenever $O_0 \subset O_1$ ([7]). When considering upper regular convergences, we will often identify a filter γ on $C(X,\$)$ and its upper regularization $\mathcal{O}_X^\natural(\gamma)$. With this convention, the previous observation becomes

$$O_0 \subset O_1 \Longrightarrow \mathcal{F}^-(O_0) \leq \mathcal{F}^-(O_1). \tag{4.1}$$

Proposition 4.1. *Each lower topology on $C(X,\$)$ is upper regular.*

Proof. It is enough to show that if $\mathcal{G} \subset C(X,\$)$ is open then $\mathcal{G} = \mathcal{O}_X(\mathcal{G})$. Let $A \supset G \in \mathcal{G}$. Then the principal ultrafilter $\{A\}^\uparrow$ of A converges to A and therefore to G, because the topology is lower. Because $\mathcal{G}$ is open, $\mathcal{G} \in \{A\}^\uparrow$ so that $A \in \mathcal{G}$. Hence $\mathcal{G} = \mathcal{O}_X(\mathcal{G})$. □

Lemma 4.2. *If $X \neq \varnothing$ and $p(X,\$) \leq \tau \leq [X,\$]$, then*

$$\mathrm{cl}_\tau \{A\} = \{O \in C(X,\$) : O \subset A\}$$

for each $A \in C(X,\$)$.

Proof. To see that $\mathrm{cl}_\tau \{A\} = \{O \in C(X,\$) : O \subset A\}$, note first that

$$\{O \in C(X,\$) : O \subset A\} \subset \mathrm{cl}_{[X,\$]} \{A\} \subset \mathrm{cl}_\tau \{A\} \subset \mathrm{cl}_{p(X,\$)} \{A\},$$

where the first inclusion follows from the fact that $[X,\$]$ is lower, and the others from the assumption $p(X,\$) \leq \tau \leq [X,\$]$. Moreover, if $O \in \mathrm{cl}_{p(X,\$)} \{A\}$ then every $p(X,\$)$-open neighborhood of O contains A. In particular, $A \in \mathcal{O}_X(x)$ for each $x \in O$, so that $O \subset A$. □

Proposition 4.3. *If $X \neq \varnothing$ and $p(X,\$) \leq \tau \leq [X,\$]$, then τ is T_0 but is not T_1.*

Proof. By Lemma 4.2, if $A_1 \neq A_0$, say, there is $x \in A_1 \setminus A_0$, then $\mathrm{cl}_\tau \{A_0\} = \{O \in C(X,\$) : O \subset A_0\}$ is τ-closed and contains A_0 but not A_1 and the convergence is therefore T_0. As $X \neq \varnothing$ and $\varnothing \in \mathrm{cl}_\tau \{X\}$, the convergence τ is not T_1. □

We say that a convergence τ on $C(X,\$)$ *respects directed sups* if whenever $\{\gamma_i : i \in I\}$ and $\{B_i : i \in I\}$ are two directed families of filters on $C(X,\$)$ and elements of $C(X,\$)$ respectively, such that $B_i \in \lim_\tau \gamma_i$ for each i in I, we have that $\bigcup_{i\in I} B_i \in \lim_\tau \bigvee_{i\in I} \gamma_i$.

[7]Indeed, if $O_0 \subset O_1$ then $f^-(O_0) \subset f^-(O_1)$, hence $\bigcup_{f\in F} \{P \in C(X,\$) : P \supset f^-(O_0)\} \supset \bigcup_{f\in F} \{P \in C(X,\$) : P \supset f^-(O_1)\}$.

A compact, lower, upper regular pseudotopology τ on $C(X,\$)$ that respects directed sups is called a *solid hyperconvergence* ([8]).

Note that in a solid hyperconvergence, every filter converges. Indeed, every ultrafilter is convergent by compactness, so that every ultrafilter converges to $\varnothing$ because the convergence is lower. As the convergence is pseudotopological, *every filter converges to $\varnothing$ in a solid hyperconvergence.*

Proposition 4.4. *$[X,\$]$ and $\alpha(X,\$)$ are solid hyperconvergences provided that $\alpha \subset \kappa(X)$.*

Proof. $[X,\$]$ is well known to be pseudotopological (e.g., [9], [17]). In view of (3.7), it is lower, and compact because every filter converges to $\varnothing$. It is upper regular by Proposition 5.2.

It respects directed sups because if $B_i \in \lim_{[X,\$]} \gamma_i$ for each $i \in I$, where the family $\{\gamma_i : i \in I\}$ is directed, then for each $x \in \bigcup_{i\in I} B_i$ there is i such that $x \in B_i \in \lim_{[X,\$]} \gamma_i$, so that there is $\mathcal{G} \in \gamma_i$ with $x \in \operatorname{int}\left(\bigcap_{G\in\mathcal{G}} G\right)$. As $\mathcal{G} \in \gamma_i \leq \bigvee_{i\in I} \gamma_i$, we have $\bigcup_{i\in I} B_i \subset \bigcup_{\mathcal{G}\in\bigvee_{i\in I}\gamma_i} \operatorname{int}\left(\bigcap_{G\in\mathcal{G}} G\right)$.

We have seen that $\alpha(X,\$) \leq [X,\$]$ whenever $\alpha \subset \kappa(X)$ because $T[X,\$] = \kappa(X,\$)$, so that $\alpha(X,\$)$ is compact because $[X,\$]$ is. It is lower (and therefore upper regular by Proposition 4.1) because $\mathcal{A} = \mathcal{O}_X(\mathcal{A})$ for each $\mathcal{A} \in \alpha$. To see that it respects directed sups, assume that $B_i \in \lim_{\alpha(X,\$)} \gamma_i$ for each $i \in I$, where the families $\{B_i : i \in I\}$ and $\{\gamma_i : i \in I\}$ are directed, and consider $\mathcal{A} \in \alpha$ containing $\bigcup_{i\in I} B_i$. By compactness of $\mathcal{A}$ there is a finite subset F of I such that $\bigcup_{i\in F} B_i \in \mathcal{A}$. Since $\{B_i : i \in I\}$ is directed, there is $i_F \in I$ such that $\bigcup_{i\in F} B_i \subset B_{i_F} \in \mathcal{A}$. Since $B_{i_F} \in \lim_{\alpha(X,\$)} \gamma_{i_F}$, the open set $\mathcal{A}$ belongs to γ_{i_F}, hence to $\bigvee_{i\in I} \gamma_i$. Therefore $\bigcup_{i\in I} B_i \in \lim_{\alpha(X,\$)} \bigvee_{i\in I} \gamma_i$. □

Proposition 4.5. *If τ is a solid hyperconvergence, $\mathcal{B}$ is an up-directed basis for the topology of Z, and $\mathcal{C}$ is a down-directed basis of closed sets in Z, then $f \in \lim_{\tau^{\Uparrow}} \mathcal{F}$ if and only if*

$$\forall_{B\in\mathcal{B}}\ f^-(B) \in \lim_\tau \mathcal{F}^-(B),$$

if and only if

$$\forall_{C\in\mathcal{C}}\ f^-(C) \in \lim_{c\tau} \mathcal{F}^-(C).$$

Proof. We only need to show the first equivalence. Assume that $\forall_{B\in\mathcal{B}}\ f^-(B) \in \lim_\tau \mathcal{F}^-(B)$. In view of (2.1), it is enough to show that $f^-(O) \in \lim_\tau \mathcal{F}^-(O)$

[8]Notions of *upper convergence*, *lower regularity* and respecting directed sups for a convergence on $cC(X,\$)$ are defined dually, and a compact lower regular upper pseudotopology on $cC(X,\$)$ that respects directed sups is also called *solid hyperconvergence*.

whenever $O \in C(Z,\$)$. Consider a family $\{B_i : i \in I\} \subset \mathcal{B}$ such that $O = \bigcup_{i\in I} B_i$. Because $\mathcal{B}$ is up-directed, $\{f^-(B_i) : i \in I\}$ is as well. Moreover, $f^-(B_i) \in \lim_\tau \mathcal{F}^-(B_i)$ for each $i \in I$ and in view of (4.1), the family of filters $\{\mathcal{F}^-(B_i) : i \in I\}$ is directed. Since τ respects directed sup,

$$f^-(O) = \bigcup_{i\in I} f^-(B_i) \in \lim_\tau \bigvee_{i\in I} \mathcal{F}^-(B_i).$$

Moreover, $\mathcal{F}^-(O) \geq \bigvee_{i\in I} \mathcal{F}^-(B_i)$ by (4.1) so that $f^-(O) \in \lim_\tau \mathcal{F}^-(O)$, which concludes the proof. □

5. Interplay between hyperconvergences and the underlying topologies

Recall that for a family $\mathcal{P}$ of subsets of X, we denote

$$\mathcal{O}_X^\natural(\mathcal{P}) := \{\mathcal{O}_X(P) : P \in \mathcal{P}\}.$$

Two families $\mathcal{A}$ and $\mathcal{B}$ of subsets of the same set X *mesh*, in symbols $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \varnothing$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We write $A\#\mathcal{B}$ for $\{A\}\#\mathcal{B}$.

Proposition 5.1. *The following are equivalent:*

(1) $\mathcal{R}\#\mathcal{O}_X^\natural(\mathcal{P})$ *in* $C(X,\$)$;
(2) $\mathcal{P}$ *is a refinement of* $\mathcal{R}$;
(3) $\mathcal{O}_X^\natural(\mathcal{P}) \leq \mathcal{O}_X^\natural(\mathcal{R})$.

Proof. By definition, $\mathcal{R}\#\mathcal{O}_X^\natural(\mathcal{P})$ if and only if for each $P \in \mathcal{P}$ there is $R \in \mathcal{R}$ with $P \subset R$, which means that $\mathcal{P}$ is a refinement of $\mathcal{R}$. Equivalently, for each $P \in \mathcal{P}$ there is $R \in \mathcal{R}$ such that $\mathcal{O}_X(R) \subset \mathcal{O}_X(P)$, that is, $\mathcal{O}_X^\natural(\mathcal{P}) \leq \mathcal{O}_X^\natural(\mathcal{R})$. □

A family $\mathcal{P}$ is said to be an *ideal subbase* if it is up-directed by inclusion, that is, if for each finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ there is $P \in \mathcal{P}$ such that $P \supset \bigcup \mathcal{P}_0$. We chose this terminology because if $\mathcal{P}$ is an ideal subbase then the family $\mathcal{P}^\cup$ of finite unions of elements of $\mathcal{P}$ is an *ideal base*, in the sense that the family $(\mathcal{P}^\cup)^\downarrow$ of subsets of elements of $\mathcal{P}^\cup$ is an ideal. Note that $\mathcal{O}_X^\natural(\mathcal{P})$ is a filter base if and only if $\mathcal{P}$ is an ideal subbase ([9]).

If γ is a filter on $C(X,\$)$ then the *reduced ideal of* γ is the ideal for which

$$\gamma^{\downarrow\downarrow} := \{\textstyle\bigcap_{G\in\mathcal{G}} G : \mathcal{G} \in \gamma\} \tag{5.1}$$

is an ideal subbase.

[9]In fact, if $P_0 \cup P_1 \subset P$, then $\mathcal{O}_X(P_0) \cap \mathcal{O}_X(P_1) = \mathcal{O}_X(P_0 \cup P_1) \supset \mathcal{O}_X(P)$.

As usual, we extend, in an obvious way, the convergence of filters to that of their filter bases ([10]). A set of filters $\mathbb{B}$ is a *convergence base* of a convergence τ on Y if for every $y \in Y$ and each $\mathcal{F}$ with $y \in \lim_\tau \mathcal{F}$ there is $\mathcal{B} \in \mathbb{B}$ such that $\mathcal{B} \leq \mathcal{F}$ and with $y \in \lim_\tau \mathcal{B}$.

Proposition 5.2. $[X, \$]$ *admits a convergence base generated by* $\mathcal{O}_X^\natural(\mathcal{P})$*, where the families* $\mathcal{P}$ *are ideal subbases.*

Proof. If $Y \in \lim_{[X,\$]} \gamma$ then, by (3.7), the family

$$\mathcal{P} := \left\{\operatorname{int}(A) : A \in \gamma^{\downarrow\downarrow}\right\}$$

is an open cover of Y. Clearly, $\mathcal{P}$ is an ideal subbase, hence $\mathcal{O}_X^\natural(\mathcal{P})$ is a filter base. As for each $P \in \mathcal{P}$ there is $\mathcal{G} \in \gamma$ such that $P = \operatorname{int}\left(\bigcap_{G \in \mathcal{G}} G\right)$ we infer that $\mathcal{G} \subset \mathcal{O}_X^\natural(P)$, that is, $\mathcal{O}_X^\natural(\mathcal{P})$ is coarser than γ. Finally $Y \in \lim_{[X,\$]} \mathcal{O}_X^\natural(\mathcal{P})$, because $\bigcap \mathcal{O}_X^\natural(P) = P$ for each $P \in \mathcal{P}$, and thus (3.7) holds. □

Proposition 5.3. *If* $\mathcal{P} \subset C(X, \$)$ *is an ideal subbase and* τ *is an upper regular convergence on* $C(X, \$)$ *then*

$$\operatorname{adh}_\tau \mathcal{P} = \lim_\tau \mathcal{O}^\natural(\mathcal{P}).$$

Proof. As $\mathcal{O}^\natural(\mathcal{P}) \# \mathcal{P}$, it is clear that $\lim_\tau \mathcal{O}^\natural(\mathcal{P}) \subset \operatorname{adh}_\tau \mathcal{P}$. Conversely, if $U \in \operatorname{adh}_\tau \mathcal{P}$ there is a filter $\eta = \mathcal{O}^\natural(\eta)$ meshing with $\mathcal{P}$ such that $U \in \lim_\tau \eta$. In other words, for each $\mathcal{A} = \mathcal{O}(\mathcal{A}) \in \eta$ there is $P \in \mathcal{P} \cap \mathcal{A}$. Thus $\mathcal{O}(P) \subset \mathcal{A}$ and $\mathcal{O}^\natural(\mathcal{P}) \geq \eta$ so that $U \in \lim_\tau \mathcal{O}^\natural(\mathcal{P})$. □

If $\mathcal{P} \subset C(X, \$)$, we denote by $\mathcal{P}^\cup$ the ideal base generated by $\mathcal{P}$.

Proposition 5.4. *Let* τ *be an upper regular convergence on* $C(X, \$)$ *such that* $p(X, \$) \leq \tau \leq [X, \$]$ *and let* $\mathcal{P} \subset C(X, \$)$*. Then* $\mathcal{P}$ *is a cover of* U *if and only if* $U \in \operatorname{adh}_\tau \mathcal{P}^\cup$.

Proof. If $\mathcal{P}$ is a cover of U so is the ideal base $\mathcal{P}^\cup$, so that $U \in \lim_{[X,\$]} \mathcal{O}^\natural(\mathcal{P}^\cup)$ by Proposition 5.2. Moreover, $\mathcal{O}^\natural(\mathcal{P}^\cup) \# \mathcal{P}^\cup$ so that $U \in \operatorname{adh}_{[X,\$]} \mathcal{P}^\cup \subset \operatorname{adh}_\tau \mathcal{P}^\cup$. Conversely, if $U \in \operatorname{adh}_\tau \mathcal{P}^\cup$ then by Proposition 5.3, $U \in \lim_\tau \mathcal{O}^\natural(\mathcal{P}^\cup) \subset \lim_{p(X,\$)} \mathcal{O}^\natural(\mathcal{P}^\cup)$. Therefore, by definition of $p(X, \$)$, for each $x \in U$ there is $S \in \mathcal{P}^\cup$ such that $\mathcal{O}(S) \subset \mathcal{O}(x)$, that is, $x \in S$. Thus there is $P \in \mathcal{P}$ containing x and $\mathcal{P}$ is a cover of U. □

[10] If $\mathcal{B}$ is a filter base and τ is a convergence, then $y \in \lim_\tau \mathcal{B}$ if $y \in \lim_\tau \mathcal{B}^\uparrow$, where $\mathcal{B}^\uparrow$ is the filter generated by $\mathcal{B}$.

Corollary 5.5. *If $\mathcal{P} \subset C(X,\$)$ is an ideal base and τ is an upper regular convergence on $C(X,\$)$ such that $p(X,\$) \leq \tau \leq [X,\$]$ then*

$$\mathrm{adh}_\tau \mathcal{P} = \lim_\tau \mathcal{O}^\natural(\mathcal{P}) = \lim_{[X,\$]} \mathcal{O}^\natural(\mathcal{P}) = \mathrm{adh}_{[X,\$]}\mathcal{P}$$

consists of those $U \in C(X,\$)$ for which $\mathcal{P}$ is a cover of U.

Even though pretopologies only depend on adherences of subsets, Corollary 5.5 does not mean that all the pretopological solid hyperconvergences between $p(X,\$)$ and $[X,\$]$ coincide! But their adherences of ideal bases are the same.

Example 5.6. Let X be an infinite countable set with the discrete topology. In this case $p(X,\$) = [X,\$]$. The hyperset $\mathcal{P} := \{\{x\} : x \in X\}$ is an open cover of X. By definition, $Y \in \mathrm{adh}_{p(X,\$)}\mathcal{P}$ if for each finite subset F of Y there is $A \in \mathcal{O}_X(F) \cap \mathcal{P}$. Hence there is $x \in X$ such that $F \subset \{x\}$, which means that the only finite subsets of Y are singletons, that is, Y is a singleton. On the other hand, $X \in \mathrm{adh}_{p(X,\$)}\mathcal{P}^\cup$, because $F \in \mathcal{O}_X(F) \cap \mathcal{P}^\cup$ for each finite subset F of Y.

If α is a collection of openly isotone families of subsets of X, we call $\mathcal{P} \subset C(X,\$)$ an (open) *α-cover* if $\mathcal{P} \cap \mathcal{A} \neq \varnothing$ for every $\mathcal{A} \in \alpha$. Of course, if $p(X) \subset \alpha$ then every open α-cover of X is also an open cover of X. Note that the notion of $p(X)$-cover coincides with the traditional notion of ω-cover, and that the notion of $k(X)$-cover coincides with the traditional notion of k-cover (see e.g., [33]). It follows immediately from the definitions that

Proposition 5.7. *Let $\mathcal{P} \subset C(X,\$)$ and let α be a topology on $C(X,\$)$. Then $U \in \mathrm{adh}_{\alpha(X,\$)}\mathcal{P}$ if and only if $\mathcal{P}$ is an α-cover of U.*

6. Transfer of filters

We shall confer particular attention to the convergence of a filter to the *zero function* for the convergence $\tau^\Uparrow$ on $C(X,\mathbb{R})$ that is preimage-wise with respect to a solid hyperconvergence τ on $C(X,\$)$. To that effect, consider a decreasing base of bounded open neighborhoods of 0 in $\mathbb{R}$:

$$\{W_n : n < \omega\}, \tag{6.1}$$

for instance, let us fix $W_n := \left\{r \in \mathbb{R} : |r| < \frac{1}{n}\right\}$.

Lemma 6.1. $\bar{0} \in \lim_{\tau^\Uparrow} \mathcal{F}$ *if and only if $X \in \lim_\tau \mathcal{F}^-(W_n)$ for each $n < \omega$.*

Proof. As $\bar{0}^-(O)$ is equal either to X (when $0 \in O$) or to $\varnothing$ (when $0 \notin O$), it follows from (2.1) that the condition is necessary. Conversely, if an open subset O of $\mathbb{R}$ contains 0, then there is $n < \omega$ such that $W_n \subset O$, hence $X \in \lim_\tau \mathcal{F}^-(W_n)$

implies that $X \in \lim_\tau \mathcal{F}^-(O)$, because $\mathcal{F}^-(W_n) \le \mathcal{F}^-(O)$. If now $0 \notin O$ then $\bar{0}^-(O) = \varnothing \in \lim_\tau \mathcal{F}^-(O)$, because τ is a solid hyperconvergence (hence every filter converges to $\varnothing$). □

This special case is important, because it is much easier to compare local properties of $\tau^{\Uparrow}$ at $\bar{0}$ with local properties of τ at X than to study analogous properties at an arbitrary $f \in C(X,\mathbb{R})$. Moreover, often a study of the mentioned special case is sufficient for the understanding of this local property at each $f \in C(X,\mathbb{R})$. This is feasible whenever all the translations are continuous for $\tau^{\Uparrow}$, that is, whenever $\tau^{\Uparrow}$ is translation-invariant. It is known that the topology of pointwise convergence, the compact-open topology, the natural convergence and thus the natural topology are translation-invariant. Translations are not always continuous for the Isbell topology (see [16], [27]), but for each topological space X, there exists the finest translation-invariant topology of the form $\alpha(X,\mathbb{R})$ that is coarser than the Isbell topology $\kappa(X,\mathbb{R})$ [15].

Lemma 6.1 suggests that local properties of $\tau^{\Uparrow}$ at $\bar{0}$ "correspond" to local properties of τ at X. The remainder of the paper is devoted to making this statement clear and exploring applications.

If α is a filter on $C(X,\$)$ then, for each (open) subset W of $\mathbb{R}$,

$$[\alpha, W] := \{[\mathcal{A}, W] : \mathcal{A} \in \alpha\} \tag{6.2}$$

is a filter base on $C(X,\mathbb{R})$, called the W-*erected filter* of α. Note that

$$\alpha \le \gamma, W \supset V \Longrightarrow [\alpha, W] \le [\gamma, V]. \tag{6.3}$$

The filter on $C(X,\mathbb{R})$ generated by the filter base $\mathcal{V}$ of the neighborhood filter of 0 ([11])

$$\bigcup_{V \in \mathcal{V}} [\alpha, V]$$

does not depend on the choice of a particular neighborhood base of 0 in $\mathbb{R}$ ([12]). We denote it by $[\alpha, \mathcal{N}(0)]$ and call it the *erected filter* of α. In particular, if a base is of the form (6.1), $[\alpha, W_n] \le [\alpha, W_{n+1}]$ and

$$[\alpha, \mathcal{N}(0)] = \bigvee_{n<\omega} [\alpha, W_n]. \tag{6.4}$$

[11] Indeed, if $B_0, B_1 \in \bigcup_{V \in \mathcal{V}(0)} [\alpha, V]$, then there are $V_0, V_1 \in \mathcal{V}(0)$ and $\mathcal{A}_0, \mathcal{A}_1 \in \alpha$ such that $[\mathcal{A}_0, V_0] \subset B_0$ and $[\mathcal{A}_1, V_1] \subset B_1$, and thus $[\mathcal{A}_0 \cap \mathcal{A}_1, V_0 \cap V_1] \subset [\mathcal{A}_0, V_0] \cap [\mathcal{A}_1, V_1] \subset B_0 \cap B_1$.

[12] In fact, if $\mathcal{V}, \mathcal{W}$ are open bases (of the neighborhood filter of 0) then for each $W \in \mathcal{W}$ there is $V \in \mathcal{V}$ such that $V \subset W$, hence $[\alpha, W] \le [\alpha, V]$, and conversely.

We shall see that if α converges to X in τ then its erected filter converges to the null function in $\tau^{\Uparrow}$. We shall in fact consider a more general case.

Lemma 6.2. *If $\{\alpha_n : n < \omega\}$ is a sequence of filters on $C(X,\$)$ such that $\alpha_n = O_X^{\natural}(\alpha_n)$, then the sequence of filters $([\alpha_n, W_n])_{n<\omega}$ admits a supremum.*

Proof. If $S_1, \ldots, S_k \in \bigcup_{n<\omega}[\alpha_n, W_n]$, then there are $n_1, \ldots, n_k < \omega$, say, $n_1 \leq \ldots \leq n_k$ and $\mathcal{A}_j \in \alpha_{n_j}$ for $1 \leq j \leq k$ such that $[\mathcal{A}_j, W_{n_j}] \subset S_j$, and thus $[\mathcal{A}, W_{n_k}] \subset \bigcap_{1\leq j\leq k}[\mathcal{A}_j, W_{n_j}] \subset \bigcap_{1\leq j\leq k} S_j$, where $\mathcal{A} := \bigcap_{1\leq j\leq k} \mathcal{A}_j$. As each α_n is based in openly isotone families $\mathcal{A}$, and $[\mathcal{A}, W] \neq \varnothing$ provided that $W \neq \varnothing$, the family $\bigcup_{n<\omega}[\alpha_n, W_n]$ is a filter subbase and generates $\bigvee_{n<\omega}[\alpha_n, W_n]$. □

Theorem 6.3. *If $X \in \lim_\tau \alpha_n$ for each $n < \omega$, then $\bar{0} \in \lim_{\tau^{\Uparrow}} \bigvee_{n<\omega}[\alpha_n, W_n]$.*

Proof. We use Lemma 6.1 to check that $\bar{0} \in \lim_{\tau^{\Uparrow}} \bigvee_{n<\omega}[\alpha_n, W_n]$. Let O be an open subset of $\mathbb{R}$ with 0. Then there is $n < \omega$ such that $O \supset W_k$ for $k \geq n$. Then $[\mathcal{A}, W_n]^-(O) = \{f^-(O) : f^-(W_n) \in \mathcal{A}\} \subset \mathcal{A}$ for every $\mathcal{A}$. It follows that, $[\alpha_n, W_n]^-(O) \geq \alpha_n$ so that $X \in \lim_\tau [\alpha_n, W_n]^-(O) \subset \lim_\tau \left(\bigvee_{n<\omega}[\alpha_n, W_n]\right)^-(O)$. □

Corollary 6.4. *If $X \in \lim_\tau \alpha$ then $\bar{0} \in \lim_{\tau^{\Uparrow}} [\alpha, \mathcal{N}(0)]$.*

In view of (4.1), we have $\bigvee_{n<\omega}[\alpha, W_n] \leq [\alpha, \{0\}]$ ([13]). Thus:

Corollary 6.5. *If $X \in \lim_\tau \alpha$ then $\bar{0} \in \lim_{\tau^{\Uparrow}} [\alpha, \{0\}]$.*

7. Construction of classes of filters corresponding to local properties of convergences

In the present section, we introduce the relevant terminology, as well as examples of local properties to be considered.

If $\mathbb{D}$ denotes a class of filters, then $\mathbb{D}(X)$ designs the set of filters on X of the class $\mathbb{D}$. The class of principal filters is denoted by $\mathbb{F}_0$ and the class of countably based filters is denoted by $\mathbb{F}_1$. More generally, $\mathbb{F}_\kappa$ stands for the class of filters that admit a base of cardinality less than $\aleph_\kappa$.

A convergence (in particular, a topological) space X is called $\mathbb{D}$-*based at* x if whenever for $x \in \lim \mathcal{F}$ there is $\mathcal{D} \in \mathbb{D}(X)$, $\mathcal{D} \leq \mathcal{F}$ with $x \in \lim \mathcal{D}$, and $\mathbb{D}$-*based* if it is $\mathbb{D}$-based at each $x \in X$.

If $*$ is a relation on $\mathbb{F}(X)$ and $\mathbb{D} \subset \mathbb{F}(X)$, then

$$\mathbb{D}^* := \bigcap_{\mathcal{D}\in\mathbb{D}} \{\mathcal{F} \in \mathbb{F}(X) : \mathcal{F} * \mathcal{D}\}. \tag{7.1}$$

[13] Here $f^-(0)$ is not open, so that we must use the general definition $[\mathcal{A}, \{0\}] := \{f : \exists_{A\in\mathcal{A}}\, A \subset f^-(0)\}$.

Many local topological properties of a space X can be expressed by: X is $\mathbb{D}^*$-based, for $\mathbb{D} = \mathbb{F}_0$ or $\mathbb{D} = \mathbb{F}_1$ and an appropriate relation $*$. Such a representation is essential for our approach.

Let us list several such properties. Recall that a topological space is said to be

(1) *first-countable* if each neighborhood filter has a countable filter-base, that is, if it is $\mathbb{F}_1$-based; The *character* $\chi(X)$ of X is the least ordinal λ such that each neighborhood filter has a filter-base of cardinality at most λ;
(2) *Fréchet* ([14]) if whenever $x \in \operatorname{adh} A$, there is a sequence (equivalently, a countably based filter) on A that converges to x;
(3) *strongly Fréchet* if whenever $x \in \operatorname{adh} \mathcal{H}$ for a countbly based filter $\mathcal{H}$, there exists a sequence (equivalently, a countably based filter) finer than $\mathcal{H}$ that converges to x;
(4) *productively Fréchet* if its product with every strongly Fréchet space is Fréchet;
(5) *countably tight* if whenever $x \in \operatorname{adh} A$ there is a countable subset B of A with $x \in \operatorname{adh} B$; More generally, the *tightness* $t(X)$ of X is the least ordinal λ such that whenever $x \in \operatorname{adh} A$, there is a subset B of A of cardinality at most λ such that $x \in \operatorname{adh} B$;
(6) *countably fan-tight* if whenever $x \in \bigcap_{n\in\omega} \operatorname{adh} A_n$, there are finite subsets B_n of A_n with $x \in \operatorname{adh} \bigcup_{n<\omega} B_n$;
(7) *strongly countably fan-tight* if whenever $x \in \bigcap_{n\in\omega} \operatorname{adh} A_n$, there is for each n a point $a_n \in A_n$ with $x \in \operatorname{adh} \{a_n : n < \omega\}$.

The *tightness* $t(X, x)$ of a topological space X at $x \in X$ is defined as the least ordinal λ such that whenever $x \in \operatorname{adh} A$ there is a subset B of A of cardinality at most λ such that $x \in \operatorname{adh} B$.

The *fan-tightness* $\operatorname{vet}(X, x)$ of a topological space X at $x \in X$ is defined as the least ordinal λ such that whenever $x \in \bigcap_{i\in I} \operatorname{adh} A_i$ where I has cardinality at most λ, there are subsets B_i of A_i of cardinality less than λ such that $x \in \operatorname{adh} \bigcup_{i\in I} B_i$.

The *strong fan-tightness* $\operatorname{vet}^*(X, x)$ of a topological space X at $x \in X$ is defined as the least ordinal λ such that whenever $x \in \bigcap_{i\in I} \operatorname{adh} A_i$ where I has cardinality at most λ, there are points a_i of A_i such that $x \in \operatorname{adh}\{a_i : i \in I\}$.

The *tightness* $t(X)$ of a topological space X is $\bigvee_{x\in X} t(X, x)$; the *fan-tightness* $\operatorname{vet}(X)$ of a topological space X is $\bigvee_{x\in X} \operatorname{vet}(X, x)$; the *strong fan-tightness* $\operatorname{vet}^*(X)$ of a topological space X is $\bigvee_{x\in X} \operatorname{vet}^*(X, x)$.

[14] often called Fréchet-Urysohn, but we use the shorter term *Fréchet*.

First-countability is readily interpreted in terms of being based in a certain class of filters. Each of the other properties listed above can also easily be interpreted in those terms. A filter $\mathcal{F}$ is:

(1) *Fréchet* if whenever $A \in \mathcal{F}^{\#}$ there is a countably based filter $\mathcal{H} \geq \mathcal{F} \vee A$;
(2) *strongly Fréchet* if whenever $\mathcal{A} \in \mathbb{F}_1$ and $\mathcal{A}\#\mathcal{F}$ there exists a countably based filter $\mathcal{H} \geq \mathcal{F} \vee \mathcal{A}$;
(3) *productively Fréchet* if whenever $\mathcal{A}$ is a strongly Fréchet filter and $\mathcal{A}\#\mathcal{F}$ there exists a countably based filter $\mathcal{H} \geq \mathcal{F} \vee \mathcal{A}$;
(4) *countably tight* if whenever $A \in \mathcal{F}^{\#}$ there is a countable subset B of A with $B \in \mathcal{F}^{\#}$;
(5) *countably fan-tight* if whenever $A_n \in \mathcal{F}^{\#}$ for each $n < \omega$, there are finite subsets B_n of A_n with $\bigcup_{n<\omega} B_n \in \mathcal{F}^{\#}$;
(6) *strongly countably fan-tight* if whenever $A_n \in \mathcal{F}^{\#}$ for each $n < \omega$, there are points a_n of A_n with $\{a_n : n < \omega\} \in \mathcal{F}^{\#}$.

It is easily seen that a topological space is Fréchet, strongly Fréchet, countably tight, countably fan-tight, and strongly countably fan-tight respectively, if and only if each neighborhood filter has the filter property of the same name. This is also true of productive Fréchetness by [28]. In other words, a topological space has one of these properties if and only if it is $\mathbb{D}$-based, for the class $\mathbb{D}$ of filters satisfying the filter property of the same name. It turns out that these classes of filters can be represented under the form $\mathbb{F}_0^*$ or $\mathbb{F}_1^*$.

If $\mathbb{D}$ and $\mathbb{J}$ are two classes of filters, we say that $\mathbb{D}$ is $\mathbb{J}$-*steady* if

$$\mathcal{D} \in \mathbb{D}, \mathcal{J} \in \mathbb{J}, \mathcal{D}\#\mathcal{J} \Longrightarrow \mathcal{D} \vee \mathcal{J} \in \mathbb{D}.$$

As usual, if $R \subset X \times Y$ and $D \subset X$ then $RD := \{y \in Y : \exists x \in D, : (x, y) \in R\}$ and $\mathcal{RD} := \{RD : R \in \mathcal{R}, D \in \mathcal{D}\}$.

A class $\mathbb{D}$ is $\mathbb{J}$-*composable* if

$$\mathcal{D} \in \mathbb{D}(X), \mathcal{R} \in \mathbb{J}\,(X \times Y) \Longrightarrow \mathcal{RD} \in \mathbb{D}(Y).$$

By convention, we consider that each class $\mathbb{D}$ contains every degenerate filter. In the sequel, classes that are $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady will be of particular interest.

For each set X, we consider the following relations $\Diamond_\kappa$, $\dagger$ and $\triangle$ on $\mathbb{F}(X)$: we write $\mathcal{F}\Diamond_\kappa\mathcal{H}$ if

$$\mathcal{F}\#\mathcal{H} \Longrightarrow \exists A \in [X]^{\leq\kappa} : A\#\left(\mathcal{F} \vee \mathcal{H}\right);$$

we denote by $\mathcal{F}\triangle_\kappa\mathcal{H}$ the following relation

$$\mathcal{F}\#\mathcal{H} \Longrightarrow \exists \mathcal{L} \in \mathbb{F}_\kappa : \mathcal{L} \geq \mathcal{F} \vee \mathcal{H}.$$

Finally, we write $\mathcal{F}\dagger_1\mathcal{H}$ if $\mathcal{F}\vee\mathcal{H}\in\mathbb{T}_1$ where $\mathcal{T}\in\mathbb{T}_1$ if

$$(A_n)_{n<\omega}\#\mathcal{T}\Longrightarrow\exists B_n\in[A_n]^{<\omega}:\left(\bigcup_{n<\omega}B_n\right)\#\mathcal{T},$$

and $\mathcal{F}\dagger_0\mathcal{H}$ if $\mathcal{F}\vee\mathcal{H}\in\mathbb{T}_0$ where $\mathcal{T}\in\mathbb{T}_0$ if

$$(A_n)_{n<\omega}\#\mathcal{T}\Longrightarrow\exists a_n\in A_n:\{a_n:n\in\omega\}\#\mathcal{T}.$$

Hence, a topological space, and by extension, a convergence space, is respectively *Fréchet* , *strongly Fréchet*, *productively Fréchet*, of *κ-tightness*, *countably fan-tight*, *strongly countably fan-tight*, if and only if it is $\mathbb{F}_0^{\triangle}$-based, $\mathbb{F}_1^{\triangle}$-based, $\mathbb{F}_1^{\triangle\triangle}$-based, $\mathbb{F}_1^{\Diamond_\kappa}$-based, $\mathbb{F}_1^{\dagger_1}$-based, $\mathbb{F}_1^{\dagger_0}$-based respectively. Here we gather the just mentioned equivalences:

(7.2)

class	based
Fréchet	$\mathbb{F}_0^{\triangle}$-based
strongly Fréchet	$\mathbb{F}_1^{\triangle}$-based
productively Fréchet	$\mathbb{F}_1^{\triangle\triangle}$-based
κ-tight	$\mathbb{F}_1^{\Diamond_\kappa}$-based
countably fan-tight	$\mathbb{F}_1^{\dagger_1}$-based
strongly countably fan-tight	$\mathbb{F}_1^{\dagger_0}$-based

Examples of $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady classes include the class $\mathbb{F}_n$ of filters with a filter-base of cardinality less than $\aleph_n$ for $n\geq 1$, as well as $\mathbb{F}_1^{\triangle}$, $\mathbb{F}_1^{\triangle\triangle}$, $\mathbb{F}_1^{\Diamond_\kappa}$, $\mathbb{F}_1^{\dagger_1}$ and $\mathbb{F}_1^{\dagger_0}$. The class $\mathbb{F}_0^{\triangle}$ of *Fréchet filters* is $\mathbb{F}_0$-composable but not $\mathbb{F}_1$-steady, and the class $\mathbb{F}_1^{\Diamond\Diamond}$ of *steadily countably tight filters* is $\mathbb{F}_1$-steady but not $\mathbb{F}_0$-composable.

See [29] for a systematic study of these concepts and applications to product theorems.

8. Transfer of classes of filters

Local properties of a topological space depend on properties of its neighborhood filters. More generally, local properties of a convergence space depend on properties of the sets of filters convergent to every point. To understand how τ and $\tau^{\Uparrow}$ are related, we first need to understand how the properties of the filter α relate to those of the filter $[\alpha,\mathcal{N}(0)]$ in Corollary 6.4.

We notice that the erected filter $[\alpha,\mathcal{N}(0)]$ of α can be reconstructed from α with the aid of compositions of relations as follows. Let $\Delta:=\{(f,A,k):A\subset f^-(W_k)\}$ and let Δ_j be the j-th projection of Δ. Let $\mathcal{N}$ stand for the cofinite filter on ω.

Proposition 8.1.

$$[\alpha, \mathcal{N}(0)] = \Delta_1(\Delta_2^- \alpha \vee \Delta_3^- \mathcal{N}).$$

Proof. If $\mathcal{A} \in \alpha$ then $\Delta_2^- \mathcal{A} = \{(f, A, k) : f^-(W_k) \in \mathcal{A}\}$. If $n < \omega$ then $\Delta_3^- \{k : k \geq n\} = \{(f, A, k) : \exists_{k \geq n}\, A \subset f^-(W_k)\}$. Hence

$$\Delta_2^- \mathcal{A} \vee \Delta_3^- \{k : k \geq n\} = \bigcup_{k \geq n} \left\{(f, A, k) : f^-(W_k) \in \mathcal{A}\right\},$$

and thus $\Delta_1 \left(\Delta_2^- \mathcal{A} \vee \Delta_3^- \{k : k \geq n\}\right) = \bigcup_{k \geq n} \{f : f^-(W_k) \in \mathcal{A}\} = \bigcup_{k \geq n} [\mathcal{A}, W_n]$. Because $W_k \subset W_n$ if $k \geq n$, hence $[\mathcal{A}, W_k] \subset [\mathcal{A}, W_n]$. Consequently,

$$\Delta_1(\Delta_2^- \alpha \vee \Delta_3^- \mathcal{N}) = \{[\mathcal{A}, W_n] : \mathcal{A} \in \alpha, n < \omega\} = [\alpha, (W_n)_n].$$

□

Corollary 8.2. *If $\mathbb{B}$ is an $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady class of filters and $\alpha \in \mathbb{B}$ then $[\alpha, \mathcal{N}(0)] \in \mathbb{B}$.*

Consider for each n, the relation $[\cdot, W_n] : C(X, \$) \to C(X, \mathbb{R})$. Note that the filter $\bigvee_{n<\omega} [\alpha_n, W_n]$ of Lemma 6.2 is the supremum of the images of the filters α_n under these relations. A class $\mathbb{B}$ of filters is *countably upper closed* if it is closed under countable suprema of increasing sequences. In particular:

Proposition 8.3. *If $\mathbb{B}$ is an $\mathbb{F}_0$-composable and countably upper closed class of filters, and if each $\alpha_n \in \mathbb{B}$, then $\bigvee_{n<\omega} [\alpha_n, W_n] \in \mathbb{B}$.*

Let $\mathcal{W} := \{W_n : n \leq \omega\}$ be a fixed base of $\mathcal{N}(0)$ in $\mathbb{R}$. Define

$$\mathcal{F}^{\mathcal{N}(0)} := \bigvee_{n<\omega} [\mathcal{F}^-(W_n), W_n], \tag{8.1}$$

on $C(X, \mathbb{R})$, associated with a filter $\mathcal{F}$ on $C(X, \mathbb{R})$. As $F \subset [F^-(W), W]$ for every $F \subset C(X, \mathbb{R})$ and $W \subset \mathbb{R}$,

$$\mathcal{F}^{\mathcal{N}(0)} \leq \mathcal{F}. \tag{8.2}$$

Proposition 8.4. *For each symmetric open intervals V, W that contain 0, there is a strictly increasing linear map h such that*

$$[\mathcal{F}^-(W), W] = h\left([\mathcal{F}^-(V), V]\right).$$

Proof. A base of $[\mathcal{F}^-(W), W]$ is of the form

$$G_F(W) := \left\{g : g^-(W) \in \left\{f^-(W) : f \in F\right\}\right\} : F \in \mathcal{F}$$

is a base of $[\mathcal{F}^-(W), W]$. Let h be a strictly increasing (linear) map such that $h(V) = W$. Then if $(h \circ g)^- (W) = g^-(V)$. Therefore $g \in G_F(V)$ if and only if $h \circ g \in G_F(W)$, that is, $G_F(W) = h(G_F(V))$. □

It follows that, if $W_n := r_n W$, where $W := (-1, 1)$ and $\{r_n\}_n$ is a decreasing sequence tending to 0, then

$$\mathcal{F}^{\mathcal{N}(0)} = \bigvee_{n<\omega} r_n \mathcal{H},$$

where $\mathcal{H} := [\mathcal{F}^-(W), W]$ and $r\mathcal{H} := \{rH : H \in \mathcal{H}\}$.

Corollary 8.5. *Let $\mathbb{B}$ be a class of filters.*

1. *If $\mathbb{B}$ is $\mathbb{F}_0$-composable and countably upper closed, and τ is $\mathbb{B}$-based at X, then $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$.*
2. *If $\mathbb{B}$ is $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady, and τ is a pretopology that is $\mathbb{B}$-based at X, then $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$.*

Proof. 1. If $\overline{0} \in \lim_{\tau^{\Uparrow}} \mathcal{F}$ then $X \in \lim_\tau \mathcal{F}^-(W_n)$ for each n. Therefore, for each n, there is $\mathcal{B}_n \in \mathbb{B}$ with $X \in \lim_\tau \mathcal{B}_n$ and $\mathcal{B}_n \leq \mathcal{F}^-(W_n)$. In view of Theorem 6.3, $\overline{0} \in \lim_{\tau^{\Uparrow}} \bigvee_{n<\omega}[\mathcal{B}_n, W_n]$. By Proposition 8.3, $\bigvee_{n<\omega}[\mathcal{B}_n, W_n] \in \mathbb{B}$. Moreover,

$$\bigvee_{n<\omega} [\mathcal{B}_n, W_n] \leq \mathcal{F}^{\mathcal{N}(0)} \leq \mathcal{F},$$

which concludes the proof.

2. If τ is pretopological, then in the proof above, for each n we can take $\mathcal{B}_n = \mathcal{V}_\tau(X)$, so that $\bigvee_{n<\omega}[\mathcal{B}_n, W_n] = [\mathcal{V}_\tau(X), \mathcal{N}(0)]$. By Proposition 8.1, $[\mathcal{V}_\tau(X), \mathcal{N}(0)] \in \mathbb{B}$. □

A filter α on $C(X, \$)$ valued in openly isotone families, can be reconstructed from its erected filter $[\alpha, \mathcal{N}(0)]$ with the aid of compositions of relations, provided that a separation condition by real-valued continuous functions holds:

A family $\mathcal{A} = \mathcal{O}_X(\mathcal{A})$ is *functionally separated* if for every $O \in \mathcal{A}$, there is $A \in \mathcal{A}$ and $h \in C(X, [0, 1])$ such that $h(A) = \{0\}$ and $h(X \setminus O) = \{1\}$. A filter on $C(X, \$)$ is called *functionally separated* if it admits a base of functionally separated sets. A solid hyperconvergence on $C(X, \$)$ is *functionally separated* if it is based in functionally separated filters.

It follows from [16, Lemma 2.5] that compact families on a completely regular space are functionally separated. Therefore:

Proposition 8.6. *If $\alpha \subset \kappa(X)$ then $\alpha(X, \$)$ is functionally separated.*

Lemma 8.7. *If X is normal, then $[X,\$]$ is functionally separated. Moreover, for each bounded open neighborhood W of 0 in $\mathbb{R}$, $[X,\$]$ has a base of filters α such that*

$$\alpha = \mathcal{O}^{\natural}\left(\alpha^{\downarrow\downarrow}\right) = \mathcal{O}^{\natural}\left(\left([\alpha,\mathcal{N}(0)]^{-}(W)\right)^{\downarrow\downarrow}\right)$$

where $\alpha^{\downarrow\downarrow}$ is the reduced ideal of α (5.1).

Proof. In view of (3.7), if $O \in \lim_{[X,\$]} \alpha$ then for each $x \in O$ there exists $\mathcal{A}_x \in \alpha$ such that $x \in \operatorname{int}_X(\bigcap_{U\in\mathcal{A}_x} U)$. By regularity, there is a closed neighborhood V_x of x such that $V_x \subset \operatorname{int}_X(\bigcap_{U\in\mathcal{A}_x} U)$. As the family $\mathcal{P} := \{\bigcup_{x\in S} V_x : S \in [O]^{<\infty}\}$ is an ideal base, $\mathcal{O}_X^{\natural}(\mathcal{P})$ is a filter-base on $C(X,\$)$; moreover, $\bigcap_{x\in S}\mathcal{A}_x \subset \mathcal{O}_X(\bigcup_{x\in S} V_x)$ for each $S \in [O]^{<\infty}$. Therefore $\alpha \geq \mathcal{O}_X^{\natural}(\mathcal{P})$ and $O \in \lim_{[X,\$]} \mathcal{O}_X^{\natural}(\mathcal{P})$. Finally, since $\mathcal{P}$ consists of closed sets and X is normal then $\mathcal{O}_X^{\natural}(\mathcal{P})$ is functionally separated, which completes the proof.

As shown in the first part of the proof, $[X,\$]$ has a base composed of filters $\alpha = \mathcal{O}_X^{\natural}(\mathcal{P})$ where $\mathcal{P}$ is an ideal base of closed sets. For each $P \in \mathcal{P}$, each $n \in \mathbb{N}$ consider the corresponding element

$$R := \bigcap_{f\in[P,W_n]} f^{-}(W)$$

of $([\alpha,\mathcal{N}(0)]^{-}(W))^{\downarrow\downarrow}$. Then $\mathcal{O}(P) \subset \mathcal{O}(R)$ so that $\alpha \geq \mathcal{O}^{\natural}\left(([\alpha,\mathcal{N}(0)]^{-}(W))^{\downarrow\downarrow}\right)$. Indeed, if $\mathcal{O}(P) \nsubseteq \mathcal{O}(R)$ then $R \nsubseteq P$ and there is $x \in R \setminus P$. By complete regularity, there is a continuous map $h \in C(X,\mathbb{R})$ such that $h(x) = 1 + \sup W$ and $h(P) = \{0\}$. Then $h \in [P,W_n]$ but $h(R) \nsubseteq W$; a contradiction. □

Let us call $\$$-*compatible* a class $\mathbb{B}$ of filters satisfying

$$\beta \in \mathbb{B}(C(X,\$)) \Longrightarrow \mathcal{O}^{\natural}(\beta^{\downarrow\downarrow}) \in \mathbb{B}(C(X,\$)).$$

Theorem 8.8. *If α is a filter on $C(X,\$)$ and W is an open bounded neighborhood of 0, then $\alpha \leq [\alpha,\mathcal{N}(0)]^{-}(W)$. If moreover α is functionally separated, then*

$$\alpha = [\alpha,\mathcal{N}(0)]^{-}(W).$$

Proof. 1. If n is such that $W_n \subset W$, then $[\mathcal{A},W_n]^{-}(W) \subset \mathcal{A}$ for each $\mathcal{A} \in \alpha$. Indeed, if $G \in [\mathcal{A},W_n]^{-}(W)$ then there is $A \in \mathcal{A}$ and $f \in C(X,\mathbb{R})$ such that $G = f^{-}(W)$ and $f(A) \subset W_n$. As $W_n \subset W$, we infer that $A \subset G$, so that $G \in \mathcal{A}$. Consequently $\alpha \leq [\alpha,\mathcal{N}(0)]^{-}(W) = \bigvee_{n<\omega}[\mathcal{A},W_n]^{-}(W)$.

2. If $G \in \mathcal{A}$ then, by the functional separation of $\mathcal{A}$, there is $A \in \mathcal{A}$ and $h \in C(X,\mathbb{R})$ such that $h(A) = \{0\}$ and $h(X \setminus G) = \{\sup W\}$. Therefore, $h \in [\mathcal{A},W_n]$ for each $n < \omega$, and $h^{-}(W) \subset G$ so that $G \in \mathcal{O}^{\natural}([\mathcal{A},W_n]^{-}(W))$, hence $[\alpha,W_n]^{-}(W) \leq \alpha$ for each $n < \omega$. □

In particular, $\alpha \leq \bigwedge_{n<\omega} [\alpha, \mathcal{N}(0)]^{-}(W_n)$ and if α is functionally separated, then the equality holds. On the other hand, if α is an ultrafilter then $\alpha = [\alpha, \mathcal{N}(0)]^{-}(W)$ for any open bounded neighborhood W of 0.

Consider the function $W_* : C(X, \mathbb{R}) \to C(X, \$)$ (defined by (2.2)). It follows from Theorem 8.8 that if α is functionally separated, then $\alpha = W_* [\alpha, \mathcal{N}(0)]$, that is, α is the image of $[\alpha, \mathcal{N}(0)]$ by a relation. This observation constitutes a considerable simplification of a construction proposed in [26] for the finite-open topologies and extended to α-topologies ([15]) in [12].

If $\mathbb{B}$ is a class of filters, let $\mathbb{B}^{\wedge}$ denote the class of filters that can be represented as an infimum of filters of the class $\mathbb{B}$.

Corollary 8.9. *Let $\mathbb{B}$ be an $\mathbb{F}_0$-composable class of filters and let τ be a solid hyperconvergence.*

1. *If τ is functionally separated and $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$ then τ is $\mathbb{B}$-based at X.*
2. *If $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$ then $P\tau$ is $\mathbb{B}^{\wedge}$-based at X.*
3. *If $\mathbb{B}$ is \$-compatible and $[X, \mathbb{R}]$ is $\mathbb{B}$-based at $\overline{0}$, then $[X, \$]$ is $\mathbb{B}$-based at X.*

Proof. (1). Let α be a functionally separated filter on $C(X, \$)$ such that $X \in \lim_{\tau} \alpha$. By Corollary 6.4, $\overline{0} \in \lim_{\tau^{\Uparrow}} [\alpha, \mathcal{N}(0)]$. Therefore, there is $\mathcal{G} \in \mathbb{B}$ such that $\overline{0} \in \lim_{\tau^{\Uparrow}} \mathcal{G}$ and $\mathcal{G} \leq [\alpha, \mathcal{N}(0)]$, hence $X \in \lim_{\tau} \mathcal{G}^{-}(W)$. In view of Theorem 8.8,

$$\alpha = [\alpha, \mathcal{N}(0)]^{-}(W) \geq \mathcal{G}^{-}(W),$$

and $\mathcal{G}^{-}(W) \in \mathbb{B}$ by $\mathbb{F}_0$-composability.

(2). If, in the proof above, α is an ultrafilter, then the assumption of functional separation is not needed. Now the vicinity filter of X for $P\tau$ is

$$\begin{aligned} \mathcal{V}_{\tau}(X) &= \bigwedge \{\alpha : \alpha \in \mathbb{U}(C(X, \$)), X \in \lim_{\tau} \alpha\} \\ &= \bigwedge \left\{\mathcal{G}^{-}(W) : \alpha \in \mathbb{U}(C(X, \$)), X \in \lim_{\tau} \alpha\right\}. \end{aligned}$$

Therefore $\mathcal{V}_{\tau}(X) \in \mathbb{B}^{\wedge}$.

(3). In the proof of (1) above, if $\tau = [X, \$]$ then by Lemma 8.7, we can assume $\alpha = \mathcal{O}^{\natural}([\alpha, \mathcal{N}(0)]^{-}(W))^{\downarrow\downarrow} \geq \mathcal{O}^{\natural}(\mathcal{G}^{-}(W))^{\downarrow\downarrow}$. By \$-compatibility and $\mathbb{F}_0$-composability, $\mathcal{O}^{\natural}(\mathcal{G}^{-}(W))^{\downarrow\downarrow} \in \mathbb{B}$ and $X \in \lim_{[X,\$]} \mathcal{O}^{\natural}(\mathcal{G}^{-}(W))^{\downarrow\downarrow}$ by Proposition 5.2. □

Combining Corollaries 8.5 and 8.9, we obtain:

[15] where α is a collection of compact families including all the finitely generated ones

Corollary 8.10. *Let $\mathbb{B}$ be an $\mathbb{F}_0$-composable class of filters.*

(1) *Let τ be a functionally separated solid hyperconvergence.*
   (a) *If $\mathbb{B}$ is countably upper closed then $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$ if and only if τ is $\mathbb{B}$-based at X.*
   (b) *If $\mathbb{B}$ is $\mathbb{F}_1$-steady and if τ is pretopological, then $\tau^{\Uparrow}$ is $\mathbb{B}$-based at $\overline{0}$ if and only if τ is $\mathbb{B}$-based at X.*

(2) *If $\mathbb{B}$ is countably upper closed and \$-compatible, then $[X,\mathbb{R}]$ is $\mathbb{B}$-based if and only if $[X,\$]$ is $\mathbb{B}$-based at X.*

In view of Lemma 8.7, we have in particular:

Corollary 8.11. *Let $\mathbb{B}$ be an $\mathbb{F}_0$-composable class of filters.*

(1) *If $\mathbb{B}$ is $\mathbb{F}_1$-steady and $\alpha \subset \kappa(X)$, then $C_\alpha(X,\mathbb{R})$ is $\mathbb{B}$-based at $\overline{0}$ if and only if $C_\alpha(X,\$)$ is $\mathbb{B}$-based at X.*

(2) *If $\mathbb{B}$ is countably upper closed and either X is normal or $\mathbb{B}$ is \$-compatible, then $[X,\mathbb{R}]$ is $\mathbb{B}$-based if and only if $[X,\$]$ is $\mathbb{B}$-based at X.*

In particular, if $\mathcal{D}$ is a hereditarily closed ([16]) compact network on a completely regular space X, we consider $\alpha_{\mathcal{D}} := \mathcal{O}_X^{\natural}(\mathcal{D})$.Then $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ is a topological group (e.g., [33, Theorem 1.1.7]) and if γ is a cardinal function corresponding to a $\mathbb{F}_1$-steady and $\mathbb{F}_0$-composable class of filters, like *character* χ, *tightness* t, *fan-tightness* vet, and *strong fan-tightness* vet*, then

$$\gamma(C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})) = \gamma(C_{\alpha_{\mathcal{D}}}(X,\$),X). \tag{8.3}$$

As mentioned before, translations need not be continuous for the Isbell topology on $C(X,\mathbb{R})$. However, the fine Isbell topology $\overline{\kappa}(X,\mathbb{R})$ is always translation-invariant and the neighborhood filter of f for the fine Isbell topology is $f+\mathcal{N}_\kappa(\overline{0})$ [16, Theorem 4.1], which implies that the translations are continuous for the Isbell topology if and only if this Isbell topology coincides with the fine Isbell topology.

On the other hand, for every X there exists the finest translation-invariant topology $\Sigma(X,\mathbb{R})$ that is an $\mathbb{R}$-dual topology of $\Sigma(X) \subset \kappa(X)$, hence coarser than the Isbell topology $\kappa(X,\mathbb{R})$ [15]. Therefore

$$\gamma(C_{\overline{\kappa}}(X,\mathbb{R})) = \gamma(C_\kappa(X,\$),X) \text{ and } \gamma(C_\Sigma(X,\mathbb{R})) = \gamma(C_\Sigma(X,\$),X).$$

We will see in the next section that calculating invariants for $C_{\alpha_{\mathcal{D}}}(X,\$), C_\kappa(X,\$)$ and $[X,\$]$ in terms of X is often easy. This way, we will recover a large number of known results, as well as obtain new ones.

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[16]that is, if $D \in \mathcal{D}$ and B is a closed subset of D, then $B \in \mathcal{D}$.

9. Character and tightness

Theorem 9.1. *(e.g.,* [35]*) The tightness and the character of* $[X,\$]$ *coincide.*

Proof. As the tightness is not greater than character, we need only prove that $\chi([X,\$],Y) \le t([X,\$],Y)$. Assume that $t([X,\$],Y) = \lambda$ and let $Y \in \lim_{[X,\$]} \gamma$. By Proposition 5.2, there exists an ideal subbase $\mathcal{P}$ that is an open cover of Y such that $Y \in \lim_{[X,\$]} \mathcal{O}_X^\natural(\mathcal{P})$ and $\mathcal{O}_X^\natural(\mathcal{P}) \le \gamma$. It is clear that $\mathcal{P}\#\mathcal{O}_X^\natural(\mathcal{P})$, hence there is a family $\mathcal{S}_0 \subset \mathcal{P}$ such that $\operatorname{card}\mathcal{S}_0 \le \lambda$ and $\mathcal{S}_0\#\mathcal{O}_X^\natural(\mathcal{P})$. The family $\mathcal{S} := \mathcal{S}_0^\cup$ is a subfamily of $\mathcal{P}$, because $\mathcal{P}$ is an ideal, $\operatorname{card}\mathcal{S} \le \lambda$ and, a fortiori $\mathcal{S}\#\mathcal{O}_X^\natural(\mathcal{P})$. In view of Proposition 5.1, $\mathcal{O}_X^\natural(\mathcal{P}) \le \mathcal{O}_X^\natural(\mathcal{S})$ and $\mathcal{O}_X^\natural(\mathcal{S})$ is a filter-base, so that $Y \in \lim_{[X,\$]} \mathcal{O}_X^\natural(\mathcal{S})$. Moreover $\mathcal{O}_X^\natural(\mathcal{P}) \ge \mathcal{O}_X^\natural(\mathcal{S})$ because $\mathcal{S} \subset \mathcal{P}$, so that $\mathcal{O}_X^\natural(\mathcal{P}) = \mathcal{O}_X^\natural(\mathcal{S})$ has a filter base of cardinality not greater than λ. □

An immediate consequence of Corollary 5.5 and Theorem 9.1 is (the known fact [35]) that

$$t([X,\$],U) = \chi([X,\$],U) = L(U) \tag{9.1}$$

at each $U \in C(X,\$)$, where $L(U)$ is the Lindelöf degree of U.

The *α-Lindelöf number* $\alpha L(U)$ of a subset U of X is the smallest cardinal λ such that every open α-cover of U has an α-subcover of U of cardinality not greater than λ. In view of Corollary 5.5, we have if $p(X) \subset \alpha \subset \kappa(X)$, then an ideal base $\mathcal{P} \subset C(X,\$)$ is an open cover of $U \in C(X,\$)$ if and only if it is an α-cover of U. Therefore

$$L(U) \le \alpha L(U) \tag{9.2}$$

for each open subset U of X.

It follows immediately from Proposition 5.7 that

$$\alpha L(U) = t(\alpha(X,\$),U). \tag{9.3}$$

In view of Corollary 8.11 (1) and of the fact that the class $\mathbb{F}_1^\Diamond$ is $\mathbb{F}_1$-steady and $\mathbb{F}_0$-composable, we obtain:

Theorem 9.2. *Let α be a topology on $C(X,\$)$ such that $p(X) \subset \alpha \subset \kappa(X)$. Then*

$$\alpha L(X) = t(\alpha(X,\$),X) = t(C_\alpha(X,\mathbb{R}),\overline{0}).$$

A similar result was announced in [12, Corollary 3.3], but the provided proof was not correct. In particular, if $\alpha = \alpha_{\mathcal{D}}$ where $\mathcal{D}$ is a network of compact subsets of X, then $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ is a topological group and

$$\alpha_{\mathcal{D}} L(X) = t\left(C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})\right). \tag{9.4}$$

This is exactly [33, Theorem 4.7.1]. Indeed, in [33], a $\mathcal{D}$-cover of X ([17]) is a family of subsets of X such that every member of $\mathcal{D}$ is contained in some members of this family. McCoy and Ntantu define the $\mathcal{D}$-Lindelöf degree of X as the least cardinality λ such that every open $\mathcal{D}$-cover has a $\mathcal{D}$-subcover of cardinality not greater than λ, and establish that $t\,(C_{\alpha_{\mathcal{D}}}(X,\mathbb{R}))$ is equal to the $\mathcal{D}$-Lindelöf degree of X [33, Theorem 4.7.1]. It is immediate that the $\mathcal{D}$-Lindelöf degree of X is equal to $\alpha_{\mathcal{D}}L(X)$. Instances include:

Corollary 9.3. *(e.g., R. McCoy and I. Ntantu* [33, Corollary 4.7.2]*) $C_k(X,\mathbb{R})$ is countably tight if and only if every open k-cover has a countable k-subcover.*

Corollary 9.4. *(e.g., A. V. Arhangel'skii* [3]*) The following are equivalent:*

(1) *$C_p(X,\mathbb{R})$ is countably tight;*
(2) *every open ω-cover has a countable ω-subcover;*
(3) *X^n is Lindelöf for every $n\in\omega$.*

Note that (2) $\Longleftrightarrow$ (3) in the corollary above uses the observation that

$$pL(X)=\sup\{L(X^n):n\in\omega\}, \tag{9.5}$$

a proof of which can be found for instance in [33, Corollary 4.7.3.].

Proposition 9.5. $\kappa L(U)=t(\kappa(X,\$),U)=t([X,\$],U)=L(U)$.

Proof. In view of $T[X,\$]=\kappa(X,\$)$ and of (9.1),

$$t(\kappa(X,\$),U)=t(T[X,\$],U)\le t([X,\$],U)=\chi([X,\$],U)=L(U),$$

because $t(X)\ge t(PX)\ge t(TX)$ for any convergence space X [35, Proposition 2.1]. In view of Theorem 9.2 and (9.2)

$$L(U)\le\kappa L(U)=t(\kappa(X,\$),U).$$

$\square$

In particular $L(X)=\kappa L(X)$, hence for the Isbell topology $\kappa(X,\mathbb{R})$ and fine Isbell topology $\overline{\kappa}(X,\mathbb{R})$, we conclude that

Corollary 9.6.

$$L(X)=t(C_\kappa(X,\mathbb{R}),\overline{0})=t(C_{\overline{\kappa}}(X,\mathbb{R})).$$

[17] where $\mathcal{D}$ is a network of closed subsets of X, that is, a family of subsets of X such that every member of $\mathcal{D}$ is contained in some member of this family.

It was shown by R. Ball and A. Hager in [5] that if X is Čech-complete then $t(C_k(X,\mathbb{R})) = L(X)$. We are now in a position to refine this result.

A topological space is called *consonant* [14] if the compact-open and Scott topologies coincide on $C(X,\$)$:

$$T[X,\$] = C_k(X,\$).$$

In other words, X is consonant if and only if every compact family on X is a union of families of the form $\mathcal{O}(K_i)$ for some compact subsets K_i of X. Every Čech-complete space is consonant [14, Theorem 4.1], but not conversely.

Corollary 9.7. *If X is a (completely regular) consonant topological space then*

$$t(C_k(X,\mathbb{R})) = L(X).$$

Proof. In view of (8.3), we have $t(C_k(X,\mathbb{R})) = t\,(C_k(X,\$),X)$. But $t\,(C_k(X,\$),X) = t\,(T[X,\$],X) = L(X)$, which concludes the proof. □

The natural convergence $[X,\mathbb{R}]$ is a convergence group, in particular translation-invariant. Therefore, in view of Corollary 8.11 (2),

$$\chi([X,\mathbb{R}]) = \chi([X,\$],X), \tag{9.6}$$

because the class $\mathbb{F}_\lambda$ is \$-compatible, $\mathbb{F}_0$-composable, and countably upper closed for every cardinal λ. Although the class of countably tight filters is not countably upper closed, we are in a position to see that $t\,([X,\mathbb{R}]) = t([X,\$],X)$. Indeed, $t([X,\mathbb{R}]) \leq \chi([X,\mathbb{R}])$ and, in view of Corollary 8.9 (2), $t(P[X,\$],X) \leq t([X,\mathbb{R}])$, because $(\mathbb{F}_1^{\Diamond})^{\wedge} = \mathbb{F}_1^{\Diamond}$. Therefore

$$\begin{aligned} L(X) &= t(T[X,\$],X) \leq t(P[X,\$],X) \leq t([X,\mathbb{R}]) \\ &\leq \chi([X,\mathbb{R}]) = \chi([X,\$],X) = L(X). \end{aligned}$$

Corollary 9.8.

$$\begin{aligned} L(X) &= \chi([X,\$],X) = t([X,\$],X) = t(T[X,\$],X) \\ &= \chi([X,\mathbb{R}]) = t([X,\mathbb{R}]). \end{aligned}$$

Note that $L(X) = \chi([X,\mathbb{R}])$ is a corollary of [19, Theorem 1] of W. Feldman. However, the surprising fact that $\chi([X,\mathbb{R}]) = t([X,\mathbb{R}])$ seems to be entirely new.

As we have seen, character and tightness coincide for $[X,\$]$ as well as for $[X,\mathbb{R}]$, but they do not for $\alpha(X,\$)$ (and therefore not for $\alpha(X,\mathbb{R})$). By definition the character of $C_\alpha(X,\$)$ at U does not exceed λ if there is $\{\mathcal{A}_\beta : \beta \leq \lambda\} \subset \alpha$ such that $U \in \mathcal{A}_\beta$ for each β and for each $\mathcal{A} \in \alpha$ such that $U \in \mathcal{A}$, there is $\beta \leq \lambda$ such that $\mathcal{A}_\beta \subset \mathcal{A}$. In particular $\chi(C_\alpha(X,\$),X) \leq \lambda$ if there is a subset γ of

α of cardinality at most λ such that each element of α contains an element of γ. In the particular case where $\alpha = \alpha_{\mathcal{D}}$ for a network $\mathcal{D}$ of closed subsets of X, the condition above translates to: $\chi(C_{\alpha_{\mathcal{D}}}(X,\$),X) \leq \lambda$ if there is $\mathcal{S} \subset \mathcal{D}$ with $|\mathcal{S}| \leq \lambda$ such that every element of $\mathcal{D}$ is contained in an element of $\mathcal{S}$, that is, if $\mathcal{D}$ contains a $\mathcal{D}$-cover (in the sense of [33]) of cardinality at most λ. In other words,

$$\chi(C_{\alpha_{\mathcal{D}}}(X,\$),X) = \mathcal{D}a(X),$$

where $\mathcal{D}a(X)$ is the $\mathcal{D}$-Arens number of X, as defined in [33]. In view of Corollary 8.11 (1), we recover [33, Theorem 4.4.1] of R. McCoy and I. Ntantu:

Corollary 9.9. *If $\mathcal{D}$ is a network of compact subsets of X then:*

$$\chi(C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})) = \chi(C_{\alpha_{\mathcal{D}}}(X,\$),X) = \mathcal{D}a(X).$$

Since $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ is a topological group it is metrizable whenever it is first-countable, by the Birkhoff-Kakutani theorem. Therefore, instances of this result include that $C_p(X,\mathbb{R})$ is metrizable if and only if X is countable, and that $C_k(X,\mathbb{R})$ is metrizable if and only if X is hemicompact.

We can more generally define, for $\alpha \subset \kappa(X)$, the α-Arens number $\alpha a(X)$ of X as the least cardinal λ such that there is a subset γ of α of cardinality at most λ such that each element of α contains an element of γ, and we have

$$\chi(C_\alpha(X,\mathbb{R}),\overline{0}) = \chi(C_\alpha(X,\$),X) = \alpha a(X).$$

The α-Arens number seems however somewhat intractable unless $\alpha = \alpha_{\mathcal{D}}$ for a network $\mathcal{D}$ of closed subsets of X.

10. Fan-tightness and strong fan-tightness

Fan-tightness vet and strong fan-tightness vet* have been defined in Section 7. We have seen that they correspond to classes of filters that are $\mathbb{F}_1$-steady and $\mathbb{F}_0$-composable. Hence, Corollary 8.11 (1) applies to the effect that

$$\begin{aligned} \text{vet}(C_\alpha(X,\$),X) &= \text{vet}(C_\alpha(X,\mathbb{R}),\overline{0}); \\ \text{vet}^*(C_\alpha(X,\$),X) &= \text{vet}^*(C_\alpha(X,\mathbb{R}),\overline{0}). \end{aligned} \tag{10.1}$$

Let α be a topology on $C(X,\$)$. We define the

(1) *α-Menger number* $\alpha M(X)$ of X to be the least ordinal λ such that if for each family $\{\mathcal{P}_\gamma : \gamma < \lambda\}$ of open α-covers of X there are subsets $\mathcal{V}_\gamma \subset \mathcal{P}_\gamma$ of cardinality less than λ for each $\gamma < \lambda$, such that $\bigcup_{\gamma<\lambda} \mathcal{V}_\gamma$ is an α-cover of X.

(2) *α-Rothberger number $\alpha R(X)$ of X* to be the least ordinal λ such that for each family $\{\mathcal{P}_\gamma : \gamma < \lambda\}$ of open α-covers of X there are $P_\gamma \in \mathcal{P}_\gamma$ for each $\gamma < \lambda$, such that $\{P_\gamma : \gamma < \lambda\}$ is an α-cover of X.

In this terminology, in view of Proposition 5.7, the definitions readily rephrase as:

$$\mathrm{vet}(C_\alpha(X,\$),U) = \alpha M(U), \tag{10.2}$$

$$\mathrm{vet}^*(C_\alpha(X,\$),U) = \alpha R(U), \tag{10.3}$$

for each open subset U of X. In particular, [32, Theorem 1] and [32, Theorem 2] of G. Di Maio, L. Kočinac and E. Meccariello, stating that $cC_p(X,\$)$ and $cC_k(X,\$)$ have countable strong fan-tightness if and only if $pR(U) = \omega$ and $kR(U) = \omega$ for each open subset U of X, respectively, are instances of (10.3) for $\alpha = p(X)$ and $\alpha = k(X)$. Similarly, [32, Theorem 9] and [32, Theorem 10] characterizing countable fan-tightness of $cC_p(X,\$)$ and $cC_k(X,\$)$ respectively, are instance of (10.2) for $\alpha = p(X)$ and $\alpha = k(X)$ respectively.

Combining (10.1) and (10.2), we have:

$$\mathrm{vet}(C_\alpha(X,\$),X) = \mathrm{vet}(C_\alpha(X,\mathbb{R}),\overline{0}) = \alpha M(X); \tag{10.4}$$

$$\mathrm{vet}^*(C_\alpha(X,\$),X) = \mathrm{vet}^*(C_\alpha(X,\mathbb{R}),\overline{0}) = \alpha R(X). \tag{10.5}$$

Let $s = \{\mathcal{O}(x) : x \in X\}$. Note that $C_s(X,\$) = C_p(X,\$)$. An infinite topological space X has the Menger property (also sometimes called Hurewicz property, e.g. [4]) if and only if $sM(X) := M(X) = \omega$ and X has the Rothberger property (e.g., [34], [39]) if and only if $sR(X) := R(X) = \omega$. An argument similar to [33, Corollary 4.7.3.] was used to show (9.5) and can be adapted to show that

$$\begin{aligned} pM(X) &= \sup\{M(X^n) : n \in \omega\}; \\ pR(X) &= \sup\{R(X^n) : n \in \omega\}. \end{aligned} \tag{10.6}$$

Note that (10.4) particularizes to [31, Theorem 1] of S. Lin when $\alpha = \alpha_{\mathcal{D}}$ where $\mathcal{D}$ is a network of compact subsets of X. Combined with (10.6), we obtain:

Corollary 10.1. *(A.V. Arhangel'skii* [4]*, S. Lin* [31, Theorem 2]*)*

$$\mathrm{vet}(C_p(X,\mathbb{R})) = \sup\{M(X^n) : n \in \omega\},$$

so that $C_p(X,\mathbb{R})$ is countably fan-tight if and only if X^n has the Menger property for each $n < \omega$.

Corollary 10.2. *(M. Sakai* [38]*)*

$$\mathrm{vet}^*(C_p(X,\mathbb{R})) = \sup\{R(X^n) : n \in \omega\},$$

so that $C_p(X,\mathbb{R})$ is countably strongly fan-tight if and only if X^n has the Rothberger property for each $n<\omega$.

On the other hand, for $\alpha = k(X)$, we obtain in particular:

Corollary 10.3. *(L. Kočinac [30])*

(1) *$C_k(X,\mathbb{R})$ is countably fan-tight if and only if for every sequence $(\mathcal{P}_n)_{n<\omega}$ of k-covers, there are finite subsets $\mathcal{V}_n \subset \mathcal{P}_n$ for each n such that $\bigcup_{n<\omega}\mathcal{V}_n$ is a k-cover.*

(2) *$C_k(X,\mathbb{R})$ is countably strongly fan-tight if and only if for every sequence $(\mathcal{P}_n)_{n<\omega}$ of k-covers, there are $P_n \in \mathcal{P}_n$ for each n such that $\{P_n : n<\omega\}$ is a k-cover.*

11. Fréchet properties

An obstacle to applying the results of Section 8 to the Fréchet property is that the class of Fréchet filters, while $\mathbb{F}_0$-composable, fails to be $\mathbb{F}_1$-steady. The results apply to the strong Fréchet property though, whose associated class of filters is both $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady. We have seen that tightness and character coincide for $[X,\$]$ and $[X,\mathbb{R}]$. Therefore these spaces are Fréchet if and only if they are strongly Fréchet if and only if they are countably tight if and only if they are first-countable. On the other hand,

Theorem 11.1. *Let $p(X) \subset \alpha \subset \kappa(X)$. The following are equivalent:*

(1) *$C_\alpha(X,\mathbb{R})$ is strongly Fréchet at $\overline{0}$;*

(2) *$C_\alpha(X,\$)$ is strongly Fréchet at X;*

(3) *For every decreasing sequence $(\mathcal{P}_n)_{n\in\omega}$ of open α-covers, for each $n<\omega$ there exists $P_n \in \mathcal{P}_n$ so that each $\mathcal{A}\in\alpha$ contains all but finitely many of the elements of $(P_n)_{n\in\omega}$.*

Proof. The equivalence between (1) and (2) follows from Corollary 8.11 (1), and the equivalence between (2) and (3) follows immediately from the definition of strongly Fréchet and Proposition 5.7. □

The Fréchet property for function spaces can nevertheless be characterized with our results in the special case of $\alpha = \alpha_{\mathcal{D}}$ for a network $\mathcal{D}$.

Following [24], we call a topological space X *Fréchet-Urysohn for finite sets at* $x\in X$, or FU_{fin} at x, if for any $\mathcal{P}\subset [X]^{<\infty}$ such that each $U\in\mathcal{O}_X(x)$ contains an element of $\mathcal{P}$, there is a sequence $(P_n)_{n\in\omega}\subset\mathcal{P}$ such that each $U\in\mathcal{O}_X(x)$ contains all but finitely many elements of $(P_n)_{n\in\omega}$.

We call a filter $\mathcal{F}$ an *FU_{fin}-filter* if for any $\mathcal{P} \subset [X]^{<\infty}$ such that $\mathcal{P} \geq \mathcal{F}$, there is a sequence $(P_n)_{n\in\omega} \subset \mathcal{P}$ such that $(P_n)_{n\in\omega} \geq \mathcal{F}$. Let $\mathbb{FU}_{fin}$ denote the corresponding class of filters. Clearly, a space is FU_{fin} at x if it is $\mathbb{FU}_{fin}$-based at x.

Theorem 11.2. *Let $\mathcal{D}$ be a network of compact subsets of X and $Y \in C(X,\$)$. If $C_{\alpha_{\mathcal{D}}}(X,\$)$ is Fréchet at Y then $C_{\alpha_{\mathcal{D}}}(X,\$)$ is Fréchet-Urysohn for finite sets at Y.*

Proof. Let β be a family of finite subsets of $C(X,\$)$ such that for each $D \in \mathcal{D}$ containing Y, there is $\mathcal{P} \in \beta$ with $\mathcal{P} \subset \mathcal{O}_X(D)$. In other words, $D \subset \bigcap_{P\in\mathcal{P}} P$. Since the intersection is finite, $\bigcap_{P\in\mathcal{P}} P \in \mathcal{O}_X(D)$. Therefore, $Y \in \mathrm{cl}_{\alpha_{\mathcal{D}}} \left\{\bigcap_{P\in\mathcal{P}} P : \mathcal{P} \in \beta\right\}$. As $C_{\alpha_{\mathcal{D}}}(X,\$)$ is Fréchet at Y, there is a sequence $(\mathcal{P}_n)_{n\in\omega}$ of elements of β such that $Y \in \lim_{\alpha_{\mathcal{D}}} \left(\bigcap_{P\in\mathcal{P}_n} P\right)_{n\in\omega}$. In other words, for each $Y \subset D \in \mathcal{D}$, there is n_D such that $\bigcap_{P\in\mathcal{P}_n} P \in \mathcal{O}_X(D)$ for each $n \geq n_D$, so that $\mathcal{P}_n \subset \mathcal{O}_X(D)$ for each $n \geq n_D$, which proves that $C_{\alpha_{\mathcal{D}}}(X,\$)$ is FU_{fin} at Y. $\square$

The method of proof does not work for general topologies $\alpha(X,\$)$ with $\alpha \subset \kappa(X)$, because compact families do not need to be filters. In particular, there remains the following problem (of course, for dissonant X):

Problem 11.3. *Does the Fréchet property and the FU_{fin} property coincide for the Scott topology $C_\kappa(X,\$)$?*

It is known (e.g., [37]) that a FU_{fin} topological space is α_2 in the sense of [2] ([18]). Therefore Theorem 11.2 implies that in $C_{\alpha_{\mathcal{D}}}(X,\$)$ the Fréchet property implies α_2, and *a fortiori* α_3 and α_4, in particular implies the strong Fréchet property.

Lemma 11.4. *The class $\mathbb{FU}_{fin}$ is $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady.*

Proof. Let $\mathcal{F} \in \mathbb{FU}_{fin}(X)$, $A \subset X \times Y$ and let $\mathcal{P} \subset [Y]^{<\infty}$ such that $\mathcal{P} \geq A\mathcal{F}$. In other words, for each $F \in \mathcal{F}$ there is $P_F \in \mathcal{P}$ such that $P_F \subset AF$. Hence for each $y \in P_F$ there is $x_y \in F$ such that $(x_y, y) \in A$. Let $Q_F := \{x_y : y \in P_F\}$ and let $\mathcal{Q} := \{Q_F : F \in \mathcal{F}\}$. Then $\mathcal{Q} \subset [X]^{<\infty}$ such that $\mathcal{Q} \geq \mathcal{F}$. Therefore there is a sequence $(F_n)_{n\in\omega} \subset \mathcal{F}$ such that $(Q_{F_n})_{n\in\omega} \geq \mathcal{F}$. It is easy to see that $(P_{F_n})_{n\in\omega} \geq A\mathcal{F}$, which shows that $\mathbb{FU}_{fin}$ is $\mathbb{F}_0$-composable.

[18] A topological space X has property α_2 (at x) if for each sequence $(\sigma_n)_{n\in\omega}$ of sequences converging to x, there is a sequence σ convergent to x such that for each $n \in \omega$, the set $\sigma_n \cap \sigma$ is infinite.

The class $\mathbb{FU}_{fin}$ is $\mathbb{F}_0$-steady because if $\mathcal{P} \geq A \vee \mathcal{F}$ there is $\mathcal{P}_0 \subset \mathcal{P}$ such that $\mathcal{P}_0 \geq A \vee \mathcal{F}$ and $\mathcal{P}_0 \subset [A]^{<\infty}$. Moreover, by [37] or [24, Theorem 20], $\mathbb{FU}_{fin} \times \mathbb{F}_1 \subset \mathbb{FU}_{fin}$ (in the sense of [29]). By [29, Theorem 20(1)], $\mathbb{FU}_{fin}$ is also $\mathbb{F}_1$-steady. □

Theorem 11.5. *Let $\mathcal{D}$ be a network of compact subsets of X. The following are equivalent:*

(1) $C_{\alpha_{\mathcal{D}}}(X,\$)$ *is Fréchet at X;*
(2) $C_{\alpha_{\mathcal{D}}}(X,\$)$ *is FU_{fin} at X;*
(3) $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ *is FU_{fin};*
(4) $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ *is Fréchet;*
(5) *For every open $\mathcal{D}$-cover $\mathcal{C}$ of X, there exists a countable subfamily $\mathcal{S}$ of $\mathcal{C}$ such that every $D \in \mathcal{D}$ is contained in all but finitely many elements of $\mathcal{S}$.*

Proof. (1) $\iff$ (2) follows from Theorem 11.2. (1) $\iff$ (5) follows immediately from the definitions. (2) $\iff$ (3) follows from Corollary 8.11 (1), because the class of FU_{fin} filters is $\mathbb{F}_0$-composable and $\mathbb{F}_1$-steady and $C_{\alpha_{\mathcal{D}}}(X,\mathbb{R})$ is a topological group. (3) $\implies$ (4) and (2) $\implies$ (1) are obvious, and (4) $\implies$ (1) follows from Corollary 8.9 (2), because $\mathbb{F}_0^{\triangle} = (\mathbb{F}_0^{\triangle})^{\wedge}$. □

Note that the equivalence (4) $\iff$ (5) is [33, Theorem 4.7.4] of R. McCoy and I. Ntantu. In the case $\alpha_{\mathcal{D}} = p(X)$, the equivalences (3) $\iff$ (4) $\iff$ (5) are due to J. Gerlitz and Z. Nagy [20].

The case $\alpha_{\mathcal{D}} = p(X)$ generalizes [10, Proposition 5 (1)] of G. Di Maio, L. Kočinac and T. Nogura, stating that $cC_p(X,\$)$ is α_2 whenever it is Fréchet. On the other hand, when $\alpha_{\mathcal{D}} = k(X)$, [10, Proposition 5 (2)] is generalized in two ways: we only need to assume that $cC_k(X,\$)$ is Fréchet (rather than the more stringent condition of strict Fréchetness) and we obtain that $cC_k(X,\$)$ is FU_{fin} rather than α_2.

Note however that while the Fréchet property is equivalent to sequentiality and even to being a k-space for $C_p(X,\mathbb{R})$ and $C_k(X,\mathbb{R})$ (e.g., E. G. Pytkeev [36]), these properties are not equivalent for the corresponding hyperspaces. For instance, an example of a space X for which $C_k(X,\$)$ is sequential but not Fréchet is given in [8, p. 275] by L. Holá, C. Costantini and P. Vitolo. Therefore, the results of Section 8 in general do not apply to sequentiality.

12. Appendix: dual convergences

We have seen that each non-degenerate $\alpha \subset C(X,\$)$ composed of openly isotone families defines a *Z-dual topology* $\alpha(X,Z)$ on $C(X,Z)$ via (3.2). Note that $f \in \lim_{\alpha(X,Z)} \mathcal{F}$ if and only if

$$\forall_{O \in \mathcal{O}_Z} \forall_{\mathcal{A} \in \alpha}\ f \in [\mathcal{A}, O] \Longrightarrow [\mathcal{A}, O] \in \mathcal{F}. \tag{12.1}$$

In view of the characterization (3.5) of the natural convergence, it is natural to consider for each collection α of (openly isotone) families on X the *Z-dual convergence* $[\alpha, Z]$ defined by: $f \in \lim_{[\alpha,Z]} \mathcal{F}$ if and only if

$$\forall_{O \in \mathcal{O}_Z} \forall_{\mathcal{A} \in \alpha}\ f \in [\mathcal{A}, O] \Longrightarrow \exists_{A \in \mathcal{A}}\ [A, O] \in \mathcal{F}. \tag{12.2}$$

Distinct collections α of families of open sets generate distinct topologies on $C(X,Z)$ provided that the elements of $C(X,Z)$ separate these families in X. Such a separation is assured for example by the Z-regularity of X and the compactness of the elements of α (see [15, Proposition 2.1]). In contrast, all the collections α including $p(X)$ and included in $\kappa(X)$ give rise the same convergence, which turns out to be the *natural convergence*.

Theorem 12.1. *The dual convergence $[\alpha, Z]$ is equal to the natural convergence $[X, Z]$ for each collection α such that $p(X) \leq \alpha \leq \kappa(X)$.*

Proof. We first show that $[X,Z] \geq [\kappa(X), Z]$. To this end, assume that $f_0 \in \lim_{[X,Z]} \mathcal{F}$ and let $f_0 \in [\mathcal{A}, O]$ where O is Z-open and $\mathcal{A} \in \kappa(X)$. It follows that $f_0^-(O) \in \mathcal{A}$. If $x \in f_0^-(O)$ then there is $V_x \in \mathcal{O}(x)$ such that $V_x \subset f_0^-(O)$ and $[V_x, O] \in \mathcal{F}$. By the compactness of $\mathcal{A}$, there is a finite subset B of $f_0^-(O)$ such that $V := \bigcup_{x \in B} V_x \in \mathcal{A}$. On the other hand, $[V, O] = \bigcap_{x \in B} [V_x, O] \in \mathcal{F}$ showing that $f_0 \in \lim_{[\kappa(X),Z]} \mathcal{F}$.

As $[\kappa(X), Z] \geq [p(X), Z]$, it is now enough to show that $[p(X), Z] \geq [X, Z]$. Suppose that $f_0 \in \lim_{[p(X),Z]} \mathcal{F}$ and let $x \in X, O \in \mathcal{O}_Z$ be such that $f_0 \in [x, O]$, equivalently $f_0^-(O) \in \mathcal{O}_X(x)$, or else, $f_0 \in [\mathcal{O}_X(x), O]$. By the assumption, there is $V \in \mathcal{O}_X(x)$ such that $[V, O] \in \mathcal{F}$, that is, $f_0 \in \lim_{[X,Z]} \mathcal{F}$. □

Note that, since $[A, O] \subset [\mathcal{A}, O]$ for each $A \in \mathcal{A}$,

$$[\alpha, Z] \geq T[\alpha, Z] \geq \alpha(X, Z). \tag{12.3}$$

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